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SOME BASIC RESULTS FOR FRACTIONAL FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

BASHIR AHMAD AND S. SIVASUNDARAM

Department of Mathematics, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia

Department of Mathematics, Embry-Riddle Aeronautical University
Daytona Beach, FL 32114, USA

ABSTRACT. This paper studies some existence and comparison results for initial value problems involving fractional functional integro-differential equations.

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1. INTRODUCTION

Integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, visco elasticity, visco elastic panel in super sonic gas flow, fluid dynamics, electro dynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors and mathematical modeling of a hereditary phenomena.

Recently the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics, with few examples of applications in bio-engineering are highlighted in the literatures. The methods of fractional calculus, when defined as a Laplace, Sumudu or Fourier convolution product, are suitable for solving many problems in emerging biomedical research. The electrical properties of nerve cell membranes and the propagation of electrical signals are well characterized by differential equations of fractional order. The fractional derivative accurately

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describes natural phenomena that occur in such common engineering problems as heat transfer, electrode/electrolyte behavior, and sub-threshold nerve propagation. Application of fractional derivatives to viscoelastic materials establishes, in a natural way, hereditary integrals and the power law stress-strain relationship for modeling biomaterials. Fractional operations by following the original approach of Heaviside, demonstrate the basic operations of fractional calculus on well-behaved functions such as step, ramp, pulse, and sinusoidal of engineering interest, and can easily be applied in electrochemistry, physics, bioengineering, and biophysics.

The differential equations involving Riemann-Liouville differential operators of fractional order occur in the mathematical modelling of several phenomena in the fields of physics, chemistry, engineering, etc. For details, see [3,4,9,11,12] and the references therein. In consequence, the subject of fractional differential equations is gaining much importance and attention, see for example [1, 5, 6, 7, 10] and the references therein.

The definition of Riemann-Liouville fractional derivative, which did certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. It was Caputo's definition of fractional derivative:

$$^{c}D^{q}x(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-s)^{n-q-1}x^{(n)}(s)ds, n-1 < q < n,$$

which solved this problem. In fact, Caputo'derivative becomes the conventional n-th derivative as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero.

Fractional functional differential equations appear in the mathematical modeling of real world problems in which the fractional rate of change depends on the influence of their hereditary effects. Recently, Lakshmikantham [6] obtained some basic results for fractional functional differential equations. In this paper, we establish some existence results for fractional functional integro-differential equations.

2. PRELIMINARY RESULTS

For any $\tau > 0$, let $\mathrm{C} = C([-\tau, 0], \mathbb{R})$ denote the set of all continuous functions on $[-\tau, 0]$. For any $\phi \in \mathrm{C}$, we define $|\phi|_0 = \max_{-\tau\le s\le 0}|\phi(s)|$. For any $t \ge t_0 \ge 0$, $x \in C([t_0-\tau, \infty), \mathbb{R})$, let x_t denote a translation of the restriction of x to the interval $[t-\tau, t]$, that is, $x_t \in \mathrm{C}$ be defined by $x_t(s) = x(t+s), -\tau \le s \le 0$. For $\rho > 0$, let $\mathrm{C}_\rho = [\phi \in \mathrm{C} : |\phi|_0 < \rho]$.

Consider an initial value problem (IVP) involving a functional fractional integro-differential equation given by

$$\begin{cases} {}^cD^q x(t) = f(t, x_t) + \int_{t_0}^t H(t, s, x_s)ds, \ 0 < q < 1, \\ x_{t_0} = \phi_0 \in \mathrm{C}, \end{cases} \tag{2.1}$$

where ${}^cD^q$ denotes Caputo fractional derivative of order q, $f \in C(J \times \mathrm{C}, \mathbb{R})$ and $H \in C(J \times J \times \mathrm{C}, \mathbb{R})$, $J = [t_0, T]$. In view of continuity of f and H, the IVP (2.1) is equivalent to the following Volterra fractional integral equation with delay

$$x(t) = \phi_0(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1}[f(s, x_s) + \int_s^t H(\sigma, s, x_s)d\sigma]ds, \ x_{t_0} = \phi_0, \tag{2.2}$$

where Γ is Gamma function.

Now we state a known lemma [7].

Lemma 2.1. *Let $m : [t_0 - \tau, T] \to \mathbb{R}$ be locally Hölder continuous such that $m(t_1) = 0$, $m(t) \leq 0$, $t_0 \leq t \leq t_1$, $t_1 \in (t_0, T]$ and $m_{t_1} \leq 0$. Then ${}^cD^q m(t_1) \geq 0$.*

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions defined on $[t_0 - \tau, T]$ satisfying*

$$\begin{cases} {}^cD^q x_\epsilon(t) = f(t, x_{\epsilon,t}) + \int_{t_0}^t H(t, s, x_{\epsilon,s})ds, \ 0 < q < 1, \\ x_{\epsilon,t_0} = \phi_0. \end{cases}$$

Further, $|f(t, x_{\epsilon,t})| \leq M_1$ for $t \in J$, $|H(t, s, x_{\epsilon,s})| \leq M_2, (t, s) \in J \times J$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on J.

Proof. For $t_1, t_2 \in J$, we have

$$\begin{aligned}
&|x_\epsilon(t_1) - x_\epsilon(t_2)| \\
&= \frac{1}{\Gamma(q)} \Big| \int_{t_0}^{t_1} (t_1 - s)^{q-1}[f(s, x_{\epsilon,s}) + \int_s^{t_1} H(\sigma, s, x_{\epsilon,s})d\sigma]ds \\
&- \int_{t_0}^{t_2} (t_2 - s)^{q-1}[f(s, x_{\epsilon,s}) + \int_s^{t_2} H(\sigma, s, x_{\epsilon,s})d\sigma]ds \Big| \\
&= \frac{1}{\Gamma(q)} \Big| \int_{t_2}^{t_1} (t_1 - s)^{q-1}[f(s, x_{\epsilon,s}) + \int_s^{t_1} H(\sigma, s, x_{\epsilon,s})d\sigma]ds \\
&- \int_{t_0}^{t_2} [(t_2 - s)^{q-1} \int_{t_2}^{s} H(\sigma, s, x_{\epsilon,s})d\sigma - (t_1 - s)^{q-1} \int_{t_1}^{s} H(\sigma, s, x_{\epsilon,s})d\sigma]ds \Big| \\
&+ \frac{1}{\Gamma(q)} \Big| \int_{t_0}^{t_2} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(s, x_{\epsilon,s})ds \Big| \\
&\leq \frac{M_1}{\Gamma(q+1)} |2(t_1 - t_2)^q + (t_2 - t_0)^q - (t_1 - t_0)^q| \\
&+ \frac{qM_2}{\Gamma(q+2)} |2(t_1 - t_2)^{q+1} + (t_2 - t_0)^{q+1} - (t_1 - t_0)^{q+1}| \\
&\leq \frac{2M_1}{\Gamma(q+1)} |t_1 - t_2|^q + \frac{2qM_2}{\Gamma(q+2)} |t_1 - t_2|^{q+1} < \epsilon,
\end{aligned}$$

provided $|t_1 - t_2| < \delta$, where δ satisfies

$$\frac{2M_1}{\Gamma(q+1)}\,\delta^q + \frac{2qM_2}{\Gamma(q+2)}\,\delta^{q+1} < \epsilon.$$

This completes the proof. □

Theorem 2.3. *Let $v, w : [t_0-\tau, T] \to \mathbb{R}$ be locally Hölder continuous, $f \in C(J\times \mathrm{C}, \mathbb{R})$ is such that $f(t,\phi)$ is nondecreasing in ϕ for each t and $H \in C(J \times J \times \mathrm{C}, \mathbb{R})$ is such that $H(t,s,\phi)$ is nondecreasing in ϕ for each (t,s). Moreover, for $t_0 \le t \le T$, assume that*

- **(i)** ${}^cD^q v(t) \le f(t, v_t) + \int_{t_0}^t H(t,s,v_s)ds,$
- **(ii)** ${}^cD^q w(t) \ge f(t, w_t) + \int_{t_0}^t H(t,s,w_s)ds,$

with one of the inequalities (i), (ii) being strict. Then $v_{t_0} < w_{t_0}$ implies that $v(t) < w(t)$, $t_0 \le t \le T$.

Proof. For the sake of contradiction, let us suppose that the conclusion is not true and the inequality (ii) is strict. In view of $v_{t_0} < w_{t_0}$ and the continuity of the functions f and H, there exists a $t_1 \in (t_0, T]$ such that $v(t_1) = w(t_1)$, $v(t) < w(t)$, $t \in [t_0, t_1]$, which implies that $v_{t_1} < w_{t_1}$. Defining $m(t) = v(t) - w(t)$, we note that $m(t_1) = 0$ and $m(t) \le 0, t \in [t_0, t_1]$. Thus, by Lemma 2.1, we obtain ${}^cD^q m(t_1) \ge 0$, that is, ${}^cD^q v(t_1) \ge {}^cD^q w(t_1)$. On the other hand, using the assumptions (i), (ii) and the monotone character of $f(t,\phi)$ and $H(t,s,\phi)$ in ϕ, we obtain

$$\begin{aligned} f(t_1, v_{t_1}) + \int_{t_0}^{t_1} H(t_1,s,v_s)ds &\ge {}^cD^q v(t_1) \ge {}^cD^q w(t_1) \\ &> f(t_1, w_{t_1}) + \int_{t_0}^{t_1} H(t_1,s,w_s)ds \ge f(t_1, v_{t_1}) + \int_{t_0}^{t_1} H(t_1,s,v_s)ds, \end{aligned}$$

which is a contradiction. Hence the proof is complete. □

Theorem 2.4. *Assume that the hypothesis of Theorem 2.3 holds with nonstrict inequalities (i) and (ii). Further*

$$f(t,\phi) - f(t,\psi) \le \frac{L_1}{1+t^q} \max_{-\tau\le s\le 0}(\phi(s) - \psi(s)), L_1 > 0,$$
$$H(t,s,\phi) - H(t,s,\psi) \le \frac{L_2}{1+t^q} \max_{-\tau\le s\le 0}(\phi(s) - \psi(s)), L_2 > 0,$$

whenever $\phi(s) \ge \psi(s)$. Then $v_{t_0} \le w_{t_0}$ implies that $v(t) \le w(t)$, $t_0 \le t \le T$ provided $(L_1 + L_2(T-t_0)) \le \frac{1}{T^q\Gamma(1-q)}$.

Proof. Define $w_\epsilon(t) = w(t) + \epsilon(1+t^q)$, $\epsilon > 0$ so that

$$w_{\epsilon,t} = w_t + \epsilon(1+(t+s)^q),\ w_{\epsilon,t} \ge w_t,\ t_0 \le t \le T. \tag{2.3}$$

Also, we have

$$\begin{aligned} {}^cD^q w_{\epsilon,t} &= {}^cD^q w_t + \epsilon {}^cD^q(1+t^q) \geq f(t,w_t) + \int_{t_0}^t H(t,s,w_s)ds \\ &\quad + \epsilon[\frac{1}{t^q\Gamma(1-q)} + \Gamma(1+q)] \\ &\geq f(t,w_{\epsilon,t}) + \int_{t_0}^t H(t,s,w_{\epsilon,s})ds - \frac{L_1}{1+t^q}\max_{-\tau\leq s\leq 0}(w_{\epsilon,t}-w_t) \\ &\quad - \int_{t_0}^t \frac{L_2}{1+s^q}\max_{-\tau\leq s_1\leq 0}(w_{\epsilon,s}-w_s)ds + \epsilon[\frac{1}{t^q\Gamma(1-q)} + \Gamma(1+q)] \\ &> f(t,w_{\epsilon,t}) + \int_{t_0}^t H(t,s,w_{\epsilon,s})ds - (L_1 + L_2(T-t_0))\epsilon + \epsilon\frac{1}{t^q\Gamma(1-q)} \\ &\geq f(t,w_{\epsilon,t}) + \int_{t_0}^t H(t,s,w_{\epsilon,s})ds. \end{aligned}$$

Now, by applying Theorem 2.3 to v and w_ϵ, we conclude that $v(t) < w_\epsilon(t)$, $t_0 \leq t \leq T$. Since ϵ is arbitrary, therefore, the conclusion of the theorem follows. This completes the proof. □

3. EXISTENCE OF EXTREMAL SOLUTIONS

Theorem 3.1. *Let $f \in C(J_a \times \mathrm{C}_\rho, \mathbb{R})$ and $H \in C(J_a \times J_a \times \mathrm{C}_\rho, \mathbb{R})$, where $J_a = [t_0, t_0 + a]$. Then there exists a solution $x(t_0, \phi_0)$ of (2.1) on $[t_0 - \tau, t_0 + \alpha]$ for some $\alpha > 0$, where $\phi_0 \in \mathrm{C}$ is an initial function at $t = t_0$.*

Proof. Let us define $y \in C([t_0 - \tau, t_0 + a], \mathbb{R})$ as

$$y(t) = \begin{cases} \phi_0(t-t_0), \; t_0 - \tau \leq t \leq t_0, \\ \phi_0(0), \; t_0 \leq t \leq t_0 + a. \end{cases}$$

Then $f(t, y_t)$ and $H(t, s, y_s)$ are continuous functions for $t, s \in J_a$ and hence $|f(t, y_t)| \leq M_1$, $|H(t, s, y_s)| \leq M_2$. Now we show that there exists a constant $b \in (0, \phi - |\phi_0(0)|)$ such that $|f(t, \psi) - f(t, y_t)| < 1$, $|H(t, s, \psi) - H(t, s, y_s)| < 1$ whenever $t, s \in J_a$, $\psi \in \mathrm{C}_\rho$ and $|\psi - y_t|_0 < b$. On the contrary, suppose that our claim is false. Then there would exist $t_k, s_k \in J_a$ and $\psi_k \in \mathrm{C}_\rho$ for each $k = 1, 2, \ldots$ such that $|\psi_k - y_{t_k}| < \frac{1}{k}$ but $|f(t_k, \psi_k) - f(t_k, y_{t_k})| \geq 1$, $|H(t_k, s_k, \psi_k) - H(t_k, s_k, y_{s_k})| \geq 1$. Choosing a subsequence $\{t_{k_p}\}$ with $\lim_{p\to\infty} t_{k_p} = t_1$ yields a contradiction to the continuity of f at (t_1, y_{t_1}) and a sequence $\{s_{k_p}\}$ with $\lim_{p\to\infty} s_{k_p} = s_1$ yields a contradiction to the continuity of H at (t_1, s_1, y_{s_1}). Thus, it follows that $|f(t, y_t)| < M_3 + 1 = M_1$, $|H(t, s, y_s)| < M_4 + 1 = M_2$ whenever $t, s \in J_a$, $\psi \in \mathrm{C}_\rho$ and $|\psi - y_t|_0 < b$. Select $\alpha = \min(a, a_1)$ where a_1 denotes the maximum value satisfying the inequality $[\frac{2M_1}{\Gamma(q+1)} a_1^q + \frac{2M_2 q}{\Gamma(q+2)} a_1^{q+1}] \leq b$.

Let E denote the space of continuous functions from $[t_0 - \tau, t_0 + \alpha]$ to $\mathbb{R}$ with the norm define by $|x|_0 = \max\{|x(u)| : u \in [t_0 - \tau, t_0 + \alpha]\}$. Clearly E is a Banach space.

Define a set $S \subset E$ as

$$S = \begin{cases} x(u) = \phi_0(u - t_0), & t_0 - \epsilon \leq u \leq t_0, \\ |x(t_1) - x(t_2)| \leq [\frac{2M_1}{\Gamma(q+1)}(t_1 - t_2)^q + \frac{2M_2 q}{\Gamma(q+2)}(t_1 - t_2)^{q+1}], & t_1, t_2 \in [t_0, t_0 + \alpha]. \end{cases}$$

We note that S is compact as members of S are uniformly bounded and equicontinuous on $[t_0 - \tau, t_0 + \alpha]$. Moreover S is also convex. For $x \in S$, we define a mapping T on S as (i) $T(x_{t_0}) = \phi_0$,

$$(ii)\ T(x(t)) = \phi_0(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[f(s, x_s) + \int_s^t H(\sigma, s, x_s) d\sigma] ds,\ t \in [t_0, t_0 + \alpha].$$

For any $x \in S$, we have

$$\begin{aligned} |x(t) - \phi_0(0)| &\leq \frac{2M_1}{\Gamma(q+1)}(t - t_0)^q + \frac{2M_2 q}{\Gamma(q+2)}(t - t_0)^{q+1} \\ &\leq \frac{2M_1}{\Gamma(q+1)}\alpha^q + \frac{2M_2 q}{\Gamma(q+2)}\alpha^{q+1} \leq b,\ t_1, t_2 \in [t_0, t_0 + \alpha], \end{aligned}$$

which implies that $x_t \in \mathrm{C}_\rho$ and $|x_t - y_t|_0 \leq b$. Thus, $f(s, x_s)$ and $H(t, s, y_s)$ are continuous functions of s with $|f(s, x_s)| \leq M_1$ and $|H(t, s, y_s)| \leq M_2$ for $s \in [t_0, t_0+\alpha]$. Therefore, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(s, x_s) - f(s, z_s)| \leq \frac{\epsilon\Gamma(q+1)}{2(s - t_0)^q},\ |H(t, s, x_s) - H(t, s, z_s)| \leq \frac{\epsilon\Gamma(q+2)}{2q(s - t_0)^{q+1}},$$

whenever $|x_s - z_s|_0 < \delta$ for $s \in [t_0, t_0 + \alpha]$. Thus, it follows that

$$\begin{aligned} |(Tx)(t) - (Tz)(t)| &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[|f(s, x_s) - f(s, z_s)| \\ &+ \int_s^t |H(\sigma, s, x_s) - H(\sigma, s, z_s)| d\sigma] ds \\ &\leq \frac{\epsilon\Gamma(q+1)}{2(t - t_0)^q \Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} ds + \frac{\epsilon\Gamma(q+2)}{2q(t - t_0)^{q+1}\Gamma(q)} \int_{t_0}^{t} (t-s)^q ds = \epsilon \end{aligned}$$

This proves the continuity of T. Moreover, for any $x \in S$, $t_1, t_2 \in [t_0, t_0 + \alpha]$, we find that

$$\begin{aligned} &|(Tx)(t_1) - (Tx)(t_2)| \\ &= \frac{1}{\Gamma(q)} |\int_{t_0}^{t_1} (t_1 - s)^{q-1}[f(s, x_s) + \int_s^{t_1} H(\sigma, s, x_s) d\sigma] ds \\ &\quad - \int_{t_0}^{t_2} (t_2 - s)^{q-1}[f(s, x_s) + \int_s^{t_2} H(\sigma, s, x_s) d\sigma] ds| \\ &= \frac{1}{\Gamma(q)} |\int_{t_2}^{t_1} (t_1 - s)^{q-1}[f(s, x_s) + \int_s^{t_1} H(\sigma, s, x_s) d\sigma] ds \\ &\quad - \int_{t_0}^{t_2} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(s, x_s) ds \end{aligned}$$

$$
\begin{aligned}
&- \int_{t_0}^{t_2} [(t_2-s)^{q-1} \int_{t_2}^{s} H(\sigma,s,x_s)d\sigma - (t_1-s)^{q-1}\int_{t_1}^{s} H(\sigma,s,x_s)d\sigma]ds| \\
&+ \frac{1}{\Gamma(q)}|\int_{t_0}^{t_2}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x_s)ds| \\
\le\ & \frac{M_1}{\Gamma(q+1)}|2(t_1-t_2)^q+(t_2-t_0)^q-(t_1-t_0)^q| \\
&+ \frac{qM_2}{\Gamma(q+2)}|2(t_1-t_2)^{q+1}+(t_2-t_0)^{q+1}-(t_1-t_0)^{q+1}| \\
\le\ & \frac{2M_1}{\Gamma(q+1)}|t_1-t_2|^q + \frac{2qM_2}{\Gamma(q+2)}|t_1-t_2|^{q+1},
\end{aligned}
$$

which implies that T maps S into S. Hence, by an application of Schauder's fixed point theorem, there exists at least one $x \in S$ such that $T(x_{t_0}) = \phi_0$ and $(Tx)(t) = x(t),\ t \in [t_0, t_0+\alpha]$ which yields (i) $x_{t_0} = \phi_0$,

$$(ii)\ x(t) = \phi_0(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,x_s) + \int_s^t H(\sigma,s,x_s)d\sigma]ds,\ t_0 \le t \le t_0+\alpha.$$

Thus, it follows that $x(t) = x(t_0,\phi_0)$ is a solution of (2.1) on $[t_0-\tau, t_0+\alpha]$. This completes the proof. □

Theorem 3.2. *Let $f \in C(J_a \times \mathrm{C}_\rho, \mathbb{R})$ and $H \in C(J_a \times J_a \times \mathrm{C}_\rho, \mathbb{R})$ be such that $f(t,\phi)$ is nondecreasing in ϕ for each t, and $H(t,s,\phi)$ is nondecreasing in ϕ for each (t,s). Then there exists an $\alpha_1 > 0$ such that the IVP* (2.1) *admits extremal solutions on $[t_0-\tau, t_0+\alpha_1]$ for given $\phi_0 \in \mathrm{C}$ at $t = t_0$.*

Proof. As in the proof of Theorem 3.1, we choose positive constants b, ϵ with $0 < \epsilon \le b/2$ and consider the IVP

$$
\begin{cases}
{}^cD^q x(t) = f(t,x_t) + \int_{t_0}^t H(t,s,x_s)ds + \epsilon, \\
\qquad x_{t_0} = \phi_0 + \epsilon.
\end{cases}
\tag{3.1}
$$

For $t \in J_a,\ \psi \in \mathrm{C}_\rho$, we observe that $|f(t,\psi)+\epsilon| < M_1 + b/2$ whenever $|\psi - y_t - \epsilon| \le b$. By Theorem 3.1, it follows that the IVP (3.1) has a solution $x(t_0,\phi_0,\epsilon)$ on the interval $[t_0-\tau, t_0+\alpha_1]$ where $\alpha_1 = \min(a, a_2)$ where a_2 denotes the maximum value satisfying the inequality $[\frac{2M_1+b}{2\Gamma(q+1)}a_2^q + \frac{2M_2q}{\Gamma(q+2)}a_2^{q+1}] \le b$. For $0 < \epsilon_2 < \epsilon_1 \le \epsilon$, we have

$$
\begin{aligned}
&x_{t_0}(t_0,\phi_0,\epsilon_2) < x_{t_0}(t_0,\phi_0,\epsilon_1), \\
&{}^cD^q x(t_0,\phi_0,\epsilon_2)(t) \le f(t, x_t(t_0,\phi_0,\epsilon_2)) + \int_{t_0}^t H(t,s,x_s(t_0,\phi_0,\epsilon_2))ds + \epsilon_2, \\
&{}^cD^q x(t_0,\phi_0,\epsilon_1)(t) > f(t, x_t(t_0,\phi_0,\epsilon_1)) + \int_{t_0}^t H(t,s,x_s(t_0,\phi_0,\epsilon_1))ds + \epsilon_2.
\end{aligned}
$$

In view of Theorem 2.3, we obtain

$$x(t_0,\phi_0,\epsilon_2)(t) < x(t_0,\phi_0,\epsilon_1)(t),\ t \in [t_0, t_0+\alpha_1].$$

For a family of continuous functions $\{x(t_0, \phi_0, \epsilon)(t)\}$ on $[t_0, t_0 + \alpha_1]$, we find that

$$
\begin{aligned}
|x(t_0, \phi_0, \epsilon)(t) - \phi_0(0) - \epsilon| &\le \frac{2M_1 + b}{\Gamma(q+1)}(t - t_0)^q + \frac{2M_2 q}{\Gamma(q+2)}(t - t_0)^{q+1} \\
&\le \frac{2M_1 + b}{2\Gamma(q+1)}\alpha_1^q + \frac{2M_2 q}{\Gamma(q+2)}\alpha_1^{q+1} \le b,
\end{aligned}
$$

which implies that the family is uniformly bounded. Since $|f(t, x_t) + \int_{t_0}^t H(t, s, x_s)ds + \epsilon| \le (2M_1 + b)/2 + \alpha_1 M_2$, therefore, by Lemma 2.2, the family is equicontinuous. Hence there exists a decreasing sequence $\{\epsilon_n\}$ such that $\lim_{n\to\infty} \epsilon_n \to 0$, implying the existence of uniform limit $\eta(t_0, \phi_0)(t) = \lim_{n\to\infty} x(t_0, \phi_0, \epsilon_n)(t)$ on $[t_0, t_0 + \alpha_1]$. Also, we have $\eta_{t_0}(t_0, \phi_0) = \phi_0$. Thus, passing onto the limit $n \to \infty$, we get

$$
\begin{aligned}
&f(t, x_t(t_0, \phi_0, \epsilon_n)) + \int_{t_0}^t H(t, s, x_s(t_0, \phi_0, \epsilon_n))ds \\
&\qquad \to f(t, \eta_t(t_0, \phi_0)) + \int_{t_0}^t H(t, s, \eta_s(t_0, \phi_0))ds
\end{aligned}
$$

uniformly. As argued in the proof of Theorem 3.1, it follows that $\eta(t_0, \phi_0)(t)$ is a solution of IVP (2.1) on $[t_0 - \tau, t_0 + \alpha_1]$.

Now we show that $\eta(t_0, \phi_0)(t)$ is the maximal solution of IVP (2.1). For that, let $x(t_0, \phi_0)(t)$ be any solution of IVP (2.1) on $[t_0 - \tau, t_0 + \alpha_1]$. Then, for $\epsilon > 0$, we have

$$
\begin{aligned}
&\phi_0 < \phi_0 + \epsilon, \\
&{}^cD^q x(t_0, \phi_0)(t) < f(t, x_t(t_0, \phi_0)) + \int_{t_0}^t H(t, s, x_s(t_0, \phi_0))ds + \epsilon, \\
&{}^cD^q x(t_0, \phi_0, \epsilon)(t) \ge f(t, x_t(t_0, \phi_0, \epsilon)) + \int_{t_0}^t H(t, s, x_s(t_0, \phi_0, \epsilon))ds + \epsilon.
\end{aligned}
$$

It follows by Theorem 2.3 that $x(t_0, \phi_0)(t) < x(t_0, \phi_0, \epsilon)(t), t_0 \in [t_0, t_0 + \alpha_1]$ for every $\epsilon > 0$. The uniqueness of the maximal solution shows that $x(t_0, \phi_0, \epsilon)(t) \to \eta(t_0, \phi_0)(t)$ on $[t_0, t_0 + \alpha_1]$ as $\epsilon \to 0$. In a similar manner, we can show the existence of minimal solution of IVP (2.1). This completes the proof. □

4. GLOBAL EXISTENCE OF SOLUTIONS

In order to establish the global existence of any solution of (2.1), we need to prove the following comparison result.

Theorem 4.1. *Let $m \in C([t_0 - \tau, \infty)$ satisfy the inequality*

$$
{}^cD^q m(t) \le g(t, |m_t|_0) + \int_{t_0}^t K(t, s, |m_s|_0)ds, \quad t > t_0, \tag{4.1}
$$

where $g \in C([t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$ *and* $K \in C([t_0, \infty) \times [t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$. *Let* $\eta(t) = \eta(t, t_0, u_0)$ *be the maximal solution existing on* $[t_0, \infty)$ *for the following IVP*

$$\begin{cases} {}^cD^q u(t) = g(t, u(t)) + \int_{t_0}^{t} K(t, s, u(s))ds, \ 0 < q < 1, \\ u(t_0) = u_0 \geq 0. \end{cases} \tag{4.2}$$

Then,

$$m(t) \leq \eta(t) \quad if \quad |m_{t_0}|_0 \leq u_0, \ t \in [t_0, \infty). \tag{4.3}$$

Proof. As $\lim_{\epsilon \to \infty} u(t, t_0, u_0, \epsilon) = \eta(t, t_0, u_0)$, so it is enough to show that

$$m(t) < u(t, t_0, u_0, \epsilon), \ t \in [t_0, \infty), \tag{4.4}$$

where $u(t, t_0, u_0, \epsilon)$ is any solution of

$$\begin{cases} {}^cD^q u(t) = g(t, u(t)) + \int_{t_0}^{t} H(t, s, u(s))ds + \epsilon, \\ u(t_0) = u_0 + \epsilon, \end{cases}$$

where ϵ is an arbitrarily small positive quantity.

On the contrary, suppose that (4.4) is false. Then there would exist a $t_1 > t_0$ such that

$$m(t_1) = u(t_1, t_0, u_0, \epsilon), \ m(t) < u(t, t_0, u_0, \epsilon), \quad t \in [t_0, t_1),$$

where $|m_{t_0}|_0 \leq u_0 + \epsilon = u(t_0, t_0, u_0, \epsilon)$. Then, In view of Lemma 2.1, we have

$$\begin{aligned} {}^cD^q m(t_1) &\geq^c D^q u(t_1, t_0, u_0, \epsilon) \\ &= g(t_1, u(t_1, t_0, u_0, \epsilon)) + \int_{t_0}^{t_1} K(t_1, s, u(s, t_0, u_0, \epsilon))ds + \epsilon. \end{aligned} \tag{4.5}$$

Since $g(t, u(t)) \geq 0, H(t, s, u(s)) \geq 0$, therefore $u(t, t_0, u_0, \epsilon)$ is nondecreasing in t and consequently the preceding arguments yields $|m_{t_1}|_0 = u(t_1, t_0, u_0, \epsilon) = m(t_1)$. This, in turn, leads to the inequality

$$\begin{aligned} {}^cD^q m(t_1) &\leq g(t_1, |m_{t_1}|_0) + \int_{t_0}^{t_1} K(t_1, s, |m_s|_0)ds \\ &= g(t_1, u(t_1, t_0, u_0, \epsilon)) + \int_{t_0}^{t_1} K(t_1, s, u(s, t_0, u_0, \epsilon))ds + \epsilon, \end{aligned}$$

which contradicts (4.5). Hence our supposition is not true and (4.4) is valid. This completes the proof. □

Theorem 4.2. *Let* $f \in C(([t_0, \infty) \times \mathrm{C}), \mathbb{R})$ *and* $H \in C(([t_0, \infty) \times [t_0, \infty) \times \mathrm{C}), \mathbb{R})$ *be such that*

$$\begin{cases} |f(t, \phi)| \leq g(t, |\phi|_0), \quad for \ (t, \phi) \in [t_0, \infty) \times \mathrm{C}, \\ |H(t, s, \phi)| \leq K(t, s, |\phi|_0), \quad for \ (t, s, \phi) \in [t_0, \infty) \times [t_0, \infty) \times \mathrm{C}, \end{cases} \tag{4.6}$$

where $g \in C([t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$ *and* $K \in C([t_0, \infty) \times [t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$ *are nondecreasing in* u *for each* t *and* (t, s) *respectively. Assume that the maximal solution* $\eta(t) = \eta(t, t_0, u_0)$ *of* (4.2) *exists for* $t \geq t_0$. *Then the largest interval of existence of any solution* $x(t_0, \phi_0)(t)$ *of* (2.1) *is* $[t_0 - \tau, \infty)$.

Proof. Suppose that $x(t_0,\phi_0)(t)$ is a solution of (2.1) on $[t_0-\tau,\beta)$, where $0<\beta<\infty$ and the value of β cannot be increased. We define $m(t)=|x(t_0,\phi_0)(t)|$ with $m_t=|x_t(t_0,\phi_0)|$ for $t\in[t_0-\tau,\beta)$. In view of (4.6), it can easily be obtained that

$$^{c}D^{q}m(t) \ \leq \ g(t,|m_t|_0)+\int_{t_0}^{t}K(t,s,|m_s|_0)ds,\ t\in[t_0-\tau,\beta).$$

Setting $|m_{t_0}|_0=|\phi_0|_0\leq u_0$, by Theorem 4.1, we get

$$|x(t_0,\phi_0)(t)|\leq\eta(t,t_0,u_0),\ t\in[t_0,\beta). \tag{4.7}$$

Since $g(t,u(t))\geq 0, K(t,s,u(s))\geq 0$, therefore, $\eta(t,t_0,u_0)$ is nondecreasing and hence (4.7) yields

$$|x_t(t_0,\phi_0)|_0\leq\eta(t,t_0,u_0),\ t\in[t_0,\beta). \tag{4.8}$$

Since $\eta(t,t_0,u_0)$ is assumed to exist on $[t_0,\infty)$, therefore, it follows that

$$\begin{aligned}&^{c}D^{q}x(t_0,\phi_0)(t)\leq g(t,|m_t|_0)+\int_{t_0}^{t}K(t,s,|m_s|_0)ds\\&\leq g(t,\eta(t,t_0,u_0))+\int_{t_0}^{t}K(t,s,\eta(s,t_0,u_0))ds\leq M_1+(\beta-t_0)M_2,\ t\in[t_0,\beta).\end{aligned}$$

By Lemma 2.2, we have

$$|x(t_0,\phi_0)(t_1)-x(t_0,\phi_0)(t_2)|\leq\frac{2M_1}{\Gamma(q+1)}|t_1-t_2|^q+\frac{2qM_2}{\Gamma(q+2)}|t_1-t_2|^{q+1}.$$

Taking $t_1,\ t_2\to\beta^-$, and using Cauchy criterion, we find that $\lim_{t\to\beta^-}x(t_0,\phi_0)(t)$ exists and denote it by $x(t_0,\phi_0)(\beta)$. Consider the new IVP

$$^{c}D^{q}x(t)=f(t,x_t)+\int_{t_0}^{t}H(t,s,x_s)ds,\ x_\beta=x_\beta(t_0,\phi_0).$$

In view of the local existence, it follows that $x(t_0,\phi_0)(t)$ can be continued beyond β, which contradicts our supposition. Hence every solution $x(t_0,\phi_0)(t)$ of (2.1) exists on $[t_0-\tau,\infty)$. This completes the proof. □

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