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MULTIPLE POSITIVE SOLUTIONS FOR DYNAMIC m-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we consider a second order m-point boundary value problem for dynamic equations on time scales. First, we establish criteria for the existence of one or more than one positive solution of a non-eigenvalue problem. Second, we consider the existence and multiplicity of positive solutions for an eigenvalue problem. We shall also obtain criteria which lead to nonexistence of positive solutions. In both problems, we will use fixed point theorems for operators on a Banach space.

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1. INTRODUCTION

We are interested in proving the existence and multiplicity results for positive solutions to the dynamic second-order m-point boundary value problem

$$(1.1)\qquad \begin{cases} -[p(t)y^{\Delta}(t)]^{\nabla}+q(t)y(t)=f(t,y(t)), \ \ t\in[a,b], \\ \alpha y(\rho(a))-\beta y^{[\Delta]}(\rho(a))=\Sigma_{i=1}^{m-2}a_i y(\xi_i), \\ \gamma y(b)+\delta y^{[\Delta]}(b)=\Sigma_{i=1}^{m-2}b_i y(\xi_i), \end{cases}$$

and the eigenvalue problem $-[p(t)y^{\Delta}(t)]^{\nabla}+q(t)y(t)=\lambda f(t,y(t))$ with the same boundary conditions for $t\in[a,b]\subset\mathbb{T}$, where $\mathbb{T}$ is time scale, $\alpha,\beta,\gamma,\delta,\xi_i,a_i,b_i$ (for $i\in\{1,2,\dots m-2\}$) are complex constants such that $|\alpha|+|\beta|\neq 0, |\gamma|+|\delta|\neq 0$ and $\xi_i\in(a,b), q:\mathbb{T}\to\mathbb{C}$ is a continuous function, $p:\mathbb{T}\to\mathbb{C}$ is $\nabla-$ differentiable on $\mathbb{T}_k$, $p(t)\neq 0$ for all $t\in\mathbb{T}$, $p^{\nabla}:\mathbb{T}_k\to\mathbb{C}$ is continuous.

Ma and Thompson [8] applied Kraskonelskii fixed point theorem to determine values of λ for which there exists at least one positive solution of the nonlinear eigenvalue problem

$$(1.2)\qquad \begin{cases} [p(t)u']'-q(t)u+\lambda h(t)f(u)=0, \ \ t\in[0,1], \\ au(0)-bp(0)u'(0)=\Sigma_{i=1}^{m-2}\alpha_i u(\xi_i), \\ cu(1)+dp(1)u'(1)=\Sigma_{i=1}^{m-2}\beta_i u(\xi_i). \end{cases}$$

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Later Ma [7] used Kraskonelskii fixed point theorem for the existence of positive solutions for the m-point boundary value problem

$$\begin{cases} [p(t)u']' - q(t)u + f(t,u) = 0, \ \ t \in [0,1], \\ au(0) - bp(0)u'(0) = \Sigma_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + dp(1)u'(1) = \Sigma_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases} \tag{1.3}$$

In this article, dynamic analogues of the BVP's (1.2) and (1.3) are considered. Besides our results include criteria for existence of one or more than one positive solution of our eigenvalue problem in terms of superlinear or sublinear behavior of $f(t,y)$. Moreover we also obtain criteria which lead to nonexistence of positive solutions.

The main tool in this paper is the following well-known Krasnoselskii fixed point theorem.

Theorem 1.1. *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1, \bar{\Omega_1} \subset \Omega_2$, and let*

$$A : P \cap (\bar{\Omega_2} \setminus \Omega_1) \to P$$

be a completely continuous operator such that, either

(i) $\|Ay\| \le \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \ge \|y\|$, $y \in P \cap \partial\Omega_2$; *or*
(ii) $\|Ay\| \ge \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \le \|y\|$, $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega_2} \setminus \Omega_1)$.

Boundary value problems for dynamic equations on time scales have been studied recent years [1, 3, 6, 9]. We cite some appropriate references here [6, 9].

Peterson, Raffoul and Tisdell [6] employed the method of upper and lower solutions and topological degree theory to establish existence criteria for second-order three point boundary value problems on time scales.

Tisdell, Drâbek and Henderson [9] used the techniques of maximum principles on time scales and additivity property of Leray-Schauder degree theory and by using two pairs of upper and lower solutions, showed the existence of multiple (at least three) solutions for second-order two point boundary value problems subject to Sturm-Liouville boundary conditions, periodic boundary conditions and homogenous Drichlet boundary conditions.

To understand this so-called dynamic equation (1.1) on a time scale $\mathbb{T}$ we need some preliminary definitions. For the details of basic notions connected to time scales we refer to [4, 5].

Definition 1.2. Define the interval in $\mathbb{T}$

$$[a,b] := \{t \in \mathbb{T} \ \text{ such that } \ a \le t \le b\}.$$

Other types of intervals are defined similarly.

Definition 1.3. Let $\mathbb{T}$ be a nonempty closed subset of the real numbers $\mathbb{R}$ and define the forward jump operator $\sigma(t)$ at t for $t < sup\mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : \ s \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ at t for $t > \inf \mathbb{T}$ by

$$\rho(t) := \sup\{s < t : \ s \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$.

We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$ we say t is right dense, while if $\rho(t) = t$ we say t is left dense.

If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 1.4. If $f : \mathbb{T} \to \mathbb{C}$ is a function and $t \in \mathbb{T}^k$, then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t.

It can be shown that if $f : \mathbb{T} \to \mathbb{C}$ is continuous at $t \in \mathbb{T}$, $t \in \mathbb{T}^k$, and t is right scattered, then

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

Note, if $\mathbb{T} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, then

$$f^\Delta(t) = \Delta f(t) := f(t+1) - f(t).$$

Moreover, if $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f'(t).$$

Definition 1.5. If $F^\Delta(t) = f(t)$, then define the Cauchy $\Delta-$integral from a to t of f by

$$\int_a^t f(\tau)\Delta\tau := F(t) - F(a), \quad \text{for all } t \in \mathbb{T}.$$

If $\Psi^\nabla(t) = f(t)$, then define the Cauchy $\nabla-$integral from a to t of f by

$$\int_a^t f(\tau)\nabla\tau := \Psi(t) - \Psi(a), \quad \text{for all} t \in \mathbb{T}.$$

Note that in the case $\mathbb{T}=\mathbb{R}$ we have

$$f^{\Delta}(t)=f^{\nabla}(t)=f'(t),\quad \int_a^b f(t)\Delta t=\int_a^b f(t)dt=\int_a^b f(t)\nabla t$$

and in the case $\mathbb{T}=\mathbb{Z}$ we have

$$f^{\Delta}(t)=f(t+1)-f(t),\quad f^{\nabla}(t)=f(t)-f(t-1)$$

$$\int_a^b f(t)\Delta t=\sum_{k=a}^{b-1} f(k),\quad \int_a^b f(t)\nabla t=\sum_{k=a+1}^{b} f(k),$$

where $a,b\in\mathbb{T}$ with $a\le b$.

2. THE PRELIMINARY LEMMAS

We will assume that the following conditions are satisfied.

(H1) $p(t)>0,\quad q(t)\ge 0.$

(H2) $\alpha,\beta,\gamma,\delta\ge 0,\quad \alpha+\beta>0,\ \gamma+\delta>0,\quad a_i,b_i\ge 0$ (for $i\in\{1,2,...m-2\}$).

(H3) If $q(t)\equiv 0$, then $\alpha+\gamma>0$.

(H4) $f:[\rho(a),b]\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to y and $f(t,y)\ge 0$ for $y\in\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

Let φ and ψ be the solutions of the linear problems

$$\begin{cases}[p(t)\varphi^{\Delta}(t)]^{\nabla}-q(t)\varphi(t)=0,\ t\in[a,b],\\ \varphi(\rho(a))=\beta,\ \varphi^{[\Delta]}(\rho(a))=\alpha\end{cases}\tag{2.1}$$

and

$$\begin{cases}[p(t)\psi^{\Delta}(t)]^{\nabla}-q(t)\psi(t)=0,\ t\in[a,b],\\ \psi(b)=\delta,\quad \psi^{[\Delta]}(b)=-\gamma\end{cases}\tag{2.2}$$

respectively. Let us set $d=-W_t(\varphi,\psi)=p(t)[\varphi^{\Delta}(t)\psi(t)-\varphi(t)\psi^{\Delta}(t)]$.

To state and prove the main results of this paper, we need following lemmas.

Lemma 2.1. [2] *Under the conditions* (H1), (H2) *the solutions $\varphi(t)$ and $\psi(t)$ posses the following properties:*

$$\varphi(t)\ge 0,\ \ t\in[\rho(a),\sigma(b)];\ \ \psi(t)\ge 0,\ \ t\in[\rho(a),b];$$

$$\varphi(t)>0,\ \ t\in(\rho(a),\sigma(b)];\ \ \psi(t)>0,\ \ t\in[\rho(a),b);$$

$$\varphi^{[\Delta]}(t)\ge 0,\ \ t\in[\rho(a),b];\ \ \psi^{[\Delta]}(t)\le 0,\quad t\in[\rho(a),b].$$

Lemma 2.2. [2] *Under the conditions* (H1), (H2)*:*

(i) *If $q(t)$ is not identically zero, then $d>0$.*

(ii) *If $q(t)$ is identically zero, then $d>0$ if and only if $\alpha+\gamma>0$.*

Let $G(t,s)$ be the Green's function for the boundary value problem

$$\begin{cases} -[p(t)y^{\Delta}(t)]^{\nabla}+q(t)y(t)=0, & t\in[a,b],\\ \alpha y(\rho(a))-\beta y^{[\Delta]}(\rho(a))=0, \\ \gamma y(b)+\delta y^{[\Delta]}(b)=0 \end{cases}$$

is given by

$$G(t,s)=\frac{1}{d}\begin{cases} \psi(t)\varphi(s), & \text{if } \rho(a)\le s\le t\le\sigma(b),\\ \psi(s)\varphi(t), & \text{if } \rho(a)\le t\le s\le\sigma(b). \end{cases}$$

Lemma 2.3. [2] *Let conditions* (H1)–(H3) *hold. Then*

(i) $G(t,s)\ge 0$ *for* $t,s\in[\rho(a),b]$.
(ii) $G(t,s)>0$ *for* $t,s\in(\rho(a),b)$.
(iii) *If* $\beta>0$ *and* $\delta>0$, *then* $G(t,s)>0$ *for* $t,s\in[a,b]$.

Set

$$\Delta:=\begin{vmatrix} -\sum_{i=1}^{m-2}a_i\varphi(\xi_i) & d-\sum_{i=1}^{m-2}a_i\psi(\xi_i)\\ d-\sum_{i=1}^{m-2}b_i\varphi(\xi_i) & -\sum_{i=1}^{m-2}b_i\psi(\xi_i)\end{vmatrix}. \tag{2.3}$$

Lemma 2.4. *Let* (H1)–(H4) *hold. Assume that*

(H5) $\Delta :\neq 0$.

Then the problem (1.1) *has a unique solution*

$$y(t)=\int_{\rho(a)}^{b}G(t,s)f(s,y(s))\nabla s+A(f)\varphi(t)+B(f)\psi(t), \tag{2.4}$$

where

$$G(t,s)=\frac{1}{d}\begin{cases} \psi(t)\varphi(s), & \text{if } \rho(a)\le s\le t\le\sigma(b),\\ \psi(s)\varphi(t), & \text{if } \rho(a)\le t\le s\le\sigma(b), \end{cases} \tag{2.5}$$

$$A(f):=\frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}a_i\int_{\rho(a)}^{b}G(\xi_i,s)f(s,y(s))\nabla s & d-\sum_{i=1}^{m-2}a_i\psi(\xi_i)\\ \sum_{i=1}^{m-2}b_i\int_{\rho(a)}^{b}G(\xi_i,s)f(s,y(s))\nabla s & -\sum_{i=1}^{m-2}b_i\psi(\xi_i)\end{vmatrix}, \tag{2.6}$$

and

$$B(f):=\frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}a_i\varphi(\xi_i) & \sum_{i=1}^{m-2}a_i\int_{\rho(a)}^{b}G(\xi_i,s)f(s,y(s))\nabla s\\ d-\sum_{i=1}^{m-2}b_i\varphi(\xi_i) & \sum_{i=1}^{m-2}b_i\int_{\rho(a)}^{b}G(\xi_i,s)f(s,y(s))\nabla s\end{vmatrix}. \tag{2.7}$$

Proof. Since φ and ψ are two linearly independent solutions of the equation

$$(p(t)y^{\Delta}(t))^{\nabla}-q(t)y(t)=0,$$

we know that any solution of

$$-(p(t)y^{\Delta}(t))^{\nabla}+q(t)y(t)=f(t,y(t))$$

can be represented by

$$y(t)=\int_{\rho(a)}^{b} G(t,s)f(s,y(s))\nabla s + A(f)\varphi(t)+B(f)\psi(t), \tag{2.8}$$

where G is as in (2.5).

Now we show that the function defined by (2.8) is a solution of (1.1) only if $A(f)$ and $B(f)$ are as in (2.6) and (2.7), respectively.

Let $y(t)=\int_{\rho(a)}^{b} G(t,s)f(s,y(s))\nabla s+A(f)\varphi(t)+B(f)\psi(t)$, be a solution of (1.1), then we have that

$$\begin{aligned} y(t) &= \frac{1}{d}\int_{\rho(a)}^{t} \varphi(s)\psi(t)f(s,y(s))\nabla s+\frac{1}{d}\int_{t}^{b}\psi(s)\varphi(t)f(s,y(s))\nabla s \\ &\quad +A(f)\varphi(t)+B(f)\psi(t), \end{aligned}$$

$$\begin{aligned} p(t)y^{\Delta}(t) &= \frac{1}{d}p(t)\psi^{\Delta}(t)\int_{\rho(a)}^{t}\varphi(s)f(s,y(s))\nabla s \\ &\quad +\frac{1}{d}p(t)\varphi^{\Delta}(t)\int_{t}^{b}\psi(s)f(s,y(s))\nabla s \\ &\quad +A(f)p(t)\varphi^{\Delta}(t)+B(f)p(t)\psi^{\Delta}(t) \end{aligned}$$

and

$$\begin{aligned} [p(t)y^{\Delta}(t)]^{\nabla} &= \frac{1}{d}[p(t)\psi^{\Delta}(t)]^{\nabla}\int_{\rho(a)}^{t}\varphi(s)f(s,y(s))\nabla s \\ &\quad +\frac{1}{d}p(t)[\psi^{\Delta}(t)\varphi(t)-\varphi^{\Delta}(t)\psi(t)]f(t,y(t)) \\ &\quad +\frac{1}{d}[p(t)\varphi^{\Delta}(t)]^{\nabla}\int_{t}^{b}\psi(s)f(s,y(s))\nabla s \\ &\quad +A(f)[p(t)\varphi^{\Delta}(t)]^{\nabla}+B(f)[p(t)\psi^{\Delta}(t)]^{\nabla}, \end{aligned}$$

so that

$$\begin{aligned} -[p(t)y^{\Delta}(t)]^{\nabla}+q(t)y(t) &= -A(f)[p(t)\varphi^{\Delta}(t)]^{\nabla}-B(f)[p(t)\psi^{\Delta}(t)]^{\nabla} \\ &\quad -\frac{1}{d}[[p(t)\varphi^{\Delta}(t)]^{\nabla}+[p(t)\psi^{\Delta}(t)]^{\nabla}] \\ &\quad +\frac{1}{d}q(t)\psi(t)\int_{\rho(a)}^{t}\varphi(s)f(s,y(s))\nabla s \\ &\quad +\frac{1}{d}q(t)\varphi(t)\int_{t}^{b}\psi(s)f(s,y(s))\nabla s \\ &\quad +A(f)q(t)\varphi(t)+B(f)q(t)\psi(t) \\ &\quad +\frac{1}{d}p(t)[\psi(t)\varphi^{\Delta}(t)-\varphi(t)\psi^{\Delta}(t)]f(t,y(t)) \\ &= -\frac{p(t)}{d}[\varphi(t)\psi^{[\Delta]}(t)-\psi(t)\varphi^{[\Delta]}(t)]f(t,y(t)) \\ &= f(t,y(t)). \end{aligned}$$

Since

$$y(\rho(a)) = \frac{1}{d}\varphi(\rho(a))\int_{\rho(a)}^{b}\psi(s)f(s,y(s))\nabla s + A(f)\varphi(\rho(a)) + B(f)\psi(\rho(a)),$$

$$\begin{aligned} p(\rho(a))y^{\Delta}(\rho(a)) &= \frac{1}{d}p(\rho(a))\varphi^{\Delta}(\rho(a))\int_{\rho(a)}^{b}\psi(s)f(s,y(s))\nabla s \\ &\quad + A(f)[p(\rho(a))\varphi^{\Delta}(\rho(a))] + B(f)[p(\rho(a))\psi^{\Delta}(\rho(a))], \end{aligned}$$

we have that

$$B(f)[\alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a))] = Y, \tag{2.9}$$

where $Y = \sum_{i=1}^{m-2} a_i[\int_{\rho(a)}^{b} G(\xi_i,s)f(s,y(s))\nabla s + A(f)\varphi(\xi_i) + B(f)\psi(\xi_i)]$.

Since

$$y(b) = \psi(b)\int_{\rho(a)}^{b}\frac{1}{d}\varphi(s)f(s,y(s))\nabla s + A(f)\varphi(b) + B(f)\psi(b),$$

$$\begin{aligned} p(b)y^{\Delta}(b) &= p(b)\psi^{\Delta}(b)\int_{\rho(a)}^{b}\frac{1}{d}\varphi(s)f(s,y(s))\nabla s + A(f)[p(b)\varphi^{\Delta}(b)] \\ &\quad + B(f)[p(b)\psi^{\Delta}(b)], \end{aligned}$$

we have that

$$B[\alpha\varphi(b) - \beta\varphi^{[\Delta]}(b)] = Z \tag{2.10}$$

where $Z = \sum_{i=1}^{m-2} b_i[\int_{\rho(a)}^{b} G(\xi_i,s)f(s,y(s))\nabla s + A\varphi(\xi_i) + B\psi(\xi_i)]$.

From (2.9) and (2.10) we get that

$$[-\sum_{i=1}^{m-2} a_i\varphi(\xi_i)]A(f) + [d - \sum_{i=1}^{m-2} a_i\psi(\xi_i)]B(f) = \sum_{i=1}^{m-2} a_i\int_{\rho(a)}^{b} G(\xi_i,s)f(s,y(s))\nabla s$$

$$[d - \sum_{i=1}^{m-2} b_i\varphi(\xi_i)]A(f) - [\sum_{i=1}^{m-2} b_i\psi(\xi_i)]B(f) = \sum_{i=1}^{m-2} b_i\int_{\rho(a)}^{b} G(\xi_i,s)f(s,y(s))\nabla s$$

which implies that $A(f)$ and $B(f)$ satisfy (2.6) and (2.7), respectively. □

Lemma 2.5. *Let* (H1)–(H4) *hold. Assume*

(H6) $\Delta < 0$, $d - \sum_{i=1}^{m-2} a_i\psi(\xi_i) > 0$, $d - \sum_{i=1}^{m-2} b_i\varphi(\xi_i) > 0$. *Then the unique solution y of the problem* (1.1) *satisfies $y(t) \geq 0$, for $t \in [\rho(a), b]$.*

Proof. It is an immediate subsequence of the facts that $G \geq 0$ on $[\rho(a), b] \times [\rho(a), b]$ and $A(f) \geq 0, B(f) \geq 0$. □

Lemma 2.6. *Let* (H1)–(H4) *and* (H6) *hold. Let* $\tau \in (\rho(a), \frac{\rho(a)+b}{2})$ *be a constant. Then the unique solution* y *of the problem* (1.1) *satisfies*

$$\min\{y(t) : t \in [\rho(a)+\tau, b-\tau]\} \geq \Gamma\|y\|$$

where $\|y\| = \max\{y(t) : t \in [\rho(a), b]\}$ *and*

$$\Gamma := \min\{\frac{\psi(b-\tau)}{\psi(\rho(a))}, \frac{\varphi(\rho(a)+\tau)}{\varphi(b)}\}. \tag{2.11}$$

Proof. We have from (2.5) that

$$0 \leq G(t,s) \leq G(s,s), \;\; t \in [\rho(a), b]$$

which implies for all $t \in [\rho(a), b]$,

$$y(t) \leq \int_{\rho(a)}^{b} G(s,s) f(s, y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t). \tag{2.12}$$

Applying (2.5), we have that for $t \in [\rho(a)+\tau, b-\tau]$,

$$\frac{G(t,s)}{G(s,s)} \geq \begin{cases} \frac{\psi(b-\tau)}{\psi(\rho(a))}, & \text{if } \rho(a) \leq s \leq t \leq b-\tau, \\ \frac{\varphi(\rho(a)+\tau)}{\varphi(b)}, & \text{if } \rho(a)+\tau \leq t \leq s \leq b, \end{cases} \geq \Gamma \tag{2.13}$$

where

$$\Gamma := \min\{\frac{\psi(b-\tau)}{\psi(\rho(a))}, \frac{\varphi(\rho(a)+\tau)}{\varphi(b)}\}.$$

Thus for $t \in [\rho(a)+\tau, b-\tau]$,

$$\begin{aligned} y(t) &= \int_{\rho(a)}^{b} G(t,s) f(s,y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t) \\ &= \int_{\rho(a)}^{b} \frac{G(t,s)}{G(s,s)} G(s,s) f(s,y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t) \\ &\geq \Gamma \int_{\rho(a)}^{b} G(s,s) f(s,y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t) \\ &\geq \Gamma[\int_{\rho(a)}^{b} G(s,s) f(s,y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t)] \\ &\geq \Gamma\|y\|. \end{aligned}$$

□

3. EXISTENCE OF ONE OR MORE POSITIVE SOLUTIONS

Denote

$$D^* = [\max_{\rho(a) \leq t \leq b} (\int_{\rho(a)+\tau}^{b-\tau} G(t,s) \nabla s)]^{-1}$$

$$D = [\max_{\rho(a) \leq t \leq b} (\int_{\rho(a)}^{b} G(t,s) \nabla s + A\varphi(t) + B\psi(t))]^{-1}$$

where $\tau \in (\rho(a), \frac{\rho(a)+b}{2})$,

$$A := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_{\rho(a)}^{b} G(\xi_i, s)\nabla s & d - \sum_{i=1}^{m-2} a_i \psi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_{\rho(a)}^{b} G(\xi_i, s)\nabla s & - \sum_{i=1}^{m-2} b_i \psi(\xi_i) \end{vmatrix}, \tag{3.1}$$

and

$$B := \frac{1}{\Delta}\begin{vmatrix} - \sum_{i=1}^{m-2} a_i \varphi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_{\rho(a)}^{b} G(\xi_i, s)\nabla s \\ d - \sum_{i=1}^{m-2} b_i \varphi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_{\rho(a)}^{b} G(\xi_i, s)\nabla s \end{vmatrix}. \tag{3.2}$$

For $\eta > 0$, set

$$F(\eta) = \max\{f(t, v) : \rho(a) \le t \le b,\ \ 0 \le v \le \eta\},$$

$$H(\eta) = \min\{f(t, v) : \rho(a) \le t \le b,\ \ \Gamma\eta \le v \le \eta\},$$

where Γ is as in (2.11).

We work in the Banach space $C[\rho(a), b]$ with the norm

$$\|y\| = \max_{\rho(a) \le t \le b} |y(t)|.$$

Let

$$K = \{y \in C[\rho(a), b] : y(t) \ge \Gamma\|y\|, \ \ t \in [\rho(a) + \tau, b - \tau]\}$$

(where Γ is as in (2.11)), then K is cone. For each $y \in K$, denote

$$Ty(t) = \int_{\rho(a)}^{b} G(t, s)f(s, y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t).$$

By Lemma 2.6, we know that $TK \subset K$. Applying Arzela-Ascoli Lemma, we can easy check that T is completely continuous.

Theorem 3.1. *Let* (H1)–(H4), *and* (H6) *hold. Suppose there exist two positive numbers* η_1 *and* η_2 *with* $\eta_1 \ne \eta_2$ *such that*

$$F(\eta_1) \le \eta_1 D, \qquad H(\eta_2) \ge \eta_2 D^*.$$

Then (1.1) *has at least one positive solution* y *satisfying*

$$\min\{\eta_1, \eta_2\} \le \|y\| \le \max\{\eta_1, \eta_2\}.$$

Proof. We only show the case that $\eta_1 < \eta_2$. The other case can be treated by the same method. From Lemma 2.4, we know that y is a solution of (1.1) if and only if y solves the fixed point problem $y = Ty$. We will apply Krasnoselskii Fixed Point Theorem (see Theorem 1.1 to prove that T has a fixed point $y \in K$ with

$$\eta_1 \le \|y\| \le \eta_2.$$

For $y \in K$ with $\|y\| = \eta_1$, we have that

$$f(t, y(t)) \le F(\eta_1) \le \eta_1 D$$

hence

$$\begin{aligned}\|Ty\| &= \max_{\rho(a)\le t\le b}\Big(\int_{\rho(a)}^{b} G(t,s)f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)\Big)\\ &\le \max_{\rho(a)\le t\le b}\Big(\int_{\rho(a)}^{b} G(t,s)\nabla s + A\varphi(t) + B\psi(t)\Big)F(\eta_1)\\ &\le \eta_1 = \|y\|\end{aligned}$$

where A and B are given as in (3.1) and (3.2), respectively. For $y \in K$ with $\|y\| = \eta_2$, we have that

$$\Gamma\eta_2 \le y(t) \le \eta_2, \qquad t \in [\rho(a)+\tau, b-\tau]$$

and

$$\min\{f(t,v) : \rho(a)+\tau \le t \le b-\tau, \Gamma\eta_2 \le v \le \eta_2\} = H(\eta_2) \ge \eta_2 D^*$$

so that

$$\begin{aligned}\|Ty\| &\ge \max_{\rho(a)\le t\le b}\Big(\int_{\rho(a)+\tau}^{b-\tau} G(t,s)f(s,y(s))\nabla s\Big)\\ &\ge \eta_2 D^* \max_{\rho(a)\le t\le b}\Big(\int_{\rho(a)+\tau}^{b-\tau} G(t,s)\nabla s\Big) = \eta_2 = \|y\|.\end{aligned}$$

Therefore, by the first part of the Krasnoselskii Fixed Point Theorem, it follows that T has a fixed point y with $\eta_1 \le \|y\| \le \eta_2$. □

Example 3.2. Let $\mathbb{T} = \{\frac{n}{3} : n \in \mathbb{N}_0\}$. Consider the following boundary value problem on $\mathbb{T}$:

$$\begin{cases} -y^{\Delta\nabla}(t) = \frac{4y^2(t)}{1+y^2(t)}, \ \ t \in [1/3, 8/3],\\ y(0) - 1/3y^{\Delta}(0) = \sum_{i=1}^{4} a_i y(\xi_i),\\ 1/2y(8/3) + 3y^{\Delta}(8/3) = \sum_{i=1}^{4} b_i y(\xi_i). \end{cases} \tag{3.3}$$

When taking $\alpha = \xi_2 = 1,\ \beta = \tau = 1/3,\ \gamma = 1/2,\ \delta = 3,\ a_1 = 1/4,\ a_2 = 1/5,\ a_3 = b_1 = 1/10,\ a_4 = 1/3,\ b_2 = 3/23,\ b_3 = 1/20,\ b_4 = 1/8,\ \xi_1 = 2/3,\ \xi_3 = 2,\ \xi_4 = 7/3,\ p(t) \equiv 1,\ q(t) \equiv 0$, and $f(t,y) = f(y) = \frac{4y^2}{1+y^2}$, we prove the solvability of the problem (3.3) by means of Theorem 3.1. The Green's function of the problem (3.3) has the form

$$G(t,s) = \frac{2}{9}\begin{cases} (\frac{13}{3} - \frac{t}{2})(s+\frac{1}{3}), \ \text{if } 0 \le s \le t \le 4,\\ (\frac{13}{3} - \frac{s}{2})(t+\frac{1}{3}), \ \text{if } 0 \le t \le s \le 4. \end{cases}$$

It is clear that the conditions (H1)-(H4) are satisfied. By using (2.3), (2.6), and (2.7), we obtain

$$\Gamma = 2/9,\ \Delta = -2.679864130,\ A = 1.887932971,\ B = 4.117190000.$$

Therefore we get

$$D^* = 0.3838862559, \quad D = 0.04669142366.$$

There exist two positive numbers 1/100 and 9/2 such that

$$F(1/100) = 0.0003999600040 \leq 0.0004669142366 = 1/100D.$$

$$H(9/2) = 2 \geq 1.727488152 = 9/2D^*.$$

Hence, from Theorem 3.1, the problem (3.3) has at least one positive solution y satisfying

$$1/100 \leq \|y\| \leq 9/2.$$

Theorem 3.3. *Let* (H1)–(H4), *and* (H6) *hold. If there exist* $j+1$ *positive numbers* $\eta_1, \eta_2, \dots \eta_{j+1}$ *with* $\eta_1 < \eta_2 < \dots < \eta_{j+1}$ *such that either*

$$\begin{cases} F(\eta_{2k-1}) < \eta_{2k-1} D \text{ for all } 2k-1 \in \{1, 2, \dots j+1\}, \\ H(\eta_{2k}) > \eta_{2k} D^* \text{ for all } 2k \in \{1, 2, \dots j+1\}, \end{cases} \tag{3.4}$$

or

$$\begin{cases} H(\eta_{2k-1}) > \eta_{2k-1} D \text{ for all } 2k-1 \in \{1, 2, \dots j+1\}, \\ F(\eta_{2k}) < \eta_{2k} D^* \text{ for all } 2k \in \{1, 2, \dots j+1\}. \end{cases} \tag{3.5}$$

Then (1.1) *has at least* j *positive solutions.*

Proof. We only prove the result under (3.4). In the case that (3.5) holds, the results can be proved by the same method. Since F and H are continuous, $0 < D \leq D^*$, we know that there exists Θ_i and τ_i with $\eta_i < \Theta_i < \tau_i < \eta_{i+1}$, $i = 1, 2, \dots, j$ such that

$$F(\Theta_{2k-1}) \leq \Theta_{2k-1} D \quad H(\tau_{2k-1}) \geq \tau_{2k-1} D^* \text{ for all } 2k-1 \in \{1, 2, \dots j+1\},$$

$$H(\Theta_{2k}) \geq \Theta_{2k} D^* \quad F(\tau_{2k}) \leq \tau_{2k} D \text{ for all } 2k \in \{1, 2, \dots j+1\}.$$

From Theorem 3.1, for each $i \in \{1, 2, \dots j\}$, (1.1) has a positive solution y_i satisfying

$$\eta_i < \Theta_i \leq \|y_i\| \leq \tau_i < \Theta_{j+1}.$$

□

Corollary 3.4. Let (H1)–(H4), and (H6) hold. Assume that there exist two sequences $\{\eta_i\}$, $\{\Theta_i\}$ of $(0, +\infty)$ such that

(i) $\lim_{i\to+\infty} \eta_i = +\infty$
(ii) $\lim_{i\to+\infty} \Theta_i = +\infty$
(iii) $\lim_{i\to+\infty} \frac{F(\eta_i)}{\eta_i} < D$
(iv) $\lim_{i\to+\infty} \frac{H(\Theta_i)}{\Theta_i} > D^*$.

Then (1.1) has a sequence of positive solutions $\{y_k\}$ such that $\|y_k\| \to \infty$ as $k \to \infty$.

4. BOUNDARY VALUE PROBLEM WITH A PARAMETER

In this section we consider the following BVP with parameter λ,

$$\begin{cases} Ly(t) := -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda f(t, y(t)), \;\; t \in [a, b], \\ \alpha y(\rho(a)) - \beta y^{[\Delta]}(\rho(a)) = \Sigma_{i=1}^{m-2} a_i y(\xi_i), \\ \gamma y(b) + \delta y^{[\Delta]}(b) = \Sigma_{i=1}^{m-2} b_i y(\xi_i). \end{cases} \tag{4.1}$$

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 := \lim_{x\to 0^+} \inf \min_{t\in[\rho(a),b]} \frac{f(t,y)}{y}, \;\; f^0 := \lim_{y\to 0^+} \sup \max_{t\in[\rho(a),b]} \frac{f(t,y)}{y},$$

$$f_\infty := \lim_{y\to \infty} \inf \min_{t\in[\rho(a),b]} \frac{f(t,y)}{y}, \;\; f^\infty := \lim_{y\to \infty} \sup \max_{t\in[\rho(a),b]} \frac{f(t,y)}{y},$$

respectively. These numbers can be regarded as generalized super or sublinear conditions on the function $f(t, y)$ at $y = 0$ and $y = \infty$. Thus, if $f_0 = f^0 = 0$ $(+\infty)$, then $f(t, y)$ is superlinear (sublinear) at $y = 0$ and if $f_\infty = f^\infty = 0$ $(+\infty)$, then $f(t, y)$ is sublinear (superlinear) at $y = +\infty$. (4.1) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) &= \lambda \int_{\rho(a)}^{b} G(t,s) f(s, y(s)) \nabla s + A(f)\varphi(t) + B(f)\psi(t) \\ &= (T_\lambda y)(t), \end{aligned}$$

where G, A, and B are defined by (2.5), (3.1) and (3.2), respectively. Denote

$$K = \{y \in C[\rho(a), b] : y(t) \geq \Gamma \|y\|, \; t \in [\rho(a) + \tau, b - \tau]\}$$

It is obvious that K is a cone in $\mathcal{C}[\rho(a), b]$. Moreover, by Lemmas 2.5 and 2.6, $T_\lambda K \subset K$. It is also easy to check that $T_\lambda : K \to K$ is completely continuous. We seek a fixed point of T_λ in the cone K. Let

$$M := \int_{\rho(a)}^{b} G(s, s)\nabla s + A\|\varphi\| + B\|\psi\|.$$

In the following, let Γ be the constant defined in (2.11) with respect to such τ. Also let $\nu \in [\rho(a), b]$ be defined by

$$\int_{\rho(a)+\tau}^{b-\tau} G(\nu, s)\nabla s = \max_{\rho(a)\leq t\leq b} \int_{\rho(a)+\tau}^{b-\tau} G(t, s)\nabla s.$$

Theorem 4.1. *Assume that* (H1)–(H4), *and* (H6) *are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{\Gamma(\int_{\rho(a)+\tau}^{b-\tau} G(\nu, s)\nabla s) f_\infty} < \lambda < \frac{1}{M f_0}, \tag{4.2}$$

there exists at least one positive solution of (4.1).

Proof. Clearly,

$$|A(f)| \leq \frac{1}{\Delta}\left| \begin{array}{ll} \sum_{i=1}^{m-2} a_i \int_{\rho(a)}^{b} G(\xi_i,s)\nabla s & d-\sum_{i=1}^{m-2} a_i\psi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_{\rho(a)}^{b} G(\xi_i,s)\nabla s & -\sum_{i=1}^{m-2} b_i\psi(\xi_i) \end{array} \right| \|f\| = A\|f\|$$

and

$$|B(f)| \leq \frac{1}{\Delta}\left| \begin{array}{ll} -\sum_{i=1}^{m-2} a_i\varphi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_{\rho(a)}^{b} G(\xi_i,s)\nabla s \\ d-\sum_{i=1}^{m-2} b_i\varphi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_{\rho(a)}^{b} G(\xi_i,s)\nabla s \end{array} \right| \|f\| = B\|f\|.$$

Let λ be given as in (4.2). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{\Gamma(\int_{\rho(a)+\tau}^{b-\tau} G(\nu,s)\nabla s)(f_\infty-\epsilon)} \leq \lambda \leq \frac{1}{M(f_0+\epsilon)}.$$

Now, turning to f_0, there exists an $H_1 > 0$ such that $f(s,y) \leq (f_0+\epsilon)y$ for $0 < y \leq H_1$. So, for $y \in K$ with $\|y\| = H_1$, we have from the fact $0 \leq G(t,s) \leq G(s,s)$ that

$$\begin{aligned} T_\lambda y(t) &= \lambda\Big[\int_{\rho(a)}^{b} G(t,s)f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)\Big] \\ &\leq \lambda\Big[\int_{\rho(a)}^{b} G(s,s)\nabla s + A\varphi(t) + B\psi(t)\Big]\|f\| \\ &\leq \lambda\Big[\int_{\rho(a)}^{b} G(s,s)\nabla s + A\|\varphi\| + B\|\psi\|\Big](f_0+\epsilon)\|y\| \leq \|y\|. \end{aligned}$$

Next, considering f_∞, there exists $\hat{H}_2 > 0$ such that $f(s,y) \geq (f_\infty-\epsilon)y$ for $y \geq \hat{H}_2$. Let $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\Gamma}\}$ Then $y \in K$ and $\|y\| = H_2$ implies

$$\min_{\rho(a)+\tau\leq t\leq b-\tau} y(t) \geq \Gamma\|y\| \geq \hat{H}_2,$$

and so

$$\begin{aligned} T_\lambda y(\nu) &= \lambda\Big[\int_{\rho(a)}^{b} G(\nu,s)f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)\Big] \\ &\geq \lambda\int_{\rho(a)+\tau}^{b-\tau} G(\nu,s)(f_\infty-\epsilon)y(s)\nabla s \\ &\geq \lambda\Gamma\int_{\rho(a)+\tau}^{b-\tau} G(\nu,s)\nabla s(f_\infty-\epsilon)\|y\| \\ &\geq \|y\|. \end{aligned}$$

Therefore, by the first part of Theorem 1.1, it follows that T_λ has a fixed point y satisfying $H_1 \leq \|y\| \leq H_2$. The proof is complete. □

Example 4.2. Let $\mathbb{T}=[-1,2]\cup[3,5]$. Consider the following boundary value problem on $\mathbb{T}$:

$$\begin{cases} -y^{\Delta\nabla}(t)+y(t)=y(t)e^{y(t)}, & t\in[0,4],\\ -y^{\Delta}(0)=2y(1)+1/12y(2)+y(3), \\ y(4)+y^{\Delta}(4)=1/3y(2)+3y(3). \end{cases} \tag{4.3}$$

Taking $\alpha=b_1=0,\ \beta=\gamma=\delta=a_3=\tau=1,\ a_1=2,\ a_2=1/12,\ b_2=1/3,\ b_3=3$, we have $d=\frac{5e^4-1}{2e}$, $\Gamma=0.05393653686$. When taking $f(t,y)=f(y)=ye^y$, we get $f_0=1,\ f_\infty=\infty$. In the case of $p(t)\equiv1,\ q(t)\equiv1,\ \alpha=0,\ \beta=\gamma=\delta=1$, the solutions of the problems (2.1) and (2.2) are

$$\varphi(t)=\begin{cases} \cosh t, & \text{if } 0\le t\le 2,\\ e^{t-1}+\sinh 2\sinh(t-3), & \text{if } 3\le t\le 4, \end{cases}$$

and

$$\psi(t)=\begin{cases} \frac{1}{2}(e^{t-1}+5e^{3-t}), & \text{if } 0\le t\le 2,\\ e^{4-t}, & \text{if } 3\le t\le 4, \end{cases}$$

respectively. Hence, the Green's function of the problem (4.3) has the form

$$G(t,s)=\frac{2e}{5e^4-1}\begin{cases} \varphi(s)\psi(t), \text{ if } 0\le s\le t\le 4,\\ \varphi(t)\psi(s), \text{ if } 0\le t\le s\le 4. \end{cases}$$

By using (2.3), (2.6) and (2.7), we get

$$\Delta=-113.8361085,\ A=0.4925499259,\ B=0.9418550020.$$

So we have $M=61.62173825$. Moreover, since

$$\max_{0\le t\le 4}\int_1^3 G(t,s)\nabla s=\int_1^3 G(\frac{5}{3},s)\nabla s=0.6053767269,$$

for each λ satisfying $0<\lambda<\frac{1}{61.62173825}$, there exists at least one positive solution of the problem (4.3).

Theorem 4.3. *Assume that* (H1)–(H4), *and* (H6) *are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{\Gamma(\int_{\rho(a)+\tau}^{b-\tau}G(\nu,s)\nabla s)f_0}<\lambda<\frac{1}{Mf_\infty}, \tag{4.4}$$

there exists at least one positive solution of (4.1)*.*

Proof. Let λ be given as in (4.4), and choose $\epsilon>0$ such that

$$\frac{1}{\Gamma(\int_{\rho(a)+\tau}^{b-\tau}G(\nu,s)\nabla s)(f_0-\epsilon)}\le\lambda\le\frac{1}{M(f_\infty+\epsilon)}.$$

Beginning with f_0, there exists an $H_1 > 0$ such that $f(s,y) \geq (f_0 - \epsilon)y$ for $0 < y \leq H_1$. So, for $y \in K$ with $\|y\| = H_1$ we have

$$\begin{aligned} T_\lambda y(\nu) &= \lambda[\int_{\rho(a)}^{b} G(\nu,s) f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)] \\ &\geq \lambda \int_{\rho(a)}^{b} G(\nu,s) f(s,y(s))\nabla s \geq \lambda \int_{\rho(a)+\tau}^{b-\tau} G(\nu,s)(f_0-\epsilon)y(s)\nabla s \\ &\geq \lambda\Gamma \int_{\rho(a)+\tau}^{b-\tau} G(\nu,s)\nabla s (f_0-\epsilon)\|y\| \geq \|y\|. \end{aligned}$$

It remains to consider f_∞. There exists an $\hat{H}_2 > 0$ such that $f(s,y) \leq (f_\infty + \epsilon)y$, for all $y \geq \hat{H}_2$. There are two cases:

(a) f is bounded, and (b) f is unbounded.

For case (a), suppose $N > 0$ is such that $f(s,y) \leq N$, for all $0 \leq y \leq \infty$. Let

$$H_2 = \max\{2H_1, \lambda N[\int_{\rho(a)}^{b} G(s,s)\nabla s + A\|\varphi\| + B\|\psi\|]\}.$$

Then, for $y \in K$ with $\|y\| = H_2$, we have

$$\begin{aligned} T_\lambda y(t) &= \lambda[\int_{\rho(a)}^{b} G(t,s) f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)] \\ &\leq \lambda(\int_{\rho(a)}^{b} G(s,s)\nabla s + A\varphi(t) + B\psi(t))N \leq \|y\| \end{aligned}$$

so that $\|T_\lambda y\| \leq \|y\|$.

For case (b), let $g(h) := \max\{f(t,y) : t \in [\rho(a), b], 0 \leq y \leq h\}$. The function g is nondecreasing and $\lim_{h\to\infty} g(h) = \infty$. Choose $H_2 = \max\{2H_1, \hat{H}_2\}$ so that $g(H_2) \geq g(h) \quad$ for $0 \leq h \leq H_2$. For $y \in K$ with $\|y\| = H_2$,

$$\begin{aligned} T_\lambda y(t) &= \lambda[\int_{\rho(a)}^{b} G(t,s) f(s,y(s))\nabla s + A(f)\varphi(t) + B(f)\psi(t)] \\ &\leq \lambda[\int_{\rho(a)}^{b} G(s,s)\nabla s + A\varphi(t) + B\psi(t))]g(H_2) \\ &\leq \lambda[\int_{\rho(a)}^{b} G(s,s)\nabla s + A\varphi(t) + B\psi(t))](f_\infty + \epsilon)H_2 \leq \|y\| \end{aligned}$$

so that $\|T_\lambda y\| \leq \|y\|$. It follows from Theorem 1.1 that T_λ has a fixed point. Hence the problem (4.1) has a positive solution. The proof is complete. □

Theorem 4.4. *In addition to* (H1)–(H4) *and* (H6) *assumptions assume* $f(s,y(s)) > 0$ *on* $[\rho(a), b] \times \mathbb{R}^+$.

(a) *If* $f^0 = 0$ *or* $f^\infty = 0$, *then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *the eigenvalue problem* (4.1) *has a positive solution on* $[\rho(a), b]$.

(b) *If* $f_0 = \infty$ *or* $f_\infty = \infty$, *then there is a* $\lambda_0 > 0$ *such that for all* $0 < \lambda \le \lambda_0$ *the eigenvalue problem* (4.1) *has a positive solution on* $[\rho(a), b]$.

Proof. We now prove the part (b) of Theorem 4.4. Let $r > 0$ be given. We can define

$$R = \max\{f(s, y(s)) : (s, y(s)) \in [\rho(a), b] \times [0, r]\} > 0.$$

If $\|y\| = r$, it follows that

$$T_\lambda y(t) \le R\lambda[\int_{\rho(a)}^{b} G(t,s)\nabla s + A\varphi(t) + B\psi(t)] \le R\lambda M \;\; s \in [\rho(a), b].$$

So we can pick $\lambda_0 > 0$ small enough so that for all $0 < \lambda \le \lambda_0$

$$\|T_\lambda y\| \le \|y\|.$$

Fix $\lambda \le \lambda_0$. Choose $L > 0$ sufficiently large so that

$$\lambda L \Gamma \int_{\rho(a)+\tau}^{b-\tau} G(t_0, s)\nabla s \ge 1, \tag{4.5}$$

where $t_0 \in [\rho(a), b]$. Since $f_0 = \infty$, there is $r_1 < r$ such that

$$\min_{t \in [\rho(a),b]} \frac{f(t,y)}{y} \ge L$$

for $0 < y \le r_1$. Hence we have that

$$f(t, y) \ge Ly$$

for $t \in [\rho(a), b]$, $0 < y \le r_1$. We next show that if $\|y\| = r_1$, then $\|T_\lambda y\| \ge \|y\|$. To show this assume $\|y\| = r_1$. Then

$$\begin{aligned} T_\lambda y(t_0) &= \lambda[\int_{\rho(a)}^{b} G(t_0,s) f(s, y(s))\nabla s + A(f)\varphi(t_0) + B(f)\psi(t_0)] \\ &\ge \lambda L \int_{\rho(a)}^{b} G(t_0,s) y(s)\nabla s \ge \lambda L \Gamma \|y\| \int_{\rho(a)+\tau}^{b-\tau} G(t_0,s) y(s)\nabla s \ge \|y\| \end{aligned}$$

by (4.5). Hence we shown that if $\|y\| = r_1$, then $\|T_\lambda y\| \ge \|y\|$.

Next, we use the assumption that $f_\infty = \infty$. Since $f_\infty = \infty$ there is a $r_2 > r$ such that

$$\min_{t \in [\rho(a),b]} \frac{f(t,y)}{y} \ge L$$

for $y \ge r_2$ and L is chosen so that (4.5) holds. It follows that

$$f(t, y) \ge Ly$$

for $t \in [\rho(a), b]$, $y \ge r_2$.

Let $r_3 := \frac{r_2}{\Gamma}$. We next show that if $\|y\| = r_3$, then $\|T_\lambda y\| \ge \|y\|$. To show this assume $\|y\| = r_3$. Thus we get

$$y(t) \ge \Gamma \|y\| = \Gamma r_3 = \Gamma \frac{r_2}{\Gamma} = r_2$$

for $t \in [\rho(a)+\tau, b-\tau]$. Using this we get that

$$\begin{aligned} T_\lambda y(t_0) &= \lambda[\int_{\rho(a)}^{b} G(t_0,s)f(s,y(s))\nabla s + A(f)\varphi(t_0) + B(f)\psi(t_0)] \\ &\geq \lambda L \int_{\rho(a)}^{b} G(t_0,s)y(s)\nabla s \geq \lambda L\Gamma\|y\| \int_{\rho(a)+\tau}^{b-\tau} G(t_0,s)\nabla s \geq \|y\| \end{aligned}$$

by (4.5). Hence we have shown that if $\|y\| = r_3$, then $\|T_\lambda y\| \geq \|y\|$.

This completes the proof of part (b) as in the previous case. Part (a) holds in an analogous way. □

Similar to the proof of Theorem 4.4 we get the next result.

Theorem 4.5. *Under the hypotheses of Theorem 4.4, the following assertions hold.*

(a) *If $f_0 = f_\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ the eigenvalue problem* (4.1) *has two positive solutions.*
(b) *If $f^0 = f^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the eigenvalue problem* (4.1) *has two positive solutions.*

Now, we give a nonexistence result as follows.

Theorem 4.6. *Under the hypotheses of Theorem 4.4, the following assertions hold.*

(a) *If there is a constant $c > 0$ such that $f(t,y) \geq cy$ for $y \geq 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (4.1) *has no positive solutions for $\lambda \geq \lambda_0$.*
(b) *If there is a constant $c > 0$ such that $f(t,y) \leq cy$ for $y \geq 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (4.1) *has no positive solutions for $0 < \lambda \leq \lambda_0$.*

Proof. We now prove the part (a) of this theorem.Assume there is a constant $c > 0$ such that $f(t,y) \geq cy$ for $y \geq 0$. Assume $y(t)$ is a positive solution of the eigenvalue problem (4.1). We will show that for λ sufficiently large that this leads to a contradiction. Since $T_\lambda y(t) = y(t)$ for $t \in [\rho(a), b]$, we have for $t_0 \in [\rho(a), b]$

$$\begin{aligned} y(t_0) &= \lambda \int_{\rho(a)}^{b} G(t_0,s)f(s,y(s))\nabla s + A(f)\varphi(t_0) + B(f)\psi(t_0) \\ &\geq c\lambda \int_{\rho(a)}^{b} G(t_0,s)y(s)\nabla s \geq c\lambda\Gamma\|y\| \int_{\rho(a)+\tau}^{b-\tau} G(t_0,s)\nabla s. \end{aligned}$$

If we pick λ_0 sufficiently large so that

$$c\lambda\Gamma \int_{\rho(a)+\tau}^{b-\tau} G(t_0,s)\nabla s > 1$$

for all $\lambda \geq \lambda_0$, then we have that $y(t_0) \geq \|y\|$ which is a contradiction.

The proof of part (b) is similar. □

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