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FUZZY DIFFERENTIAL SYSTEMS UNDER GENERALIZED METRIC SPACES APPROACH

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ABSTRACT. We study the existence and uniqueness of solution for fuzzy differential systems under the point of view of generalized metric spaces. The results obtained are applied to study the solvability of first-order fuzzy linear systems, as well as higher-order fuzzy differential equations and systems.

Key Words: Fuzzy Differential Equations, Fuzzy Differential Systems, Fixed Point Theorems, Generalized Metric Spaces, Generalized Contraction Theorem

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1. INTRODUCTION

First-order fuzzy differential equations have been considered, for instance, in [1]–[10]. A detailed analysis of first-order linear fuzzy initial value problems is included in [11], where the exact expression of the solution is obtained (whenever it exists). Higher order linear ordinary differential equations with fuzzy initial conditions are studied in [12] under two different points of view, some results on existence and uniqueness of solution for two-point boundary value problems relative to second order fuzzy differential equations are given in [6, 13, 14] and, besides, [15, 16] include some results on higher order fuzzy differential equations with crisp initial conditions. For the study of some numerical methods for fuzzy differential equations, see [2], and [17]–[20]. On the other hand, the basic theory concerning metric spaces of fuzzy sets can be found in [1]. In the following, we consider E^m the space of fuzzy subsets of $\mathbb{R}^m$

$$u : \mathbb{R}^m \longrightarrow [0, 1],$$

satisfying the following properties:

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i): u is normal: there exists $x_0 \in \mathbb{R}^m$ with $u(x_0) = 1$.

ii): u is fuzzy convex: for all x, $y \in \mathbb{R}^m$ and $\lambda \in [0,1]$,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}.$$

iii): u is upper-semicontinuous.

iv): $[u]^0 = \overline{\{x \in \mathbb{R}^m \,:\, u(x) > 0\}}$ is a compact set.

The level sets of u,

$$[u]^a = \{x \in \mathbb{R}^m \,:\, u(x) \geq a\}, \;\; a \in (0,1],$$

and $[u]^0$ are nonempty compact convex sets in $\mathbb{R}^m$ (see [1]). In E^m, we consider the distance

$$d(u,v) = \sup_{a\in[0,1]} d_H\left([u]^a, [v]^a\right), \qquad u,\, v \in E^m,$$

with d_H the usual Hausdorff distance for nonempty compact convex subsets of $\mathbb{R}^m$. The metric space (E^m, d) is complete (see [1]). The distance d satisfies the following properties:

$$d(u+w, v+w) = d(u,v), \; u,\, v,\, w \in E^m,$$

$$d(\lambda\, u, \lambda\, v) = \lambda\, d(u,v), \; u,\, v \in E^m, \; \lambda > 0,$$

$$d(u+w, v+z) \leq d(u,v) + d(w,z), \; u,\, v,\, w,\, z \in E^m.$$

For u, $v \in E^m$, if there exists $w \in E^m$ such that $u = v + w$, then w is called the Hukuhara-difference of u and v, which is denoted by $u - v$. Note that, if u, $v \in E^1$ are such that $u + v = \chi_{\{0\}}$, then u and v are crisp (real) and $u = -v$. We say that a function $f : [t_0, T] \longrightarrow E^m$ is differentiable (in the sense of Hukuhara) at $t \in [t_0, T]$ if the Hukuhara-differences

$$f(t+h) - f(t), \;\; f(t) - f(t-h)$$

exist for $h > 0$ small enough and there exists $f'(t) \in E^m$ such that

$$\lim_{h\to 0^+} \frac{f(t+h) - f(t)}{h}, \;\; \lim_{h\to 0^+} \frac{f(t) - f(t-h)}{h}$$

exist and are equal to $f'(t)$. These limits are taken in the space (E^m, d), and if t is t_0 or T, then we consider the corresponding one-sided derivative. Some other approaches to fuzzy differentiability are included in [3, 21, 22]. We say that a fuzzy function $f : [t_0, T] \longrightarrow E^m$ is strongly measurable if, forall $a \in (0,1]$, the set-valued mapping

$$f_a : [t_0, T] \longrightarrow \mathcal{P}_K(\mathbb{R}^m),$$

given by

$$f_a(t) = [f(t)]^a,$$

is Lebesgue-measurable, considering the space of nonempty, compact, convex subsets of $\mathbb{R}^m$, $\mathcal{P}_K(\mathbb{R}^m)$, endowed with the topology generated by the Hausdorff distance d_H.

The integral of f over $[t_0,T]$, denoted by $\int_{[t_0,T]} f(t)\,dt$ or $\int_{t_0}^{T} f(t)\,dt$, is defined level wise by

$$\left[\int_{[t_0,T]} f(t)\,dt\right]^a = \int_{[t_0,T]} f_a(t)\,dt$$
$$= \left\{\int_{[t_0,T]} g(t)\,dt \;:\; g:[t_0,T]\longrightarrow \mathbb{R}^m \text{ is a measurable selection for } f_a\right\},$$

for $a \in (0,1]$. We say that f is integrable over $[t_0,T]$ if $\int_{[t_0,T]} f(t)\,dt \in E^m$. The continuity of $f:[t_0,T]\longrightarrow E^m$ implies the integrability of f. Besides, for f, g integrable functions, $d(f,g)$ is integrable (see [1], and [4, 6] for details) and

$$d\left(\int f, \int g\right) \le \int d(f,g).$$

In [23], an existence and uniqueness result for fuzzy differential systems is proved, showing some applications to the solvability of first-order fuzzy linear systems and higher-order fuzzy differential equations and systems. In this reference, it is also analyzed the structure of the set of solutions for fuzzy differential systems. The approach of this reference uses the metric d_0 in $(E^m)^n$ given by

$$d_0(U,V) = \sum_{i=1}^{n} d(u_i,v_i),$$

where $U=(u_1,\ldots,u_n)$, $V=(v_1,\ldots,v_n)\in (E^m)^n$, and

$$d(x,y) = \sup_{a\in[0,1]} d_H([x]^a,[y]^a),\ \forall x,\, y\in E^m,$$

with d_H the usual Hausdorff distance between nonempty compact convex subsets of $\mathbb{R}^m$. This paper is also devoted to the study of the existence and uniqueness of solution for initial value problems associated to fuzzy differential systems, and to its application to first-order fuzzy linear systems, as well as higher-order fuzzy differential equations and systems. However, in this case, we use vector valued metric and generalized metric spaces, which makes it necessary to apply some fixed point results included in [24] as the generalized Contraction Theorem.

2. GENERALIZED FUZZY SPACES

Definition 1. Let E be a real fuzzy space. A generalized metric for E is a mapping $d: E\times E\longrightarrow \mathbb{R}^n$ such that

a): $d(x,y)=d(y,x)$.
b): $d(x,y)\ge 0$, and $d(x,y)=0 \Longleftrightarrow x=y$.
c): $d(x,z)\le d(x,y)+d(y,z)$, where x, y, z are any elements of E.

In Definition 4.5.3 [24], it is considered that E is a real vector space, but the vectorial structure of E is not essential for the validity of the existence results we use in our procedure. In our study, we consider as generalized metric space the cartesian product of a finite number of copies of the fuzzy space E^m. If $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in (E^m)^n$, then we say that $x \le y$ if and only if $x_i \le y_i$, for all $i = 1, \dots, n$, where $\le$ represents the partial ordering in E^m given by:

$$u,\, v \in E^m, \quad u \le v \Longleftrightarrow [u]^a \subseteq [v]^a,\ \forall a \in [0,1].$$

E^m is not a real vector space, but it has the structure of a cone over $\mathbb{R}$. We consider fuzzy differential systems of the type

$$\begin{aligned} &y'(t) = F(t, y(t)),\ t \ge t_0,\\ &y(t_0) = y_0, \end{aligned}$$

where $F \in C\left(\mathbb{R}_+ \times (E^m)^n, (E^m)^n\right)$, $y_0 \in (E^m)^n$.Here, $(E^m)^n = E^m \times \overset{n)}{\cdots} \times E^m$, and $y_0 = (y_{01}, y_{02}, \dots, y_{0n})$, $y_{0i} \in E^m$, $\forall i = 1, \dots, n$. We take the generalized metric space $((E^m)^n, D)$, where

$$D(U, V) = (d(u_1, v_1), d(u_2, v_2), \dots, d(u_n, v_n)) \in \mathbb{R}^n_+.$$

Note that (E^m, d) is a complete metric space, thus $((E^m)^n, D)$ is a complete generalized metric space. In this context, we say that F satisfies a generalized contractivity condition if

$$D(F(t, U), F(t, V)) \le S\, D(U, V),$$

where $S = (s_{ij})$ is an $n \times n$ matrix with $s_{ij} \ge 0$, for all i, j and, for some $k > 1$, S^k is an A-matrix, that is, $I - S^k$ is positive definite, where I is the identity matrix. If we define

$$\begin{array}{rccl} \|\cdot\|_0 : & (E^m)^n & \longrightarrow & (\mathbb{R}_+)^n \\ & x = (x_1, x_2, \dots, x_n) & \longrightarrow & \|(x_1, x_2, \dots, x_n)\|_0 = D(x, (\chi_{\{0\}})^n) \\ & & & = (d(x_1, \chi_{\{0\}}), d(x_2, \chi_{\{0\}}), \dots, d(x_n, \chi_{\{0\}})), \end{array}$$

then, by the properties of distance d, $\|\cdot\|_0$ satisfies that

- $\|x\|_0 \ge \mathbf{0}$, for every $x \in (E^m)^n$.
- $\|x\|_0 = \mathbf{0}$ if and only if $x = (\chi_{\{0\}})^n$.
- $\|\lambda x\|_0 = |\lambda| \|x\|_0$, for every $\lambda \in \mathbb{R}$ and $x \in (E^m)^n$.
- $\|x + y\|_0 \le \|x\|_0 + \|y\|_0$, for every x, $y \in (E^m)^n$.

Here, we denote by $\mathbf{0} = (0, \dots, 0)$. Thus, $\|\cdot\|_0$ is a generalized norm for $(E^m)^n$ (this space has not a vectorial structure). The following results are essential to our procedure.

Theorem 2.1 (Theorem 4.5.2 [24])**.** *Let (E, d) be a complete generalized metric space and let $T : E \longrightarrow E$ be such that $d(Tx, Ty) \le Sd(x, y)$, where S is a nonnegative* *matrix such that for some k, S^k is an A-matrix. Then T has a unique fixed point x^*. Furthermore, for any $x \in E$, $x^* = \lim_{j\to\infty} T^j x$ and*

$$d(x^*, T^j x) \le (I - S)^{-1} S^j d(Tx, x).$$

Theorem 2.2 (Corollary 4.5.1 [24])**.** *Let (E, d) be a complete generalized metric space and let $T : E \longrightarrow E$ be such that $d(Tx, Ty) \le Sd(x, y)$, where S is a nonnegative matrix. If there exists an $x_0 \in E$ such that $\sum_{j=0}^{\infty} S^j d(Tx_0, x_0)$ converges, then T has a fixed point x^* such that $x^* = \lim_{j\to\infty} T^j x_0$.*

3. FIRST ORDER FUZZY DIFFERENTIAL SYSTEMS

Consider a fuzzy differential system of the type

$$\text{(1)}\qquad \begin{cases} y_i'(t) = F_i(t, y_1(t), \ldots, y_n(t)),\ t \in [t_0, T],\ i = 1, 2, \ldots, n, \\ \\ y_i(t_0) = b_i,\ \ i = 1, 2, \ldots, n, \end{cases}$$

where $F_i : [t_0, T] \times (E^m)^n \longrightarrow E^m$, $i = 1, \ldots, n$, and $b_1, b_2, \ldots, b_n \in E^m$, which can be written as

$$\begin{cases} Y' = F(t, Y), \\ Y(t_0) = \bar{b}, \end{cases}$$

where $\bar{b} = (b_1, \ldots, b_n)$, $F : [t_0, T] \times (E^m)^n \longrightarrow (E^m)^n$,$Y = (y_1, \ldots, y_n)$, and $F(t, Y) = (F_1(t, Y), \ldots, F_n(t, Y))$. If F is 'linear', then we get

$$\text{(2)}\qquad \begin{cases} y_i'(t) = \displaystyle\sum_{j=1}^{n} \alpha_{i,j}(t) y_j(t),\ i = 1, 2, \ldots, n, \\ \\ y_i(t_0) = b_i,\ \ i = 1, 2, \ldots, n. \end{cases}$$

where

$$\alpha_{i,j} : [t_0, T] \longrightarrow \mathbb{R},\ i,\ j = 1, 2, \ldots, n.$$

Equivalently,

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = A(t) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

where

$$\text{(3)}\qquad A(t) = \begin{pmatrix} \alpha_{1,1}(t) & \cdots & \alpha_{1,n}(t) \\ \vdots & \ddots & \vdots \\ \alpha_{n,1}(t) & \cdots & \alpha_{n,n}(t) \end{pmatrix},$$

and the product of a real $n \times n$ matrix by a fuzzy n-dimensional vector is defined by using the operations in E^m. Our approach uses generalized distances. In the following, we use that $((E^m)^n, D)$ is a complete generalized metric space. Next, we prove some auxiliary results.

Lemma 1. *Consider $J = [t_0, T]$ a bounded interval,*

$$C(J, E^m) = \{x : J \longrightarrow E^m \ : \ x \text{ is continuous}\},$$

and

$$\widetilde{D}(x, y) = (H(x_1, y_1), \ldots, H(x_n, y_n)), \quad x = (x_1, \ldots, x_n),\, y = (y_1, \ldots, y_n) \in (C(J, E^m))^n,$$

where

$$H(z, w) = \sup_{t \in J} \left\{ d(z(t), w(t)) e^{-\rho t} \right\},\ z,\ w \in C(J, E^m),$$

and $\rho > 0$. Then the space $((C(J, E^m))^n, \widetilde{D})$ is a complete generalized metric space.

Lemma 2. *The space of functions $x : J \longrightarrow E^m$ of class C^r $(r \in \mathbb{N})$ in the sense of Hukuhara (continuous and existing x', $\ldots$, $x^{(r}$ continuous) $(C^r(J, E^m), \bar{H})$ is a complete metric space, where*

$$\bar{H}(x, y) = \sum_{i=0}^{r} H\left(x^i, y^i\right), \quad x,\ y \in C^r(J, E^m),$$

where $x^{(0} = x$ and x^i denotes the ith-derivative of x in the sense of Hukuhara.

The proof can be derived similarly to the results in [16].

Lemma 3. *$((C^r(J, E^m))^n, \widetilde{H})$ is a complete generalized metric space, where*

$$\widetilde{H}(U, V) = (\bar{H}(u_1, v_1), \bar{H}(u_2, v_2), \ldots, \bar{H}(u_n, v_n)),$$

for $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n) \in (C^r(J, E^m))^n$.

The following result extends to fuzzy differential systems the results given in[4] for fuzzy differential equations, and our approach is based on generalized metric spaces.

Theorem 3.1. *Consider system (1), that is, $Y' = F(t, Y)$, where*

$$F : [t_0, T] \times (E^m)^n \longrightarrow (E^m)^n$$

is continuous,$Y = (y_1, \ldots, y_n)$, and $F(t, Y) = (F_1(t, Y), \ldots, F_n(t, Y))$. Suppose that there exists $S = (s_{ij})$ an $n \times n$ matrix with $s_{ij} \geq 0$, for all i, j and such that, for some $k > 1$, S^k is an A-matrix, and that the following condition holds, for $t \in J = [t_0, T]$, u, $v \in (C(J, E^m))^n$, and $j = 1, 2, \ldots, n$,

$$\int_{t_0}^{t} d\left(F_j(s, u(s)), F_j(s, v(s))\right) ds \leq \sum_{i=1}^{n} s_{ji} d(u_i(t), v_i(t)). \tag{4}$$

Then, for a given initial condition $\bar{b} = (b_1, \ldots, b_n) \in (E^m)^n$, system (1) has a unique solution.

Proof: Let $J = [t_0, T]$, and consider the complete generalized metric space $((C(J, E^m))^n, \widetilde{D})$ (see Lemma 1). Define the operator

$$G: \quad (C(J,E^m))^n \longrightarrow (C(J,E^m))^n$$
$$u \longrightarrow Gu,$$

by $Gu = (G_1 u, \ldots, G_n u)$, where

$$[G_i u](t) = b_i + \int_{t_0}^{t} F_i(s, u(s))\, ds,\ t \in J,\ i = 1, \ldots, n.$$

Here, $u = (u_1, \ldots, u_n)$, $u_i : J \longrightarrow E^m$ continuous, and $u(s) = (u_1(s), \ldots, u_n(s))$. We prove that, for an appropriate $\rho > 0$, G satisfies conditions in Theorem 2.1 (Theorem 4.5.2 [24]). The generalized contraction Theorem provides the existence of a unique fixed point u for G, which is the unique solution to problem (1), and satisfying that, for any $u_0 \in (C(J, E^m))^n$, $\lim_{j\to\infty} G^j(u_0) = u$, and

$$\tilde{D}(u, G^j u_0) \leq (I - S)^{-1} S^j \tilde{D}(G u_0, u_0).$$

To this purpose, we check that $\tilde{D}(Gu, Gv) \leq S\tilde{D}(u, v)$. Note that S is a nonnegative matrix such that, for some $k > 1$, S^k is an A-matrix, that is, nonnegative with $I - S^k$ positive definite. Since

$$\widetilde{D}(Gu, Gv) = (H(G_1 u, G_1 v), \ldots, H(G_n u, G_n v)),$$

then, for every $j = 1, 2, \ldots, n$ and u, $v \in (C(J, E^m))^n$, we get

$$\begin{aligned}
H(G_j u, G_j v) &= \sup_{t\in J} d\left(b_j + \int_{t_0}^{t} F_j(s, u(s))\, ds, b_j + \int_{t_0}^{t} F_j(s, v(s))\, ds\right) e^{-\rho t} \\
&\leq \sup_{t\in J} \int_{t_0}^{t} d\left(F_j(s, u(s)), F_j(s, v(s))\right)\, ds\, e^{-\rho t} \\
&\leq \sup_{t\in J} \sum_{i=1}^{n} s_{ji} d(u_i(t), v_i(t)) e^{-\rho t} \\
&= \sum_{i=1}^{n} s_{ji} \sup_{t\in J} \left\{d(u_i(t), v_i(t)) e^{-\rho t}\right\} = \sum_{i=1}^{n} s_{ji} H(u_i, v_i) = (S\widetilde{D}(u, v))_j.
\end{aligned}$$

This proves that G satisfies a generalized contractive condition and the result follows. □

Remark 1. Condition (4) in Theorem 3.1 can be replaced by the more general condition: There exists $\rho > 0$ such that

$$\text{(5)} \qquad \sup_{t\in J} \int_{t_0}^{t} d\left(F_j(s, u(s)), F_j(s, v(s))\right)\, ds\, e^{-\rho t} \leq \sum_{i=1}^{n} s_{ji} \sup_{t\in J} \left\{d(u_i(t), v_i(t)) e^{-\rho t}\right\},$$

for every u, $v \in (C(J, E^m))^n$ and $j = 1, 2, \ldots, n$.

Theorem 3.2. *Consider system* (1)*, that is,* $Y' = F(t, Y)$*, where*

$$F : [t_0, T] \times (E^m)^n \longrightarrow (E^m)^n$$

is continuous, $Y = (y_1, \ldots, y_n)$*, and* $F(t, Y) = (F_1(t, Y), \ldots, F_n(t, Y))$*. Suppose that there exists* $S = (s_{ij})$ *an* $n \times n$ *matrix with* $s_{ij} \geq 0$*, for all* i*,* j*, and that, for some* $k > 1$*,* S^k *is an* A*-matrix. Suppose also that the following condition holds, for* $t \in [t_0, T]$ *and* U*,* $V \in (E^m)^n$*,*

$$D\left(F(t, U), F(t, V)\right) \leq S\, D(U, V), \tag{6}$$

that is, for every $j = 1, 2, \ldots, n$*,*

$$d\left(F_j(t, U), F_j(t, V)\right) \leq \sum_{i=1}^{n} s_{ji} d(u_i, v_i).$$

Then, for a given initial condition $\bar{b} = (b_1, \ldots, b_n) \in (E^m)^n$*, system* (1) *has a unique solution.*

Proof: Following the proof of Theorem 3.1, and, using (6), we obtain, for every $j = 1, 2, \ldots, n$ and u, $v \in (C(J, E^m))^n$, that

$$\begin{aligned}
H(G_j u, G_j v) &\leq \sup_{t \in J} \int_{t_0}^{t} d\left(F_j(s, u(s)), F_j(s, v(s))\right)\, ds\, e^{-\rho t} \\
&\leq \sup_{t \in J} \int_{t_0}^{t} \sum_{i=1}^{n} s_{ji} d(u_i(s), v_i(s))\, ds\, e^{-\rho t} \\
&\leq \sum_{i=1}^{n} s_{ji} \sup_{t \in J} \left\{ d(u_i(t), v_i(t))\, e^{-\rho t} \right\} \sup_{t \in J} \int_{t_0}^{t} e^{\rho s}\, ds\, e^{-\rho t} \\
&= \sum_{i=1}^{n} s_{ji} \frac{1 - e^{-\rho(T - t_0)}}{\rho} H(u_i, v_i) \\
&= \left(S \frac{1 - e^{-\rho(T - t_0)}}{\rho} \widetilde{D}(u, v) \right)_j .
\end{aligned}$$

This shows that

$$\widetilde{D}(Gu, Gv) \leq S \left(\frac{1 - e^{-\rho(T - t_0)}}{\rho} \right) \widetilde{D}(u, v),\; u,\, v \in (C(J, E^m))^n.$$

Now, for $\alpha = \dfrac{1 - e^{-\rho(T - t_0)}}{\rho} > 0$, αS satisfies the following properties:

- αS is nonnegative.
- $(\alpha S)^k = \alpha^k S^k$ is nonnegative.
- For α small enough, $I - (\alpha S)^k = I - \alpha^k S^k$ is positive definite. Indeed,

$$x^t (I - \alpha^k S^k) x = x^t x - \alpha^k x^t S^k x.$$

Since $I - S^k$ is positive definite, then $x^t(I - S^k)x > 0$, for $x \neq 0$, which implies that $x^t x > x^t S^k x$, for $x \neq 0$, and hence

$$x^t(I - \alpha^k S^k)x = x^t x - \alpha^k x^t S^k x > x^t x - \alpha^k x^t x$$
$$= (1-\alpha^k)x^t x = (1-\alpha^k)\|x\|_2^2.$$

This expression is clearly positive if $x \neq 0$ and $\alpha^k < 1$.

Since

$$\lim_{\rho\to+\infty} \frac{1-e^{-\rho(T-t_0)}}{\rho} = 0,$$

the proof is concluded choosing $\rho > 0$ with $\alpha = \frac{1-e^{-\rho(T-t_0)}}{\rho} < 1$. □

Theorem 3.3. *Consider system* (1)*, that is,* $Y' = F(t, Y)$*, where*

$$F : [t_0, T] \times (E^m)^n \longrightarrow (E^m)^n$$

is continuous, $Y = (y_1, \dots, y_n)$*, and* $F(t, Y) = (F_1(t, Y), \dots, F_n(t, Y))$*. Suppose that*

$$D\left(F(t, U), F(t, V)\right) \leq S\, D(U, V), \tag{7}$$

for $t \in [t_0, T]$ *and* $U, V \in (E^m)^n$*, where* $S = (s_{ij})$ *is a* $n \times n$ *nonnegative matrix such that there exists* $\rho > 0$ *satisfying that*

$$S^k \left(\frac{1-e^{-\rho(T-t_0)}}{\rho}\right)^k \quad \text{is an } A\text{-matrix, for some } k > 1.$$

Then, for a given initial condition $\bar{b} = (b_1, \dots, b_n) \in (E^m)^n$*, system* (1) *has a unique solution.*

Theorem 3.4. *Consider system* (1) *and* G *the operator defined in the proof of Theorem* 3.1*. If there exists* $S = (s_{ij})$ *an* $n \times n$ *matrix with* $s_{ij} \geq 0$*, for all* i, j*, and there exists* $g \in (C(J, E^m))^n$ *such that* $\sum_{j=0}^{\infty} S^j \widetilde{D}(Gg, g)$ *converges, then* G *has a fixed point* x^* *such that* $x^* = \lim_{j\to\infty} G^j g$*.*

Remark 2. Note that

$$(\widetilde{D}(Gg, g))_i = \sup_{t\in J} d\left(b_i + \int_{t_0}^t F_i(s, g(s))\, ds, g_i(t)\right) e^{-\rho t}, \quad i = 1, 2, \dots, n.$$

Lemma 4. *If* $M = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathcal{M}_{n\times n}$ *is the constant matrix whose coefficients are equal to* 1*, and* $k \in \mathbb{N}$*, then* $M^k = \begin{pmatrix} n^{k-1} & \cdots & n^{k-1} \\ \vdots & \ddots & \vdots \\ n^{k-1} & \cdots & n^{k-1} \end{pmatrix}$.

The proof can be easily completed by induction in k. Concerning the existence of solution for linear systems (2), we can deduce some results.

Theorem 3.5. *If the maps $\alpha_{i,j} : [t_0, T] \longrightarrow \mathbb{R}, i, j = 1, 2, \ldots, n$, are continuous, then, for each fixed initial condition, system (2) has a unique solution $\bar{y} = (y_1, \ldots, y_n)$.*

Proof: System (2) can be written in terms of system (1), taking

$$F(t,Y) = \begin{pmatrix} F_1(t,Y) \\ \vdots \\ F_n(t,Y) \end{pmatrix} = A(t) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

for $A(t)$ given in (3). By hypothesis, F is a continuous function. We check that condition (5) holds. Indeed, for every $j = 1, 2, \ldots, n$,

$$F_j(t,Y) = \alpha_{j,1}(t)y_1 + \cdots + \alpha_{j,n}(t)y_n.$$

For every $t \in J = [t_0, T]$, $u, v \in (C(J, E^m))^n$, $u(s) = (u_1(s), \ldots, u_n(s))$, $v(s) = (v_1(s), \ldots, v_n(s))$, $s \in J$, and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&\sup_{t\in J} \int_{t_0}^{t} d\left(F_j(s,u(s)), F_j(s,v(s))\right)\, ds\, e^{-\rho t} \\
&\quad = \sup_{t\in J} \int_{t_0}^{t} d\left(\sum_{i=1}^{n} \alpha_{j,i}(s)u_i(s), \sum_{i=1}^{n} \alpha_{j,i}(s)v_i(s)\right) ds\, e^{-\rho t} \\
&\quad \le \sup_{t\in J} \int_{t_0}^{t} \left\{\sum_{i=1}^{n} |\alpha_{j,i}(s)| d(u_i(s), v_i(s))\right\} ds\, e^{-\rho t} \\
&\quad = \sup_{t\in J} \left\{\sum_{i=1}^{n} \int_{t_0}^{t} |\alpha_{j,i}(s)| d(u_i(s), v_i(s))\, ds\right\} e^{-\rho t} \\
&\quad \le \sum_{i=1}^{n} \sup_{t\in J} \left\{d(u_i(t), v_i(t)) e^{-\rho t}\right\} \sup_{t\in J} \left\{\int_{t_0}^{t} |\alpha_{j,i}(s)| e^{\rho s}\, ds\, e^{-\rho t}\right\} \\
&\quad \le \sum_{i=1}^{n} \sup_{t\in J} \left\{d(u_i(t), v_i(t)) e^{-\rho t}\right\} \sup_{t\in J} K \frac{1 - e^{-\rho(t-t_0)}}{\rho} \\
&\quad = \sum_{i=1}^{n} \sup_{t\in J} \left\{d(u_i(t), v_i(t)) e^{-\rho t}\right\} K \frac{1 - e^{-\rho(T-t_0)}}{\rho},
\end{aligned}$$

where $|\alpha_{i,j}(t)| \le K$, for every $t \in J$ and $i, j \in \{1, 2, \ldots, n\}$, since $\alpha_{i,j}$ is continuous in the compact interval J, for $i, j \in \{1, 2, \ldots, n\}$. Note that condition (5) is satisfied taking

$$s_{ji} = K \frac{1 - e^{-\rho(T-t_0)}}{\rho}, \quad i, j = 1, 2, \ldots, n,$$

thus S is a constant matrix, which is equal to $S = K \frac{1-e^{-\rho(T-t_0)}}{\rho}(1)$, where (1) is the matrix whose coefficients are equal to 1. It is clear that S is nonnegative. We have to find $k \in \mathbb{N}$, $k > 1$, and $\rho > 0$ such that $I - S^k$ is positive definite. We prove that there exists $\rho > 0$ such that $I - S^2$ is positive definite. Note that $S^2 = \left(K\frac{1-e^{-\rho(T-t_0)}}{\rho}\right)^2 (1)^2 = (\beta) = \beta(1)$ is a constant matrix whose coefficients are equal to $\beta := \left(K\frac{1-e^{-\rho(T-t_0)}}{\rho}\right)^2 n$, and that

$$I - S^2 = \begin{pmatrix} 1-\beta & -\beta & \cdots & -\beta \\ -\beta & 1-\beta & \ddots & -\beta \\ \vdots & \ddots & \ddots & \vdots \\ -\beta & \cdots & -\beta & 1-\beta \end{pmatrix}.$$

We check that, for $\beta > 0$ small enough ($\rho > 0$ large enough), $I - S^2$ is positive definite. Indeed,

$$\begin{aligned}
&\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} (I - S^2) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} (1-\beta)x_1 - \beta x_2 - \cdots - \beta x_n \\ -\beta x_1 + (1-\beta)x_2 - \cdots - \beta x_n \\ \vdots \\ -\beta x_1 - \beta x_2 - \cdots - \beta x_{n-1} + (1-\beta)x_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} x_1 - \beta\sum_{i=1}^n x_i \\ x_2 - \beta\sum_{i=1}^n x_i \\ \vdots \\ x_n - \beta\sum_{i=1}^n x_i \end{pmatrix} \\
&= x_1^2 - \beta x_1 \sum_{i=1}^n x_i + x_2^2 - \beta x_2 \sum_{i=1}^n x_i + \cdots + x_n^2 - \beta x_n \sum_{i=1}^n x_i \\
&= \sum_{i=1}^n x_i^2 - \beta \sum_{i=1}^n x_i \sum_{j=1}^n x_j = \sum_{i=1}^n x_i^2 - \beta\left(\sum_{i=1}^n x_i\right)^2.
\end{aligned}$$

Hence $x^t(I - S^2)x > 0$ if and only if $\left(\sum_{i=1}^n x_i\right)^2 < \frac{1}{\beta}\sum_{i=1}^n x_i^2$, that is,$\sum_{i=1}^n x_i < \frac{1}{\sqrt{\beta}}\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. Due to the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathbb{R}^n$,

$$\sum_{i=1}^n x_i \leq \sum_{i=1}^n |x_i| = \|x\|_1 \leq R\|x\|_2, \text{ where } R > 0.$$

If $\beta > 0$ is small enough, $0 < \beta < \left(\frac{1}{R}\right)^2$, then $R < \frac{1}{\sqrt{\beta}}$, and taking $x \neq 0$, then $\|x\|_2 > 0$, and

$$\sum_{i=1}^n x_i \leq \sum_{i=1}^n |x_i| = \|x\|_1 \leq R\|x\|_2 < \frac{1}{\sqrt{\beta}}\|x\|_2.$$

The proof is complete taking into account that

$$\lim_{\rho\to+\infty} K^2\left(\frac{1-e^{-\rho(T-t_0)}}{\rho}\right)^2 n = 0,$$

since n is fixed. Hence, there exists a unique solution $\bar{y}$. Note that $I-S$ is also positive definite. Besides, for any $u \in (C(J,E^m))^n$,

$$\tilde{D}(\bar{y}, G^j(u)) \leq \begin{pmatrix} 1-K\frac{1-e^{-\rho(T-t_0)}}{\rho} & \cdots & -K\frac{1-e^{-\rho(T-t_0)}}{\rho} \\ \vdots & \ddots & \vdots \\ -K\frac{1-e^{-\rho(T-t_0)}}{\rho} & \cdots & 1-K\frac{1-e^{-\rho(T-t_0)}}{\rho} \end{pmatrix}^{-1}$$
$$\times \left(\frac{K}{\rho}\right)^j (1-e^{-\rho(T-t_0)})^j \begin{pmatrix} n^{j-1} & \cdots & n^{j-1} \\ \vdots & \ddots & \vdots \\ n^{j-1} & \cdots & n^{j-1} \end{pmatrix} \tilde{D}(Gu, u),$$

where Gu is defined in the proof of Theorem 3.1. On the other hand, according to Theorem 3.3,

$$d\left(\sum_{i=1}^n \alpha_{j,i}(t)u_i, \sum_{i=1}^n \alpha_{j,i}(t)v_i\right) \leq \sum_{i=1}^n |\alpha_{j,i}(t)| d(u_i, v_i) \leq \sum_{i=1}^n K d(u_i, v_i),$$

and

$$D\left(F(t,U), F(t,V)\right) \leq \begin{pmatrix} K & \cdots & K \\ \vdots & \ddots & \vdots \\ K & \cdots & K \end{pmatrix} D(U,V),$$

for $t \in [t_0, T]$, and $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n) \in (E^m)^n$, where $K \geq 0$ is such that $|\alpha_{i,j}(t)| \leq K$, $\forall t \in [t_0, T]$, $i, j = 1, 2, \ldots, n$, and we have proved that there exists $\rho > 0$ with

$$\left(\frac{1-e^{-\rho(T-t_0)}}{\rho}\right) \begin{pmatrix} K & \cdots & K \\ \vdots & \ddots & \vdots \\ K & \cdots & K \end{pmatrix} = \frac{K}{\rho}\left(1-e^{-\rho(T-t_0)}\right) \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

is an A-matrix. □

A similar result is valid for $F(t,Y) = A(t)Y + \sigma(t)$, for $\sigma : J \longrightarrow (E^m)^n$ a continuous function, as established in [23]. Under the hypotheses of Theorem 3.5, if $\bar{b} = (b_1, \ldots, b_n)$ is the initial condition, then the sequence $\{\bar{g}_j\}_{j\in\mathbb{N}}$ defined by$\bar{g}_0(t) = \bar{b}$, and

$$\bar{g}_j(t) = \bar{b} + \int_{t_0}^t A(s)\bar{g}_{j-1}(s)\, ds, \;\; t \in [t_0, T], \;\; j = 1, 2, \ldots$$

converges towards the unique solution to problem (2) with initial condition $\bar{b}$, and the convergence is in the generalized distance $\widetilde{D}$. On the other hand, for system (1), we can define $\bar{g}_0(t) = \bar{b}$, and

$$\bar{g}_j(t) = \bar{b} + \int_{t_0}^{t} F(s, \bar{g}_{j-1}(s))\, ds, \;\; t \in J = [t_0, T], \;\; j = 1, 2, \ldots,$$

obtaining a sequence which approximates the unique solution to problem (1) relative to the initial condition $\bar{b}$.

4. HIGHER ORDER FUZZY DIFFERENTIAL EQUATIONS

In this section, we analyze higher-order fuzzy differential equations by reducing them to a first-order system. The following result refers to the 'linear' case.

Lemma 5. *If $\alpha_i : [t_0, T] \longrightarrow \mathbb{R}$ are continuous, for $i = 0, 1, \ldots, n-1$, $\sigma \in C([t_0, T], E^m)$, and $b_i \in E^m$, for $i = 0, 1, \ldots, n-1$, then equation*

$$\text{(8)} \qquad \begin{cases} y^{(n}(t) = \alpha_{n-1}(t) y^{(n-1)}(t) + \cdots + \alpha_0(t) y(t) + \sigma(t), \; t \in [t_0, T], \\ y(t_0) = b_0, \; \ldots, y^{(n-1)}(t_0) = b_{n-1}, \end{cases}$$

has a unique solution.

Proof: It follows easily by taking $y_1 = y$, $y_2 = y'$, …, $y_n = y^{(n-1)}$ in equation (8), which leads to the system

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ \cdots \\ y_{n-1}' = y_n, \\ y_n' = \alpha_0(t) y_1 + \cdots + \alpha_{n-1}(t) y_n + \sigma(t), \end{cases}$$

or

$$\begin{pmatrix} y_1' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = A(t) \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} \chi_{\{0\}} \\ \vdots \\ \chi_{\{0\}} \\ \sigma(t) \end{pmatrix},$$

where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \alpha_0(t) & \alpha_1(t) & \cdots & \alpha_{n-2}(t) & \alpha_{n-1}(t) \end{pmatrix}.$$

The conclusion is derived applying Theorem 3.5. □

The following result analyzes nth-order fuzzy differential equations by reducing them to n-dimensional first-order fuzzy differential systems. Obviously, the ith-derivative of y, y^i, is considered in the sense of Hukuhara.

Corollary 1. *Suppose that* $b_0, b_1, b_2, \ldots, b_{n-1} \in E^m$*,* $f : [t_0, T] \times (E^m)^n \longrightarrow E^m$ *is continuous, and that there exist real numbers* $M_1, M_2, \ldots, M_n \geq 0$ *such that*

$$d(f(t, u_1, u_2, \ldots, u_n), f(t, v_1, v_2, \ldots, v_n)) \leq \sum_{i=1}^{n} M_i \, d(u_i, v_i), \tag{9}$$

for all $t \in [t_0, T]$*,* $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in E^m$*. Then the initial value problem for the higher-order fuzzy differential equation*

$$\begin{cases} y^{(n}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)), \; t \in [t_0, T], \\ \\ y(t_0) = b_0, \;\; y'(t_0) = b_1, \;\; \ldots, \;\; y^{(n-1)}(t_0) = b_{n-1} \end{cases} \tag{10}$$

has a unique solution on $[t_0, T]$*.*

Proof: By the change of variable $y_1 = y$, $y_2 = y'$, $\ldots$, $y_n = y^{(n-1)}$, problem (10) is written as $Y' = F(t, Y)$, $Y(t_0) = (b_0, \ldots, b_{n-1}) \in (E^m)^n$, where $Y = (y_1, \ldots, y_n)$, and F is the continuous function given by

$$F(t, Y) = F(t, y_1, \ldots, y_n) = (y_2, y_3, \ldots, y_n, f(t, y_1, y_2, \ldots, y_n)).$$

To check assumption (5), take $u, v \in (C(J, E^m))^n$, then, for $j = 1, 2, \ldots, n-1$, we get

$$\begin{aligned} &\sup_{t \in J} \int_{t_0}^{t} d\left(F_j(s, u(s)), F_j(s, v(s))\right) \, ds \, e^{-\rho t} \\ &\quad = \sup_{t \in J} \int_{t_0}^{t} d(u_{j+1}(s), v_{j+1}(s)) \, ds \, e^{-\rho t} \\ &\quad \leq \sup_{t \in J} \left\{ d(u_{j+1}(t), v_{j+1}(t)) \, e^{-\rho t} \right\} \sup_{t \in J} \int_{t_0}^{t} e^{\rho s} \, ds \, e^{-\rho t} \\ &\quad = \sup_{t \in J} \left\{ d(u_{j+1}(t), v_{j+1}(t)) \, e^{-\rho t} \right\} \sup_{t \in J} \frac{1 - e^{-\rho(t - t_0)}}{\rho} \\ &\quad = \sup_{t \in J} \left\{ d(u_{j+1}(t), v_{j+1}(t)) \, e^{-\rho t} \right\} \frac{1 - e^{-\rho(T - t_0)}}{\rho} \\ &\quad = \sum_{i=1}^{n} s_{ji} \sup_{t \in J} \left\{ d(u_i(t), v_i(t)) e^{-\rho t} \right\}, \end{aligned}$$

where

$$s_{j(j+1)} = \frac{1 - e^{-\rho(T - t_0)}}{\rho}, \qquad s_{ji} = 0, \, i \neq j+1.$$

Finally, for $j = n$,

$$\begin{aligned} &\sup_{t \in J} \int_{t_0}^{t} d\left(F_n(s, u(s)), F_n(s, v(s))\right) \, ds \, e^{-\rho t} \\ &\quad = \sup_{t \in J} \int_{t_0}^{t} d(f(s, u_1(s), \ldots, u_n(s)), f(s, v_1(s), \ldots, v_n(s))) \, ds \, e^{-\rho t} \end{aligned}$$

$$\leq \sup_{t\in J} \int_{t_0}^{t} \sum_{i=1}^{n} M_i d(u_i(s), v_i(s))\, ds\, e^{-\rho t}$$
$$\leq \sum_{i=1}^{n} M_i \sup_{t\in J} \left\{ d(u_i(t), v_i(t)) e^{-\rho t} \right\} \sup_{t\in J} \int_{t_0}^{t} e^{\rho s}\, ds\, e^{-\rho t}$$
$$= \sum_{i=1}^{n} M_i \sup_{t\in J} \left\{ d(u_i(t), v_i(t)) e^{-\rho t} \right\} \sup_{t\in J} \frac{1 - e^{-\rho(t-t_0)}}{\rho}$$
$$= \sum_{i=1}^{n} M_i \frac{1 - e^{-\rho(T-t_0)}}{\rho} \sup_{t\in J} \left\{ d(u_i(t), v_i(t)) e^{-\rho t} \right\}$$
$$\leq \sum_{i=1}^{n} \frac{K}{\rho} (1 - e^{-\rho(T-t_0)}) \sup_{t\in J} \left\{ d(u_i(t), v_i(t)) e^{-\rho t} \right\},$$

where $K = \max_{i=1,\ldots,n} M_i$. Hence, we choose

$$s_{ni} = \frac{K}{\rho}(1 - e^{-\rho(T-t_0)}),\ i = 1, 2, \ldots, n.$$

In consequence, taking $\alpha = \frac{1-e^{-\rho(T-t_0)}}{\rho} > 0$, the matrix S can be written as

$$S = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha \\ K\alpha & K\alpha & \cdots & K\alpha & K\alpha \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ K & K & \cdots & K & K \end{pmatrix}.$$

For this choice, S is nonnegative and $S^k = \alpha^k \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ K & K & \cdots & K & K \end{pmatrix}^k$ is nonnegative, for $k \in \mathbb{N}$. If $K = 0$, then $M_i = 0$, for all i, and $S = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ is such that $det(S) = 0$. However, in this case, the Lipschitz condition (9) implies that $d(f(t, u_1, u_2, \ldots, u_n), f(t, v_1, v_2, \ldots, v_n)) = 0$, for all $t \in [t_0, T]$, and u_1, $u_2, \ldots, u_n$, v_1, v_2, $\ldots$, $v_n \in E^m$, so that f is a function of the variable t (independent of $u_1, \ldots, u_n$) and the equation is easily solvable. Assume that $K > 0$. We check that $I - S^2$ is positive definite. Indeed,

$$S^2 = \alpha^2 \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ K & K & \cdots & K & K \end{pmatrix}^2$$

$$= \alpha^2 \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ K & K & \cdots & \cdots & K & K \\ K^2 & K+K^2 & \cdots & \cdots & K+K^2 & K+K^2 \end{pmatrix},$$

and

$$I - S^2 = \begin{pmatrix} 1 & 0 & -\alpha^2 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -\alpha^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & -\alpha^2 \\ -K\alpha^2 & -K\alpha^2 & \cdots & \cdots & 1-K\alpha^2 & -K\alpha^2 \\ -K^2\alpha^2 & -(K+K^2)\alpha^2 & \cdots & \cdots & -(K+K^2)\alpha^2 & 1-(K+K^2)\alpha^2 \end{pmatrix}.$$

Note that, for $n = 2$,

$$S^2 = \alpha^2 \begin{pmatrix} K & K \\ K^2 & K+K^2 \end{pmatrix},$$

and

$$I - S^2 = \begin{pmatrix} 1-K\alpha^2 & -K\alpha^2 \\ -K^2\alpha^2 & 1-(K+K^2)\alpha^2 \end{pmatrix}.$$

To prove that the matrix $I - S^2$ is positive definite,

$$x^t(I-S^2)x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} (I-S^2) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} x_1 - \alpha^2 x_3 \\ x_2 - \alpha^2 x_4 \\ \vdots \\ x_{n-2} - \alpha^2 x_n \\ x_{n-1} - K\alpha^2 \sum_{i=1}^{n} x_i \\ x_n - K^2\alpha^2 \sum_{i=1}^{n} x_i - K\alpha^2 \sum_{i=2}^{n} x_i \end{pmatrix}$$

$$= \sum_{i=1}^{n} x_i^2 - \alpha^2 \sum_{i=1}^{n-2} x_i x_{i+2} - K\alpha^2 \sum_{i=1}^{n} x_i x_{n-1} - K^2\alpha^2 \sum_{i=1}^{n} x_i x_n - K\alpha^2 \sum_{i=2}^{n} x_i x_n,$$

with the exception of the case $n = 2$, for which the term $-\alpha^2 \sum_{i=1}^{n-2} x_i x_{i+2}$ makes no sense. Last expression is positive if and only if

$$\sum_{i=1}^{n} x_i^2 > \alpha^2 \sum_{i=1}^{n-2} x_i x_{i+2} + K\alpha^2 \sum_{i=1}^{n} x_i x_{n-1} + K^2\alpha^2 \sum_{i=1}^{n} x_i x_n + K\alpha^2 \sum_{i=2}^{n} x_i x_n,$$

which can be obtained, for $x \neq 0$, if $\alpha > 0$ is small enough, that is, if $\rho > 0$ is large enough. Indeed,

$$\begin{aligned}
&\alpha^2 \sum_{i=1}^{n-2} x_i x_{i+2} + K\alpha^2 \sum_{i=1}^{n} x_i x_{n-1} + K^2\alpha^2 \sum_{i=1}^{n} x_i x_n + K\alpha^2 \sum_{i=2}^{n} x_i x_n \\
&\quad \leq \alpha^2 \left\{ \sum_{i=1}^{n-2} |x_i||x_{i+2}| + K \sum_{i=1}^{n} |x_i||x_{n-1}| + K^2 \sum_{i=1}^{n} |x_i||x_n| + K \sum_{i=2}^{n} |x_i||x_n| \right\} \\
&\quad \leq \alpha^2(1 + 2K + K^2) \left(\sum_{i=1}^{n} |x_i| \right)^2 = \alpha^2(1 + 2K + K^2)\|x\|_1^2 \\
&\quad \leq \alpha^2(1 + 2K + K^2)R^2\|x\|_2^2 = \alpha^2(1 + 2K + K^2)R^2 \sum_{i=1}^{n} |x_i|^2 \\
&\quad = \alpha^2(1 + 2K + K^2)R^2 \sum_{i=1}^{n} x_i^2,
\end{aligned}$$

where we have taken into account the existence of repeated terms of the type $x_i x_j$, and $R > 0$ is such that $\|x\|_1 \leq R\|x\|_2$. Considering $x \neq 0$, then $\|x\|_2 > 0$, and it suffices to take $0 < \alpha < \frac{1}{R\sqrt{1+2K+K^2}}$ to finish the proof, since $\alpha^2(1+2K+K^2)R^2 < 1$, and, for $x \neq 0$,

$$\alpha^2(1 + 2K + K^2)R^2 \sum_{i=1}^{n} x_i^2 < \sum_{i=1}^{n} x_i^2.$$

On the other hand, $\alpha = \dfrac{1 - e^{-\rho(T-t_0)}}{\rho} > 0$, thus we can choose $\alpha > 0$ small enough, taking $\rho > 0$ large enough. By Theorem 3.1 (and Remark 1), there exists a unique solution to $Y' = F(t, Y)$, corresponding to the initial condition $Y(t_0) = (b_0, \ldots, b_{n-1})$ and, therefore, a unique solution to problem (10). □

Note that condition (9) coincides with Condition (7) in [23].

Remark 3. If we use Theorem 3.3 in the proof of Corollary 1, we can easily check that

$$D\left(F(t, U), F(t, V)\right) \leq S\, D(U, V),$$

for $t \in [t_0, T]$ and U, $V \in (E^m)^n$, where

$$s_{j(j+1)} = 1, \qquad s_{ji} = 0,\ i \neq j + 1,\ \text{for } j = 1, 2, \ldots, n - 1,$$

$$s_{ni} = K = \max\{M_i, i = 1, 2, \ldots, n\},\ i = 1, 2, \ldots, n.$$

Indeed, for $t \in [t_0, T]$, and $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n) \in (E^m)^n$,

$$d\left(F_j(t, U), F_j(t, V)\right) = d(u_{j+1}, v_{j+1}) = \sum_{i=1}^{n} s_{ji} d(u_i, v_i),$$

for $j = 1, 2, \ldots, n-1$, and

$$d\left(F_n(t, U), F_n(t, V)\right) = d(f(t, U), f(t, V)) \leq \sum_{i=1}^{n} M_i d(u_i, v_i) \leq \sum_{i=1}^{n} K d(u_i, v_i).$$

Finally, S is nonnegative and there exists $\rho > 0$ such that

$$S\left(\frac{1 - e^{-\rho(T-t_0)}}{\rho}\right) = \left(\frac{1 - e^{-\rho(T-t_0)}}{\rho}\right) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ K & K & \cdots & K & K \end{pmatrix}$$

is an A-matrix.

Similarly to [23], and as a particular case, uniqueness of solution can be deduced for the following equation

$$\text{(11)} \qquad \left\{ \begin{array}{l} y^{(n}(t) = q_1 y(t) + q_2 y'(t) + \cdots + q_n y^{(n-1)}(t) + \sigma(t),\ t \in [t_0, T], \\ \\ \qquad y(t_0) = b_0,\ \ y'(t_0) = b_1,\ \ \ldots,\ \ y^{(n-1)}(t_0) = b_{n-1}, \end{array} \right.$$

with $\sigma \in C([t_0, T], E^m)$, $q_1,\ q_2,\ \ldots,\ q_n \in \mathbb{R}$, $b_0,\ b_1, \ldots, b_{n-1} \in E^m$.

5. HIGHER-ORDER FUZZY DIFFERENTIAL SYSTEMS

We consider the higher order fuzzy differential system

$$\left\{ \begin{array}{l} U^{(r} = F(t, U, U', \ldots, U^{(r-1)}),\ t \in [t_0, T], \\ \qquad U(t_0) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = B, \\ \qquad U'(t_0) = \begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix} = B', \\ \qquad\qquad \vdots \\ \qquad U^{(r-1}(t_0) = \begin{pmatrix} b_1^{r-1} \\ \vdots \\ b_n^{r-1} \end{pmatrix} = B^{r-1}, \end{array} \right.$$

with $r \in \mathbb{N}$, $(r \geq 2)$, B, B', $\ldots$, $B^{r-1} \in (E^m)^n$, and $F : [t_0, T] \times [(E^m)^n]^r \longrightarrow (E^m)^n$ continuous. The system is written componentwise as

$$\text{(12)} \qquad \begin{cases} u_1^{(r}(t) = F_1(t, U(t), U'(t), \ldots, U^{(r-1}(t)),\ t \in [t_0, T], \\ \qquad\vdots \\ u_n^{(r}(t) = F_n(t, U(t), U'(t), \ldots, U^{(r-1}(t)),\ t \in [t_0, T], \\ \\ \left.\begin{array}{c} u_i(t_0) = [U(t_0)]_i = b_i, \\ u_i'(t_0) = [U'(t_0)]_i = b_i', \\ \vdots \\ u_i^{(r-1}(t_0) = [U^{(r-1}(t_0)]_i = b_i^{r-1}, \end{array}\right\} i = 1, 2, \ldots, n, \end{cases}$$

where $F_i : [t_0, T] \times (E^m)^{nr} \longrightarrow E^m$, and b_i, $b_i', \ldots, b_i^{r-1} \in E^m$, $i = 1, 2, \ldots, n$. Using the change of variable

$$X_1 = U,\ X_2 = U', \ldots,\ X_r = U^{(r-1} \in (E^m)^n$$

and $\bar{X} = (X_1, \ldots, X_r) \in (E^m)^{nr}$, the higher-order fuzzy differential system is written as the nr-dimensional first-order fuzzy differential system

$$\begin{cases} \bar{X}' = \bar{F}(t, \bar{X}) = \bar{F}(t, X_1, X_2, \ldots, X_r),\ t \in [t_0, T], \\ \bar{X}(t_0) = \left(b_1, \ldots, b_n, b_1', \ldots, b_n', \ldots, b_1^{r-1}, \ldots, b_n^{r-1}\right) \in (E^m)^{nr}, \end{cases}$$

with $\bar{F} : [t_0, T] \times (E^m)^{nr} \longrightarrow (E^m)^{nr}$ a continuous function given by

$$\bar{F}(t, \bar{X}) = \bar{F}(t, X_1, X_2, \ldots, X_r) = (X_2, X_3, \ldots, X_r, F(t, X_1, X_2, \ldots, X_r)).$$

In the following result, we analyze sufficient conditions for the existence of a unique solution to this system, given a fixed initial condition.

Theorem 5.1. *If there exist constants $M_{il} \geq 0$, for $i = 1, \ldots, n$, $l = 1, \ldots, nr$ such that*

$$d\left(F_i(t, \bar{X}), F_i(t, \bar{Y})\right) \leq \sum_{l=1}^{nr} M_{il} d(\bar{X}_l, \bar{Y}_l),$$

for every $i = 1, 2, \ldots, n$, then problem (12) *has a unique solution.*

Proof: We prove that condition (6) in Theorem 3.2 holds, that is, for $t \in [t_0, T]$ and $\bar{X}$, $\bar{Y} \in (E^m)^{nr}$, then

$$D\left(\bar{F}(t, \bar{X}), \bar{F}(t, \bar{Y})\right) \leq S\, D(\bar{X}, \bar{Y}),$$

where $S = (s_{ij})$ is an $(nr) \times (nr)$ matrix with $s_{ij} \geq 0$, for all i, j, and such that, for some $k > 1$, S^k is an A-matrix, or, equivalently, for $t \in [t_0, T]$, $\bar{X}$, $\bar{Y} \in (E^m)^{nr}$, and every $j = 1, 2, \ldots, nr$,

$$d\left(\bar{F}_j(t, \bar{X}), \bar{F}_j(t, \bar{Y})\right) \leq \sum_{i=1}^{nr} s_{ji} d(\bar{X}_i, \bar{Y}_i).$$

Take $j \in \{pn+1, \ldots, (p+1)n\}$, where $p = 0, 1, \ldots, r-2$, then

$$d\left(\bar{F}_j(t,\bar{X}), \bar{F}_j(t,\bar{Y})\right) = d\left(\bar{X}_{j+n}, \bar{Y}_{j+n}\right) = \sum_{i=1}^{nr} s_{ji} d(\bar{X}_i, \bar{Y}_i),$$

by choosing

$$s_{j(j+n)} = 1, \quad s_{ji} = 0,\, i \neq j+n.$$

For $j \in \{(r-1)n+1, \ldots, rn\}$, then $j-(r-1)n \in \{1, \ldots, n\}$, and

$$\begin{aligned} d\left(\bar{F}_j(t,\bar{X}), \bar{F}_j(t,\bar{Y})\right) &= d\left(F_{j-(r-1)n}(t,\bar{X}), F_{j-(r-1)n}(t,\bar{Y})\right) \\ &\leq \sum_{i=1}^{nr} M_{(j-(r-1)n)i} d(\bar{X}_i, \bar{Y}_i) = \sum_{i=1}^{nr} s_{ji} d(\bar{X}_i, \bar{Y}_i), \end{aligned}$$

where

$$s_{ji} = M_{(j-(r-1)n)i}, \quad i = 1, \ldots, nr.$$

For simplicity, we choose $K = \max\{M_{il},\, i = 1, 2, \ldots, n,\, l = 1, 2, \ldots, nr\}$, and, thus,

$$s_{ji} = K, \text{ for } j \in \{(r-1)n+1, \ldots, rn\},\ i = 1, 2, \ldots, nr.$$

In consequence, the matrix S can be chosen as

$$S = \left(\begin{array}{c|c|c|c|c|c} \theta & I & \theta & \cdots & \theta & \theta \\ \hline \theta & \theta & I & \cdots & \theta & \theta \\ \hline \theta & \theta & \theta & \ddots & \theta & \theta \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline \theta & \theta & \theta & \cdots & \theta & I \\ \hline \hat{K} & \hat{K} & \hat{K} & \cdots & \hat{K} & \hat{K} \end{array}\right),$$

where $\hat{K}$ is the $n \times n$ constant matrix$\hat{K} = \begin{pmatrix} K & \cdots & K \\ \vdots & \ddots & \vdots \\ K & \cdots & K \end{pmatrix}$. It is clear that S is a nonnegative matrix. Accordingly to Theorem 3.3, we prove that there exist $\rho > 0$ and $k > 1$ $(k \in \mathbb{N})$ such that $S^k \left(\frac{1-e^{-\rho(T-t_0)}}{\rho}\right)^k$ is an A-matrix. Take $\alpha = \frac{1-e^{-\rho(T-t_0)}}{\rho} > 0$. We check that, for $\alpha > 0$ small enough, $(\alpha S)^2$ is an A-matrix. Indeed,$(\alpha S)^2 = \alpha^2 S^2$, and

$$S^2 = \left(\begin{array}{c|c|c|c|c|c|c} \theta & \theta & I & \theta & \cdots & \theta & \theta \\ \hline \theta & \theta & \theta & I & \cdots & \theta & \theta \\ \hline \theta & \theta & \theta & \theta & \ddots & \theta & \theta \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline \theta & \theta & \theta & \cdots & \theta & \theta & I \\ \hline \hat{K} & \hat{K} & \hat{K} & \cdots & \cdots & \hat{K} & \hat{K} \\ \hline \hat{K}^2 & \hat{K}+\hat{K}^2 & \hat{K}+\hat{K}^2 & \cdots & \cdots & \hat{K}+\hat{K}^2 & \hat{K}+\hat{K}^2 \end{array}\right),$$

where

$$\hat{K}^2 = K^2 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}^2 = K^2 \begin{pmatrix} n & \cdots & n \\ \vdots & \ddots & \vdots \\ n & \cdots & n \end{pmatrix}.$$

Hence, the matrix $I-(\alpha S)^2$ is the following:

$$\left(\begin{array}{c|c|c|c|c|c|c} I & \theta & -\alpha^2 I & \theta & \cdots & \theta & \theta \\ \hline \theta & I & \theta & -\alpha^2 I & \cdots & \theta & \theta \\ \hline \theta & \theta & I & \theta & \ddots & \theta & \theta \\ \hline \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \hline \theta & \theta & \theta & \cdots & I & \theta & -\alpha^2 I \\ \hline -\alpha^2\hat{K} & -\alpha^2\hat{K} & -\alpha^2\hat{K} & \cdots & \cdots & I-\alpha^2\hat{K} & -\alpha^2\hat{K} \\ \hline -\alpha^2 K^2 n(1) & -\alpha^2 B & -\alpha^2 B & \cdots & \cdots & -\alpha^2 B & I-\alpha^2 B \end{array}\right),$$

where$B = \hat{K} + \hat{K}^2 = K[1+Kn](1)$, and (1) represents the $n \times n$ constant matrix $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. For $r=2$, we obtain

$$S^2 = \left(\begin{array}{c|c} \hat{K} & \hat{K} \\ \hline \hat{K}^2 & \hat{K}+\hat{K}^2 \end{array}\right), \qquad I-(\alpha S)^2 = \left(\begin{array}{c|c} I-\alpha^2\hat{K} & -\alpha^2\hat{K} \\ \hline -\alpha^2 K^2 n(1) & I-\alpha^2 B \end{array}\right).$$

To check that $I-(\alpha S)^2$ is positive definite, for $\alpha > 0$ small, we take

$$\bar{X} = \left(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, \ldots, x_{n(r-1)+1}, \ldots, x_{nr}\right)^t,$$

and then, for$j = 1, 2, \ldots, n(r-2)$,

$$((I-(\alpha S)^2)\bar{X})_j = x_j - \alpha^2 x_{j+2n},$$

for$j = n(r-2)+1, \ldots, n(r-1)$,

$$((I-(\alpha S)^2)\bar{X})_j = x_j - \alpha^2 K \sum_{i=1}^{nr} x_i,$$

and, finally, for $j = n(r-1)+1, \ldots, nr$,

$$((I-(\alpha S)^2)\bar{X})_j = x_j - \alpha^2 K^2 n \sum_{i=1}^{nr} x_i - \alpha^2 K \sum_{i=n+1}^{nr} x_i.$$

This implies that

$$\begin{aligned} &\bar{X}^t(I-(\alpha S)^2)\bar{X} \\ &\quad = \sum_{j=1}^{n(r-2)} x_j\left(x_j - \alpha^2 x_{j+2n}\right) + \sum_{j=n(r-2)+1}^{n(r-1)} x_j\left(x_j - \alpha^2 K \sum_{i=1}^{nr} x_i\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=n(r-1)+1}^{nr} x_j\left(x_j-\alpha^2K^2n\sum_{i=1}^{nr}x_i-\alpha^2K\sum_{i=n+1}^{nr}x_i\right)\\
&=\sum_{i=1}^{nr}x_i^2-\alpha^2\sum_{j=1}^{n(r-2)}x_jx_{j+2n}-\alpha^2K\sum_{j=n(r-2)+1}^{n(r-1)}x_j\left(\sum_{i=1}^{nr}x_i\right)\\
&\quad-\alpha^2K^2n\sum_{j=n(r-1)+1}^{nr}x_j\left(\sum_{i=1}^{nr}x_i\right)-\alpha^2K\sum_{j=n(r-1)+1}^{nr}x_j\left(\sum_{i=n+1}^{nr}x_i\right)\\
&=\sum_{i=1}^{nr}x_i^2-H.
\end{aligned}
$$

Note that, if $r=2$, a similar expression is obtained, with the particularity that the term $-\alpha^2\sum_{j=1}^{n(r-2)}x_jx_{j+2n}$ makes no sense. The previous inequality is positive if and only if

$$
\begin{aligned}
\sum_{i=1}^{nr}x_i^2>\alpha^2\sum_{j=1}^{n(r-2)}x_jx_{j+2n}+\alpha^2K\sum_{j=n(r-2)+1}^{n(r-1)}x_j\left(\sum_{i=1}^{nr}x_i\right)&\\
+\alpha^2K^2n\sum_{j=n(r-1)+1}^{nr}x_j\left(\sum_{i=1}^{nr}x_i\right)+\alpha^2K\sum_{j=n(r-1)+1}^{nr}x_j\left(\sum_{i=n+1}^{nr}x_i\right)&=H.
\end{aligned}
$$

Since, in this expression, we find terms of the type x_ix_j with repeated indexes, then

$$
\begin{aligned}
H&\le|H|\\
&\le\alpha^2\sum_{j=1}^{n(r-2)}|x_j||x_{j+2n}|+\alpha^2K\sum_{j=n(r-2)+1}^{n(r-1)}|x_j|\left(\sum_{i=1}^{nr}|x_i|\right)\\
&\quad+\alpha^2K^2n\sum_{j=n(r-1)+1}^{nr}|x_j|\left(\sum_{i=1}^{nr}|x_i|\right)+\alpha^2K\sum_{j=n(r-1)+1}^{nr}|x_j|\left(\sum_{i=n+1}^{nr}|x_i|\right)\\
&\le\alpha^2(1+2K+K^2n)\left(\sum_{i=1}^{nr}|x_i|\right)^2=\alpha^2(1+2K+K^2n)\|x\|_1^2\\
&\le\alpha^2(1+2K+K^2n)R^2\|x\|_2^2=\alpha^2(1+2K+K^2n)R^2\sum_{i=1}^{nr}x_i^2.
\end{aligned}
$$

In consequence, if $x\neq0$, then $\|x\|_2>0$, and

$$H\le\alpha^2(1+2K+K^2n)R^2\sum_{i=1}^{nr}x_i^2<\sum_{i=1}^{nr}x_i^2,$$

provided that

$$\alpha^2(1+2K+K^2n)R^2<1,$$

that is, $0<\alpha<\frac{1}{R\sqrt{1+2K+K^2n}}$, which is obtained taking $\rho>0$ large enough. Using Theorem 3.3, we deduce the existence of a unique solution to system (12) (for each fixed initial condition). Similar conclusions can be obtained using Theorem 3.1. □

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REFERENCES

[1] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets, World Scientific, Singapore,1994.

[2] S. Abbasbandy, J.J. Nieto, M. Alavi, Tuning of reachable set in one dimensional fuzzy differential inclusions, Chaos Solitons & Fractals 26 (2005) 1337–1341.

[3] T.G. Bhaskar, V. Lakshmikantham, V. Devi, Revisiting fuzzy differential equations, Nonlinear Anal. 58 (2004) 351–358.

[4] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[5] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems 35 (1990) 389–396.

[6] V. Lakshmikantham, R.N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, Taylor and Francis,2003.

[7] V. Lakshmikantham, J.J. Nieto, Differential equations in metric spaces: an introduction and an application to fuzzy differential equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 10 (2003) 991–1000.

[8] J.J. Nieto, The Cauchy problem for continuous fuzzy differential equations, Fuzzy Sets and Systems 102 (1999) 259–262.

[9] J.J. Nieto, R. Rodríguez-López, Bounded solutions for fuzzy differential and integral equations, Chaos Solitons & Fractals 27 (2006) 1376–1386.

[10] H. Román-Flores, M. Rojas-Medar, Embedding of level-continuous fuzzy sets on Banach spaces, Information Sciences 144 (2002) 227–247.

[11] J.J. Nieto, R. Rodríguez-López, D. Franco, Linear first-order fuzzy differential equations, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 14 (2006) 687–709.

[12] J.J. Buckley, T. Feuring, Fuzzy initial value problem for nth-order linear differential equations, Fuzzy Sets and Systems 121 (2001) 247–255.

[13] V. Lakshmikantham, K.N. Murty, J. Turner, Two-point boundary value problems associated with non-linear fuzzy differential equations, Math. Inequal. Appl. 4 (2001) 527–533.

[14] D. O'Regan, V. Lakshmikantham, J.J. Nieto, Initial and boundary value problems for fuzzy differential equations, Nonlinear Anal. 54 (2003) 405–415.

[15] D.N. Georgiou, I.E. Kougias, On Cauchy problems for fuzzy differential equations, Int. J. Math. Math. Sci. 15 (2004) 799–805.

[16] D.N. Georgiou, J.J. Nieto, R. Rodríguez-López, Initial value problems for higher-order fuzzy differential equations, Nonlinear Anal. 63 (2005) 587–600.

[17] S. Abbasbandy, T.A. Viranloo, Ó. López-Pouso, J.J. Nieto, Numerical methods for fuzzy differential inclusions,Comput. Math. Appl. 48 (2004) 1633–1641.

[18] M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133–138.

[19] E. Hüllermeier, Numerical methods for fuzzy initial value problems. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 7 (1999) 439–461.

[20] E. Babolian, H. Sadeghi, S. Javadi, Numerically solution of fuzzy differential equations by Adomian method.Appl. Math. Comput. 149 (2004) 547–557.

[21] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[22] L.J. Rodríguez-Muñiz, M. López-Díaz, M.A. Gil, D.A. Ralescu, The s-differentiability of a fuzzy-valued mapping, Information Sciences 151 (2003) 283–299.

[23] D.N. Georgiou, J.J. Nieto, R. Rodríguez-López, Solvability of fuzzy differential systems, preprint.

[24] S.R. Bernfeld, V. Lakshmikantham, An introduction to Nonlinear Boundary Value Problems, Academic Press, New York, 1974.

[25] J.J. Nieto, R. Rodríguez-López, Hybrid metric dynamical systems with impulses, Nonlinear Anal. 64 (2006) 368–380.

[26] P. Prakash, J.J. Nieto, J.-H. Kim, R. Rodríguez-López, Existence of solutions of fuzzy neutral differential equations in Banach spaces, *Dynam. Systems Appl.* 14 (2005), no. 3–4, 407–417.

[27] R.P. Agarwal, D. O'Regan, V. Lakshmikantham, Viability theory and fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 563–580.

[28] R.P. Agarwal, D. O'Regan, V. Lakshmikantham, A stacking theorem approach for fuzzy differential equations, Nonlinear Anal. 55 (2003) 299–312.