Communications in Applied Analysis **12** (2008), no. 4, 459–466

STABILITY THEORY FOR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

V. LAKSHMIKANTHAM AND S. LEELA

Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901 USA
lakshmik@fit.edu

1. INTRODUCTION

The theory of fractional differential equations has been recently developed and the existing fundamental results have been presented in a unified way in a research monograph [4]. This provides us with an advantage to understand and appreciate the intricacies involved so as to pave the way for further development of this important subject as an independent branch of nonlinear analysis.

In a recent paper, [3] we have discussed the relation between the solutions of fractional differential equations and the solutions of ordinary differential equations. Since the properties of solutions of ODEs is relatively easier to investigate due to its well developed theory, the advantage of the relationship between the two systems is quite clear. In this paper, we shall consider multi-order fractional differential equations, develop needed mechanism of generalized spaces, utilize the method of vector Lyapunov functions and use the modified procedure discussed in [3] to study stability theory of multi-order fractional differential systems by relating them to the corresponding ordinary differential systems. We believe that the mechanism developed in this paper would be very useful to investigate further multi-order systems to obtain various other results.

2. MULTI-ORDER FRACTIONAL DIFFERENTIAL SYSTEMS

The systems of fractional differential equations where every component of the vector $x \in \mathbb{R}^n$, has the same arbitrary order derivative $0 < q < 1$. They are studied utilizing the Euclidean norm in $\mathbb{R}^n$ to obtain the estimates on the solutions in order to investigate the qualitative properties of solutions as a whole [4]. However, if each component has a different arbitrary order derivative such that $D^{q_1}x_1, D^{q_2}x_2, \ldots, D^{q_n}x_n$ or certain groups of components have the same order, then the foregoing investigation

Received October 1, 2008

cannot apply for such a decomposition. We need to develop a mechanism to handle the situation of multi-order fractional differential systems. For this purpose, we need to first define the concept of a generalized space with a generalized norm whose values are in R_+^k for some $k > 1$ and split the multi-order fractional differential system suitably to yield as much information as possible.

We begin with the following definition [1].

Definition 1. Let E be a real vector space. A generalized norm for E is a mapping $|\cdot|_G : E \to R_+^k$ denoted by

$$|x|_G = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)),$$

such that

(a) $|x|_G \geq 0$, that is, $\alpha_i(x) \geq 0$ for all $i = 1, 2, \ldots, k$;
(b) $|x|_G = 0$ if and only if $x = 0$, that is, $\alpha_i(x) = 0$ for all i, if and only if $x = 0$;
(c) $|\lambda x|_G = |\lambda||x|_G$, that is, $\alpha_i(\lambda x) = |\lambda|\alpha_i(x)$;
(d) $|x + y|_G \leq |x|_G + |y|_G$, which means, $\alpha_i(x + y) \leq \alpha_i(x) + \alpha_i(y)$.

One can also define $\alpha_i(x) = |x_{n_i}|$, where $\sum_{i=1}^{k} n_i = n$, when E is $\mathbb{R}^n$, to split the system in $\mathbb{R}^n$ into k-subsystems, each of which will have n_i components for each i.

For each $x \in E$ and $\epsilon \in R_+^k, \epsilon > 0$, let

$$B_\epsilon(x) = [y \in E : \|y - x\|_G < \epsilon].$$

Then $[B_\epsilon(x) : x \in E, \epsilon \in R_+^k, \epsilon > 0]$ is a basis for a topology on E.

Remark 2.1. It is not difficult to see that every generalized normed space $(E, \|\cdot\|G)$ has an equivalent (ordinary) norm. For example in $\mathbb{R}^2$, $\|x\|_G = (|x_1|, |x_2|)$ and $\|x\| = \max(|x_1|, |x_2|)$ are equivalent. For purely algebraic and topological considerations, it is immaterial whether we view E as a generalized norm space or an ordinary norm space. Such concepts as convexity, closure, completeness and compactness remain the same. We do, however, have more flexibility working with generalized spaces.

We shall need the following terminology.

Definition 2. An A-matrix is a nonnegative matrix S such that $I - S$ is positive definite.

A positive definite matrix S will be any $n \times n$ matrix such that $x.\ Sx > 0$ for all $x \in \mathbb{R}^n$. We will use the following properties of a positive definite matrix S:

(i) $\det S > 0$,
(ii) all the principal minors of S are positive definite,
(iii) if all the off-diagonal elements of S are non-positive then S^{-1} is nonnegative,

(iv) If $S \geq 0$, then $\sum_{n=0}^{\infty} S^n$ converges if and only if for some $m, I - S^m$ is positive definite in which case $(I - S)^{-1} = \sum_{n=0}^{\infty} S^n$.

We now state the Schauder fixed point theorem and contraction mapping theorem in a generalized normed space.

Theorem 2.1. *Let E be a generalized Banach space and let $F \subset E$ be closed and convex. If $T : F \to F$ is completely continuous, then T has a fixed point.*

Proof. In view of Remark 2.1, we may view E as an ordinary Banach space with an equivalent ordinary norm. Then Theorem 2.1 becomes the classical Schauder-Tychonoff theorem. □

Definition 3. Let E be a real vector space. A generalized metric for E is a mapping $d : E \times E \to R_+^k$ such that

(a) $d(x, y) = d(y, x)$
(b) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$, where x, y, z are any elements of E.

Theorem 2.2. *Let E be a complete generalized metric space and let $T : E \to E$ such that*

$$d(Tx, Ty) \leq Sd(x, y),$$

where S is a nonnegative matrix such that for some m, S^m is an A-matrix. Then T has a unique fixed point x^*. Further for any $x \in E$

$$x^* = \lim_{n \to \infty} T^n x$$

and

$$d(x^*, T^n x) \leq S^n (1 - S)^{-1} d(Tx, x).$$

We leave the proof as an exercise.

Corollary 2.3. *Let E be a complete generalized metric space and let $T : E \to E$ such that*

$$d(Tx, Ty) \leq Sd(x, y),$$

where S is a nonnegative matrix. If there is an $x_0 \in E$ such that $\sum_{n=0}^{\infty} S^n d(Tx_0, x_0)$ converges, then T has a fixed point x^* such that

$$x^* = \lim_{n \to \infty} T^n x_0.$$

We note that one needs to employ Theorems 2.1 and 2.2 in this frame work to yield existence and uniqueness results for the multi-order fractional differential systems. We do not prove these results and leave it to the readers for proving them.

Let us consider the general type of multi-order fractional differential system given by

$$^{c}D^{q}x = f(t,x), \qquad x(t_0) = x_0, \tag{1}$$

where $^{c}D^{q}$ is the Caputo derivative. Let $n = \sum_{i=1}^{k} n_i$, we split the system (1) such that for $i = 1, 2, \ldots, k$, we can represent n-system (1) into k-subsystems, each group consists of n_i components for each i. Then we have

$$^{c}D^{q_i}x_{n_i} = f_{ni}(t, x_{n1}, x_{n2}, \ldots, x_{ni}, \ldots, x_{nk}),\ x_{ni}(t_0) = x_{0ni}. \tag{2}$$

Clearly, each group consists of n_i components of the same order fractional derivative and have different number of components. We therefore have k-groups, each not necessarily having the same number of components of the vector. Hence utilizing k-valued generalized norm would help to obtain properties of each group which could be different.

If we employ the usual norm, we can not do so, because if one component has an unwanted property, we get the same for all, even though all other components may have a good property that is desired. This remark is also true even when all the components have the same arbitrary order. In fact, we cannot utilize the usual norm in the multi-order case. We can now state the following comparison result to deal with the case of (2), whose proof is similar to the proof of Theorem 4.2.2 in [4] with suitable modifications of the corresponding existence of extremal results.

Theorem 2.4. *Assume that for each $i = 1, 2, \ldots, k$,*

(i) $V_i \in C(R_+ \times R^{n_i},\ R^k_+)$, *each $V_i(t,x)$ is locally Lipschitzian in x and*

$$^{c}D^{q_i}_{+}V_i(t, x_{n_i}) \le g_i(t, V_1, V_2, \ldots, V_k),\ i = 1, 2, \ldots, k,$$

where $^{c}D^{q_i}_{+}V_i$ is the generalized derivative as usual [2] for each i and $g \in C(R_+ \times R^k_+, R^k_+)$ and $g(t,u)$ is quasimonotone nondecreasing in u for each t;

(ii) *the maximal solution $r_i(t, t_0, u_{0i})$ of the fractional differential system*

$$^{c}D^{q_i}u_i = g_i(t, u_i),\ u_i(t_0) = u_{0i} \ge 0 \tag{3}$$

exists on $[t_0, \infty)$. Then $V_i(t_o, x_{0n_i}) \le u_{0i}$ implies

$$V_i(t, x_{n_i}(t)) \le r_i(t, t_0, u_{0i}),\ t \ge t_0. \tag{4}$$

We note that, the system (3), is such that each component u_i has the fractional derivative of order q_i, which is the order for the group of n_i. components.

Having at our disposal the estimate (4), we can prove the following result so that one can extend to other situations appropriately. The type of multi-order systems defined above cover a variety of situations since they are general enough.

Theorem 2.5. *Let the conditions of Theorem* 2.4 *hold. Suppose further*

$$b_i(|x_{ni}|) \leq V_i(t, x_{ni}) \leq a_i(|x_{ni}|), \tag{5}$$

where b_i, a_i *are* $\mathcal{K}$*-class functions. Then if the comparison fractional differential system* (3) *has different qualitative property for each* u_i *component,* $i = 1, 2, \ldots, k$*, the corresponding group of* ni *components* n_i *of* $x \in \mathbb{R}^n$*, would have the same property.*

Proof. Since we employ different components of the generalized norm of any solution $x(t, t_0, x_0)$ of the IVP(2) and the corresponding $i = 1, 2, \ldots, k$, components of the comparison IVP (3), each group of n_i components being represented by a component u_i of (3), it is not difficult to mimic the proof of earlier proofs of Lyapunov method. Nonetheless, since the comparison IVP (3) is also the similar type of multi-order system, albeit easier, it is difficult to find the properties of the solutions of (3). Consequently, if we can relate such multi-order systems to a system of ODE, with order one, we can utilize the well known theory of such non-fractional ODE, it would be very helpful for the described theory of fractional differential systems, even simpler ones. This approach is discussed in the next section. □

3. STABILITY OF MULTI-ORDER SYSTEMS VIA ODEs

Let us follow the procedure outlined in [3] to the comparison fractional differential system (3), written in a vector form, for convenience,

$$^cD^q u = g(t, u) \ u(t_0) = u_0 \geq 0. \tag{6}$$

We recall that the use of Lyapunov function $V(t, x)$ demands $r(t, t_0, u_0)$, the maximal solution of the IVP (6) to be necessarily nonnegative for $t \geq t_0$, and therefore, we need to consider only the nonnegative solutions of (6). We wish to find desirable estimates on the solutions $u(t, t_0, u_0)$ of (6) in terms of solutions of certain ordinary differential systems that are comparatively easier to determine, because a lot of theory is well known for such ODEs. Our aim is therefore to obtain upper bounds for any solution $u(t, t_0, u_0)$ of the IVP (6) so that we can draw conclusions that are interesting and useful to us for the original multi-order fractional differential system (1) via the solutions of (6). As indicated above, we modify the technique developed in [3] suitably.

Let us first assume that the solutions $x(t, t_0, x_0)$ of (1) exist and unique for $t \geq t_0$. We shall also suppose that the solutions $u(t, t_0, u_0)$ of the comparison fractional differential system (6) also exist and are unique for $t \geq t_0$. Then we can utilize the relation

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \frac{d}{ds} u(s) ds, \tag{7} $$

to connect the IVP (6) to certain ODE to be obtained. Let us proceed (first temporarily) to suppose that

$$ u'(t) = u'(s) + \phi(t,s,q), \ ' = \frac{d}{dt}, \tag{8} $$

where the function $\phi(t,s,q)$ will be chosen later in an appropriate manner depending on our requirements. Then we get

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} [u'(t) - \phi(t,s,q)] ds, $$

which reduces to

$$ {}^cD^q u(t) = \frac{u'(t)(t-t_0)^{1-q}}{\Gamma(2-q)} - \eta(t,t_0,q), \tag{9} $$

where

$$ \eta(t,t_0,q) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \phi(t,s,q) ds. $$

Using the IVP (6), we arrive at

$$ u'(t) = G(t,u) + \tilde{\eta}(t,t_0,q), \ u(t_0) = u_0 \geq 0, \tag{10} $$

where

$$ \left. \begin{aligned} G(t,u) &= g(t,u)(t-t_0)^{q-1}\Gamma(2-q), \\ \tilde{\eta}(t,t_0,q) &= \eta(t,t_0,q)(t-t_0)^{q-1}\Gamma(2-q). \end{aligned} \right\} \tag{11} $$

By imposing various suitable estimates on $\tilde{\eta}(t,t_0,q)$, and choosing $g(t,u)$ appropriately, we can get bounds on $u(t,t_0,u_0)$ by means of the solutions of the corresponding ordinary differential system (10). As one possible simple choice, let us suppose that

$$ g(t,u) = -\lambda(t,t_0)u, \ \phi(t,t_0,q) \leq k(t-t_0)^q, \tag{12} $$

where $\lambda(t,t_0) \geq 0$ is continuous, and $k > 0$ is a constant. Then using (9),(10) and (11), we have a differential inequality, componentwise,

$$ u'(t) \leq \Gamma(2-q)[-\lambda(t,t_0)u(t)(t-t_0)^{q-1} + \frac{k(t-t_0)}{\Gamma(1-q)}], u(t_0) = u_0 \geq 0. \tag{13} $$

We now choose $\lambda(t,t_0) = \frac{(t-t_0)^{1-q}}{\Gamma(2-q)}\lambda_0, \ \lambda_0 > 0$, so that the corresponding comparison system becomes

$$ v'(t) = -\lambda_0 v + k_0(t-t_0), \ v(t_0) = u_0 \geq 0, \ k_0 = \frac{k\Gamma(2-q)}{\Gamma(1-q)}, \tag{14} $$

whose solution is

$$v(t) = u_0 e^{-\lambda_0(t-t_0)} + \int_{t_0}^{t} e^{-\lambda_0(t-s)} k_0(s-t_0)ds, \; t \geq t_0. \tag{15}$$

By the usual comparison theorem, Corollary 1.7.1 in [2], this implies that

$$u(t) \leq v(t), \; t \geq t_0,$$

and consequently, in view of the estimate (4) of solutions $x(t, t_0, x_0)$ of the original multi-system (1), we get

$$V(t, x_{n_i}(t)) \leq u(t) \leq v(t), \; t \geq t_0. \tag{16}$$

In view of the assumption on the vector $V(t, x)$ supposed in (5), and the fact that all solutions $v(t, t_0, u_0)$ are bounded by $\frac{k_0}{\lambda_0}$, we see that all solutions $x(t, t_0, x_0)$ are also bounded. It is not difficult to see that $v(t) \leq \frac{k_0}{\lambda_0}$, $t \geq t_0$.

If, on the other hand, we suppose that not all components of $\phi(t, t_0, q)$ satisfy the same estimate, but, we find that

$$\left.\begin{array}{rcl} \phi_i(s,t,q) & \leq & k_i(t-s)^q, \; i = 1, 2, \ldots, k_0, \\ \phi_i(s,t,q) & \leq & k_i(t)(t-s)^q, \; i = k_0+1, \ldots, k. \end{array}\right\} \tag{17}$$

where the functions $k_i(t)$ satisfy the condition

$$\lim_{t\to\infty} \int_{t}^{t+1} (\int_{t_0}^{s} k(\sigma)d\sigma)ds \to 0, \tag{18}$$

we get the same estimate for $v(t) \leq \frac{\tilde{k}}{\lambda_0}$ as before, for the first $i = 1, 2, \ldots, k_0$, components. However, for the rest of $k_0+1, k_0+2, \ldots, k$, components of $v(t)$, it follows that $|x_{n_i}(t, t_0, x_0)| \to 0$ as $t \to \infty$, where n_i ranges from n_{k_0+1} to n_k. The rest of $|x_{n_i}(t, t_0, x_0)| \leq N$, $t \geq t_0$, where n_i ranges from $n_1, \ldots, n_{k_0}$ and N is a suitable constant depending on $\frac{k_i}{\lambda_0}$ and the $\mathcal{K}$-class functions b_{ni}, a_{ni} ranging from $n_1, \ldots, n_k$. For the proof of the first part we need to follow the complicated arguments of Theorem 2.14.6 in [2], which result because of the condition (4.8.13) in [2]. Thus we see a given system could have different qualitative behavior for different groups of components, which can only be detected using the generalized norm and several Lyapunov functions. Even though such a varied behavior can happen for ordinary differential system, it is more common in multi-order systems of fractional differential equations.

REFERENCES

[1] Bernfeld, S. R. and Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[2] Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol I and II, Academic Press, New York, 1969.

[3] Lakshmikantham, V. and Leela, S., Relation between fractional and ordinary differential equations (to appear).

[4] Lakshmikantham, V., Leela, S. and Vasundhara Devi, J., *Theory of Fractional Dynamic Systems*, To be published by Cambridge Academic Publishers, U.K.