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# ASYMPTOTIC CONSTANCY FOR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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**ABSTRACT.** Sufficient conditions for the asymptotic constancy of the solutions of impulsive delay differential system

$$\begin{cases} x'(t) = A(t)\,[x(t-\sigma) - x(t-\tau)] + f(t),\ t \geq t_0,\ t \neq t_i, \\ \Delta x(t_i) = B_i x(t_i) + D_i,\ i = 1,\ 2, \dots, \end{cases}$$

are obtained. Moreover, as $t \to \infty$, the limits of the solutions of the impulsive delay differential system with $B_i = 0$ are computed in terms of the initial function and a special matrix solution of the corresponding adjoint system.

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## 1. INTRODUCTION

In this paper, we shall consider the nonhomogeneous linear impulsive delay differential system

$$\begin{cases} x'(t) = A(t)\,[x(t-\sigma) - x(t-\tau)] + f(t),\ t \geq t_0,\ t \neq t_i, \\ \Delta x(t_i) = B_i x(t_i) + D_i,\ i = 1,\ 2, \dots, \end{cases} \tag{1}$$

where $0 \leq \sigma < \tau$; $0 \leq t_0 \leq t_1 < t_2 < \dots < t_i < t_{i+1} < \dots$, $\lim_{i\to\infty} t_i = \infty$; $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+) = \lim_{t \to t_i^+} x(t)$ and $x(t_i^-) = \lim_{t \to t_i^-} x(t) = x(t_i)$.

Let $R$ and $R^n$ be the set of real numbers and the $n$-dimensional space of real column vectors, respectively. Let $\|\cdot\|$ denote any convenient norm on $R^n$ or the associated induced norm of a square matrix. We shall study system (1) together with the following hypotheses:

$(H1)$ $A : [t_0, \infty) \to R^{n\times n}$ is a continuous matrix function,

$(H2)$ $f : [t_0, \infty) \to R^n$ is a continuous vector function,

$(H3)$ $B_i \in R^{n\times n}$ such that $\det(I + B_i) \neq 0$, where $I$ is the $n \times n$ identity matrix, $i = 1, 2, \dots,$

$(H4)$ $D_i \in R^n$, $i = 1, 2, \dots$

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We denote by $PC([t_0 - \tau, t_0], R)$ the set of piecewise continuous functions on $[t_0 - \tau, t_0]$.

**Definition 1.** For any real-valued piecewise continuous $n$-dimensional vector function $\phi = (\phi_1, \dots, \phi_n)$ where $\phi_i \in PC([t_0-\tau, t_0], R)$, $x(t) = (x_1(t), \dots, x_n(t))$ is said to be a solution of (1) on $[t_0, \infty)$ and satisfies the initial condition $x(t) = \phi(t)$, $t \in [t_0 - \tau, t_0]$, if the following conditions are satisfied:

$(B1)$ $x(t)$ is absolutely continuous on each interval $(t_i, t_{i+1})$,

$(B2)$ for any $t_i \in [t_0, \infty)$, $i = 1, 2, \dots$, $x(t_i^+)$ and $x(t_i^-)$ exist and $x(t_i^-) = x(t_i)$,

$(B3)$ $x(t)$ satisfies (1) for almost everywhere in $[t_0, \infty)$ and at impulsive points $t_i$ situated in $[t_0, \infty)$ may have discontinuity of the first kind.

**Definition 2.** We say a vector function is continuous (absolutely continuous) if all of its components are continuous (absolutely continuous).

Impulsive and delay differential equations are observed in many fields of science and technology such as biology, engineering and physics. Many dynamic population models are special cases of (1). For example, in [12], Cooke and Yorke proposed the scalar equation

$$x'(t) = g(x(t)) - g(x(t - L)),$$

as a model for certain population growth if individuals have a constant life span $L$, where $x(t)$ is the size of the population at time $t$ and $g(x)$ is the birth rate. When $g(x) = ax$ ($a = const.$), then the equation above reduces to (1) with $n = 1$, $A(t) = a$, $\sigma = 0$, $\tau = L$, and $f(t) \equiv 0$. Furthermore, the scalar linear delay differential equation of the type of (1)

$$x'(t) = a(t)\,(x(t) - x(t - h))$$

is a sort of pantograph equation which arises as a mathematical model of an industrial problem involving wave motion in the overhead supply line to an electrified railway system [17]. Also, system (1) with $f \equiv 0$ has an application in number theory [28].

On the other hand, there are many works on the theory of impulsive differential equations as well as on the theory of delay differential equations. The monographs [5, 31] are for the impulsive case, and the books of Gopalsamy [18], Gyori and Ladas [20], Hale [21], Kuang [24] are good sources for delay differential equations. But, the corresponding theory of impulsive delay differential equations (IDDE) has been less developed because of significant technical and theoretical difficulties. Only a few publications have been produced in the direction of IDDE. Existence and uniqueness results for IDDE are presented in [6, 16, 25]. Results on stability for IDDE are given in [1, 2, 10, 27, 32, 35]. Boundedness results for IDDE are established in [9, 26]. Oscillation theory for IDDE is investigated in [19, 33, 34, 38, 39]. Boundary value problems and existence of periodic solutions for IDDE are studied in [22, 36, 37].

Recently, the problem of the asymptotic constancy of solutions and asymptotic convergence of solutions of linear delay differential equations without impulses has been an extensive study [3, 4, 7, 11, 13, 14, 15, 23, 29, 30] and the references cited therein. In all these works, authors generally consider the scalar linear equation with variable bounded delay of the form

$$x'(t) = a(t)\left(x(t) - x(t-\tau(t))\right),$$

which is similar to (1). Moreover, in [8], Pituk and the first author have proved some results for the asymptotic constancy of the solutions of (1) without impulses.

So, due to practical reasons and the papers mentioned above, we have been motivated to deal with system (1) and study the problem of asymptotic constancy. According to the best of our knowledge, the asymptotic constancy for IDDE has not been studied till now.

The main purpose of this work is to give sufficient conditions for asymptotic constancy of (1) and also, as $t \to \infty$, to compute the limits of the solutions of the impulsive delay differential system

$$\left\{\begin{array}{l} x'(t) = A(t)\left[x(t-\sigma) - x(t-\tau)\right] + f(t),\ t \geq t_0,\ t \neq t_i, \\ \Delta x(t_i) = D_i,\ i = 1, 2, \ldots, \end{array}\right. \tag{2}$$

in terms of a special matrix solution of the corresponding adjoint system and the initial function

$$x(t) = \phi(t),\ t \in [t_0 - \tau, t_0], \tag{3}$$

where $\phi : [t_0 - \tau, t_0] \to R^n$ is continuous and $\|\phi\|_r = \sup\limits_{t_0 - r \leq t \leq t_0} \|\phi(t)\|$ for $r \geq 0$.

We can guarantee the existence and uniqueness of the solution of the initial value problem (1), (3) as showed in [1]. In fact, it is well known that under the above conditions the problem (1), (3) with $B_i = D_i = 0$, $i = 1, 2, \ldots$, has a unique solution $x$ on $[t_0 - \tau, \infty)$ [21]. This interval includes all of the intervals $[t_i, t_{i+1}]$, $i = 1, 2, \ldots$, and so there is a unique solution of the initial value problem (1), (3). Moreover, we should note that a straightforward verification shows that the solution $x(t)$ of the initial value problem (1), (3) satisfies the following integral equation

$$x(t) = \left\{\begin{array}{ll} \phi(t), & t_0 - \tau \leq t \leq t_0, \\ x(t_0) + \int\limits_{t_0}^{t} A(s)x(s-\sigma)ds - \int\limits_{t_0}^{t} A(s)x(s-\tau)ds + \int\limits_{t_0}^{t} f(s)ds & \\ + \sum\limits_{t_0 \leq t_i < t} B_i x(t_i) + \sum\limits_{t_0 \leq t_i < t} D_i, & t \geq t_0. \end{array}\right. \tag{4}$$

This paper is organised as follows. In Section 2, the main results are presented. Section 3 contains the proof of our first main result. In Section 4, the proof of second main result is given. Finally, in Section 5, some remarks and examples are stated.

## 2. MAIN RESULTS

We can state our main results as follows.

**Theorem 1.** In addition to $(H1)-(H4)$, assume that

(i) $\int\limits_{t_0}^{\infty} \|A(s)\| \, ds \leq K_1 < \infty$,

(ii) $\int\limits_{t_0}^{\infty} \|f(s)\| \, ds \leq K_2 < \infty$,

(iii) $\prod\limits_{i=1}^{\infty} (1 + \|B_i\|) \leq L_1 < \infty$,

(iv) $\prod\limits_{i=1}^{\infty} (1 + \|D_i\|) \leq L_2 < \infty$.

Then the solution $x(t)$ of (1), (3) tends to a constant vector as $t \to \infty$.

**Theorem 2.** Assume that all assumptions, except $(H3)$ and $(iii)$, of Theorem 1 are satisfied. Let $x(t)$ be a solution of (2)–(3) and $\lim\limits_{t\to\infty} x(t) = l(\phi)$.
If

(v) $\int\limits_{t+\sigma}^{t+\tau} \|A(s)\| \, ds \leq \rho < 1, \; t \geq t_0$,

then

$$l(\phi) = Y(t_0)\phi(t_0) - \int_{t_0}^{t_0+\tau} Y(s)A(s)\phi(s-\tau)ds + \int_{t_0}^{t_0+\sigma} Y(s)A(s)\phi(s-\sigma)ds + \int_{t_0}^{\infty} Y(s)f(s)ds + \sum_{t_0 \leq t_i} Y(t_i)D_i, \tag{5}$$

where $Y$ is a special matrix solution of the integral equation

$$Y(t) = I + \int_{t+\sigma}^{t+\tau} Y(s)A(s)ds \text{ for } t \geq t_0. \tag{6}$$

## 3. THE PROOF OF THEOREM 1

To prove this theorem we consider the following well known lemma. For the proof, see [5].

**Lemma 1.** Suppose that for $t \geq t_0$ the inequality

$$u(t) \leq c + \int_{t_0}^{t} b(s)u(s)ds + \sum_{t_0 \leq \tau_k < t} \beta_k u(\tau_k)$$

holds, where $u(t) \in PC\,(R, R^+)$, $b(t) \in PC\,(R, R^+)$ and $\beta_k \geq 0$, $k \in N$ and $c \geq 0$ are constants (Here, $PC(R, R^+)$ denotes the set of functions $\psi : R \to R^+$ which are continuous for $t \in R$, $t \neq \tau_k$, are continuous from the left for $t \in R$, and have discontinuities of the first kind at the points $\tau_k \in R$).

Then for $t \geq t_0$

$$u(t) \leq c \prod_{t_0 \leq \tau_k < t} (1+\beta_k) \exp\left(\int_{t_0}^{t} b(s)ds\right).$$

**Proof of Theorem 1.** Let $x(t)$ be the solution of (1), (3). Then from (4) it follows that

$$\begin{aligned}\|x(t)\| \leq\ & \|x(t_0)\| + \int_{t_0}^{t} \|A(s)\| \, \|x(s-\sigma)\| \, ds + \int_{t_0}^{t} \|A(s)\| \, \|x(s-\tau)\| \, ds \\ & + \int_{t_0}^{t} \|f(s)\| \, ds + \sum_{t_0 \leq t_i < t} \|B_i\| \, \|x(t_i)\| + \sum_{t_0 \leq t_i < t} \|D_i\|\end{aligned}$$

for $t \geq t_0$. This inequality can be written as below:

$$\begin{aligned}\|x(t)\| \leq\ & \|x(t_0)\| + \int_{t_0-\sigma}^{t_0} \|A(s+\sigma)\| \, \|x(s)\| \, ds + \int_{t_0}^{t} \|A(s+\sigma)\| \, \|x(s)\| \, ds \\ & - \int_{t-\sigma}^{t} \|A(s+\sigma)\| \, \|x(s)\| \, ds + \int_{t_0-\tau}^{t_0} \|A(s+\tau)\| \, \|x(s)\| \, ds \\ & + \int_{t_0}^{t} \|A(s+\tau)\| \, \|x(s)\| \, ds - \int_{t-\tau}^{t} \|A(s+\tau)\| \, \|x(s)\| \, ds \\ & + \int_{t_0}^{t} \|f(s)\| \, ds + \sum_{t_0 \leq t_i < t} \|B_i\| \, \|x(t_i)\| + \sum_{t_0 \leq t_i < t} \|D_i\| \\ \leq\ & \|x(t_0)\| + \|\phi\|_\sigma \int_{t_0-\sigma}^{t_0} \|A(s+\sigma)\| \, ds + \int_{t_0}^{t} \|A(s+\sigma)\| \, \|x(s)\| \, ds \\ & + \|\phi\|_\tau \int_{t_0-\tau}^{t_0} \|A(s+\tau)\| \, ds + \int_{t_0}^{t} \|A(s+\tau)\| \, \|x(s)\| \, ds \\ & + \int_{t_0}^{t} \|f(s)\| \, ds + \sum_{t_0 \leq t_i < t} \|B_i\| \, \|x(t_i)\| + \sum_{t_0 \leq t_i < t} \|D_i\| \, .\end{aligned}$$

From (i), (ii) and (iv), we find that

$$\|x(t)\| \leq c + \int_{t_0}^{t} \left(\|A(s+\sigma)\| + \|A(s+\tau)\|\right) \|x(s)\| \, ds + \sum_{t_0 \leq t_i < t} \|B_i\| \, \|x(t_i)\|$$

for $t \geq t_0$, where $c = \|x(t_0)\| + K_1 \|\phi\|_\sigma + K_1 \|\phi\|_\tau + K_2 + L_2$.

Applying Lemma 1,

$$\|x(t)\| \leq c \prod_{t_0 \leq t_i < t} (1 + \|B_i\|) \exp\left( \int_{t_0}^{t} (\|A(s+\sigma)\| + \|A(s+\tau)\|) \, ds \right),$$

and using the conditions (i) and (iii), we have

$$\|x(t)\| \leq cL_1 e^{2K_1} = K \text{ , for } t \geq t_0.$$

Therefore the solution $x(t)$ is bounded for $t \geq t_0$ and so there is a positive constant $K'$, $K' \geq K$, such that

$$\|x(t)\| \leq K' \; for \; t \geq t_0 - \tau. \tag{7}$$

On the other hand, for $t_0 \leq s \leq t < \infty$

$$\|x(t) - x(s)\| \leq \int_s^t \|A(u)\| \, \|x(u-\sigma)\| \, du + \int_s^t \|A(u)\| \, \|x(u-\tau)\| \, du$$
$$+ \int_s^t \|f(u)\| \, du + \sum_{s \leq t_i < t} \|B_i\| \, \|x(t_i)\| + \sum_{s \leq t_i < t} \|D_i\| \, . \tag{8}$$

Now, using (7), (8) may be written as

$$\|x(t) - x(s)\| \leq 2K' \int_s^\infty \|A(u)\| \, du + \int_s^\infty \|f(u)\| \, du$$
$$+ K' \sum_{s \leq t_i} \|B_i\| + \sum_{s \leq t_i} \|D_i\| \, . \tag{9}$$

We note that because of (iii) and (iv), we have, respectively,

$$\sum_{i=1}^{\infty} \|B_i\| < \infty, \tag{10}$$

and

$$\sum_{i=1}^{\infty} \|D_i\| < \infty. \tag{11}$$

From (9) and conditions (i), (ii), (10) and (11), it is easy to see that

$$\lim_{s \to \infty} \|x(t) - x(s)\| = 0.$$

Hence the Cauchy convergence criterion implies the existence of $\lim\limits_{t \to \infty} x(t)$ in $R^n$.

## 4. THE PROOF OF THEOREM 2

The proof of Theorem 2 is based on the method presented in [8]. Therefore, we first need to prove the following results.

**Theorem 3.** Suppose $(H1)$ and (v) hold. Then there exists a unique continuous and bounded matrix function $Y : [t_0, \infty) \to R^{n\times n}$ such that (6) holds.

We omit the proof of this theorem because the equation (6) does not include any impulse, and so the proof of Theorem 3 is a repetition of the arguments in [8, Theorem 2].

Let us denote

$$C(t) = Y(t)x(t) - \int_t^{t+\tau} Y(s)A(s)x(s-\tau)ds + \int_t^{t+\sigma} Y(s)A(s)x(s-\sigma)ds \text{ for } t \geq t_0, \tag{12}$$

where $Y$ has the meaning from Theorem 3 and $x$ is the solution of (2), (3).

**Lemma 2.** Suppose that $(H1)$, $(H2)$, $(H4)$ and (v) hold. Then

$$C(t) = C(t_0) + \int_{t_0}^{t} Y(s)f(s)ds + \sum_{t_0 \leq t_i < t} Y(t_i)D_i \text{ for } t \geq t_0. \tag{13}$$

**Proof.** First, we should prove that $C(t)$ defined by (12) satisfies

$$\begin{cases} C'(t) = Y(t)f(t), & t \geq t_0, \quad t \neq t_i, \\ \Delta C(t_i) = Y(t_i)D_i, & i = 1, 2, \ldots \end{cases} \tag{14}$$

Because of Theorem 3, we know that the special matrix solution $Y$ of (6) is differentiable on $[t_0, \infty)$ and satisfies the "adjoint system"

$$Y'(t) = Y(t+\tau)A(t+\tau) - Y(t+\sigma)A(t+\sigma) \tag{15}$$

for $t \geq t_0$.

Differentiating (12) for $t \geq t_0$ and $t \neq t_i$, we get

$$C'(t) = Y'(t)x(t) + Y(t)x'(t) - Y(t+\tau)A(t+\tau)x(t) + Y(t)A(t)x(t-\tau) + Y(t+\sigma)A(t+\sigma)x(t) - Y(t)A(t)x(t-\sigma). \tag{16}$$

Using (2) and (15), Equation (16) implies that

$$C'(t) = Y(t)f(t) \text{ for } t \geq t_0 \text{ and } t \neq t_i.$$

Moreover, from (12),

$$\begin{aligned} \Delta C(t_i) &= C(t_i^+) - C(t_i^-) \\ &= Y(t_i)\Delta x(t_i) \end{aligned}$$

$$= \quad Y(t_i)D_i$$

which completes the proof of (14).

Integrating both sides of (14) with respect to $s$ from $t_0$ to $t$, we obtain

$$C(t) = C(t_0) + \int_{t_0}^{t} Y(s)f(s)ds + \sum_{t_0 \le t_i < t} \Delta C(t_i)$$

for $t \ge t_0$ which implies (13).

We are now ready to prove the second main result.

**Proof of Theorem 2.** Let $x(t)$ be the solution of equation (2), (3). It is sufficient to show that

$$\lim_{t\to\infty} x(t) = C(t_0) + \int_{t_0}^{\infty} Y(s)f(s)ds + \sum_{t_0 \le t_i} Y(t_i)D_i \tag{17}$$

where $C$ is defined by (12). We have for $t \ge t_0$,

$$\begin{aligned} & x(t) - C(t_0) - \int_{t_0}^{\infty} Y(s)f(s)ds - \sum_{t_0 \le t_i} Y(t_i)D_i \\ &\quad = \quad x(t) - \left[ C(t_0) + \int_{t_0}^{t} Y(s)f(s)ds + \sum_{t_0 \le t_i < t} Y(t_i)D_i \right] \\ &\quad\quad\quad - \int_{t}^{\infty} Y(s)f(s)ds - \sum_{t \le t_i} Y(t_i)D_i \\ &\quad = \quad x(t) - C(t) - \int_{t}^{\infty} Y(s)f(s)ds - \sum_{t \le t_i} Y(t_i)D_i. \end{aligned}$$

The last equality is a consequence of (13). This, together with (12), implies for $t \ge t_0$,

$$\begin{aligned} & x(t) - C(t_0) - \int_{t_0}^{\infty} Y(s)f(s)ds - \sum_{t_0 \le t_i} Y(t_i)D_i \\ &\quad = x(t) - Y(t)x(t) + \int_{t}^{t+\tau} Y(s)A(s)x(s-\tau)ds \\ &\quad\quad - \int_{t}^{t+\sigma} Y(s)A(s)x(s-\sigma)ds - \int_{t}^{\infty} Y(s)f(s)ds \\ &\quad\quad - \sum_{t \le t_i} Y(t_i)D_i. \end{aligned} \tag{18}$$

Multiplying (6) by $x(t)$, we obtain for $t \ge t_0$,

$$x(t) \quad = \quad Y(t)x(t) - \int_{t+\sigma}^{t+\tau} Y(s)A(s)x(t)ds$$

$$= \quad Y(t)x(t) - \int_t^{t+\tau} Y(s)A(s)x(t)ds + \int_t^{t+\sigma} Y(s)A(s)x(t)ds.$$

Substituting the last expression into (18), we find for $t \geq t_0$,

$$(19) \qquad x(t) - C(t_0) - \int_{t_0}^{\infty} Y(s)f(s)ds - \sum_{t_0 \leq t_i} Y(t_i)D_i$$

$$\begin{aligned}
= \quad & -\int_t^{t+\tau} Y(s)A(s)x(t)ds + \int_t^{t+\sigma} Y(s)A(s)x(t)ds \\
& + \int_t^{t+\tau} Y(s)A(s)x(s-\tau)ds - \int_t^{t+\sigma} Y(s)A(s)x(s-\sigma)ds \\
& - \int_t^{\infty} Y(s)f(s)ds - \sum_{t \leq t_i} Y(t_i)D_i \\
= \quad & \int_t^{t+\tau} Y(s)A(s)\left[x(s-\tau) - x(t)\right] ds \\
& + \int_t^{t+\sigma} Y(s)A(s)\left[x(t) - x(s-\sigma)\right] ds \\
& - \int_t^{\infty} Y(s)f(s)ds - \sum_{t \leq t_i} Y(t_i)D_i.
\end{aligned}$$

From (19) together with the estimate (7) and the boundedness of $Y(t)$ on $[t_0, \infty)$, we have

$$\left\| x(t) - C(t_0) - \int_{t_0}^{\infty} Y(s)f(s)ds - \sum_{t_0 \leq t_i} Y(t_i)D_i \right\|$$

$$\begin{aligned}
\leq \quad & 2K' \left\| Y \right\|_B \int_t^{t+\tau} \left\| A(s) \right\| ds + 2K' \left\| Y \right\|_B \int_t^{t+\sigma} \left\| A(s) \right\| ds \\
& + \left\| Y \right\|_B \int_t^{\infty} \left\| f(s) \right\| ds + \left\| Y \right\|_B \sum_{t \leq t_i} \left\| D_i \right\|
\end{aligned}$$

for $t \geq t_0$, where $\left\| Y \right\|_B = \sup\limits_{t \geq t_0} \left\| Y(t) \right\|$. Hence it follows that (17) is correct.

Taking into account (3) and (12), it is easily verified that the limit relation (17) reduce to (5). So the proof of Theorem 2 is completed.

## 5. REMARKS AND EXAMPLES

**Remark 1.** Our technique for the proof of Theorem 1 is different from that in [8, Theorem 1]. Because the technique used there did not work for IDDE. So, we first had to establish the boundedness of $x(t)$, and then we followed the Cauchy convergence criterion to complete the proof of Theorem 1.

**Remark 2.** Our method for the proof of Theorem 2 is a modification of that in [8, Theorem 3]. However, there is a significant difference between the two proofs. As a matter of the fact that our proof is based on the boundedness of $x(t)$ instead of the convergence of $\int_{t_0}^{\infty} |x'(t)|\, dt$ used in [8, Theorem 3].

**Remark 3.** We proved Theorem 1 for the impulse conditions

$$\Delta x(t_i) = B_i x(t_i) + D_i, \quad i = 1, 2, \ldots,$$

that are considered as general case for linear impulsive delay differential equations [see 19, 33, 38, 39]. But, we could managed to prove Theorem 2 only for the impulse conditions

$$\Delta x(t_i) = D_i, \quad i = 1, 2, \ldots,$$

which are equal to

$$x(t_i^+) = x(t_i^-) + D_i, \quad i = 1, 2, \ldots,$$

and these type of conditions are formally similar to $x(t_i^+) = B_i x(t_i^-)$ used in [1, 9].

**Example 1.** Let us consider the scalar initial value problem without impulses

$$\begin{cases} x'(t) = A(t)[x(t) - x(t-1)] + f(t), \ t \geq 0, \\ x(t) = e^{-t}, \ \ -1 \leq t \leq 0, \end{cases} \tag{20}$$

where $A(t) = \dfrac{1}{(1+t)^2}$, $f(t) = \left(-1 - \dfrac{1}{(1+t)^2} + \dfrac{e}{(1+t)^2}\right) e^{-t}$.

This problem satisfies all hypotheses of Theorem 1 and 2 with $B_i = 0$ and $D_i = 0$ for $i = 1, 2, \ldots$. So, the solution $x(t)$ of (20) tends to a real constant as $t \to \infty$ and this limit, say $l(\phi)$, can be calculated with subject to (5) as

$$l(\phi) = Y(0) - \int_0^1 Y(s)A(s)e^{-(s-1)}ds + \int_0^\infty Y(s)f(s)ds \tag{21}$$

where $Y(t)$, by (6), satisfies the following scalar integral equation

$$Y(t) = 1 + \int_t^{t+1} Y(s)A(s)ds \text{ for } t \geq 0. \tag{22}$$

On the other hand, the solution of (20) is $x(t) = e^{-t}$ for $-1 \leq t < \infty$ and $l(\phi) = \lim\limits_{t\to\infty} x(t) = 0$.

**Example 2.** Again, consider Eq. (20) with impulses as follows

$$\begin{cases} x'(t) = A(t)[x(t) - x(t-1)] + f(t), \quad t \geq 0, \quad t \neq t_i = i, \\ \Delta x(t_i) = \dfrac{1}{2^i}, \quad i = 1, 2, 3, \dots \\ x(t) = e^{-t}, \quad -1 \leq t \leq 0, \end{cases} \tag{23}$$

Here, also, all hypotheses of Theorem 1 and 2 are satisfied for impulsive delay differential equation (23). Therefore, due to Theorem 1, the solution $x(t)$ of (23) has a limit, that is, $\lim_{t\to\infty} x(t) = l(\phi) \in R$. From Theorem 2, it is clear that

$$l(\phi) = \sum_{i=1}^{\infty} \frac{1}{2^i} Y(i)$$

where $Y(t)$ is the same as (22).

To calculate $l(\phi)$ explicitly from (5) and (6), we may consider the following example.

**Example 3.** If we take, $n = 1$, $A(t) = 0$, $f(t) = 0$, $B_i = 0$, $D_i = \dfrac{1}{2^i}$ $(i = 1, 2, \dots)$, then system (1) reduces to the scalar equation

$$\begin{cases} x'(t) = 0, \; t \geq 0, \quad t \neq t_i = i, \\ \Delta x(t_i) = \dfrac{1}{2^i}, \quad i = 1, 2, 3, \dots \end{cases} \tag{24}$$

Here, in place of initial function (3), we take into account the initial condition

$$x(0) = 1. \tag{25}$$

Clearly, Theorems 1 and 2 can be applied to the problem (24)–(25). From (5) and (6), the limit of the solution $x(t)$ of (24)–(25) is calculated as

$$\lim_{t\to\infty} x(t) = l(\phi) = 2.$$

Indeed, the solution $x(t)$ is obtained as

$$x(t) = \begin{cases} 1, \quad 0 \leq t \leq 1, \\ \sum\limits_{m=0}^{k} \dfrac{1}{2^m}, \quad k < t \leq k+1, \quad k = 1, 2, \dots \end{cases}$$

and

$$\lim_{t\to\infty} x(t) = \sum_{m=0}^{\infty} \frac{1}{2^m}.$$

We note that, in the case of without impulses, the problem (24)–(25) reduces to

$$\begin{cases} x'(t) = 0, \quad t \geq 0, \\ x(0) = 1 \end{cases} \tag{26}$$

which has the solution $x(t) = 1$ on $[0, \infty)$. For this solution $\lim_{t\to\infty} x(t) = 1$ which also can be found by applying (6) and (5) with $D_i = 0$.

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## REFERENCES

[1] A. Anokhin, L. Berezansky, and E. Braverman, Exponential stability of linear delay impulsive differential equations, *J. Math. Anal. Appl.,* 193: 923–941, 1995.

[2] A. Anokhin, L. Berezansky, and E. Braverman, Stability of linear delay impulsive differential equations, *Dynam. Systems Appl.,* 4:173–187, 1995.

[3] O. Arino, and M. Pituk, More on linear differential systems with small delays, *J. Differential Equ.*, 170:381–407, 2001.

[4] F. V. Atkinson, and J. R. Haddock, Criteria for asymptotic constancy of solutions of functional differential equations, *J. Math. Anal. Appl.*, 91: 410–423, 1983.

[5] D.D. Bainov, and P.S. Simeonov, *Impulsive Differential Equations, Asymptotic Properties Of The Solutions*, p.16, World Scientific, 1995.

[6] G. Ballinger, and X. Liu, Existence and uniqueness results for impulsive delay differential equations, *DCDIS.*, 5:579–591, 1999.

[7] J. Bastinec, J. Diblik, and Z. Smarda, Convergence tests for one scalar differential equation with vanishing delay, *Arch. Math. (Brno),* 36:405–414, 2000, CDDE issue.

[8] H. Bereketoğlu, and M. Pituk, Asymptotic constancy for nonhomogeneous linear differential equations with unbounded delays, *Discrete andContinuous Dynamical Systems, A Supplement Volume*: 100–107, 2003.

[9] L. Berezansky, and E. Braverman, Exponential boundedness of solutions for impulsive delay differential equations, *Appl. Math. Lett.*, 9:91–95, 1996.

[10] L. Berezansky, and E. Braverman, Impulsive stabilization of linear delay differential equations, *Dynam. Systems Appl.*, 5:263–276, 1996.

[11] J. Cermak, Asymptotic behavior of solutions of some differential equations with an unbounded delay, *E.J. Qualitative Theory of Diff. Equ.,* no.2, 1–8, 2000.

[12] K. L. Cooke, and J. A. Yorke, Some equations modelling growth processes and gonorrhea epidemics, *Math. Biosci.*, 16:75–101, 1973.

[13] J. Diblik, Asymptotic representation of solutions of equation $y'(t) = \beta(t)\,[y(t) - y(t-\tau(t))]$, *J. Math. Anal. Appl.,* 217:200–215, 1998.

[14] J. Diblik, Behaviour of solutions of linear differential equations with delay, *Archivum Mathematicum (Brno),* 34:31–47, 1998.

[15] J. Diblik, A criterion for convergence of solutions of homogeneous delay linear differential equations, *Anal. Polon. Mat.*, LXXII.2, 115–130, 1999.

[16] F. Dubeau, and J. Karrakchou, State-dependent impulsive delay differential equations, *Applied Mathematics Letters,* 15:333–338, 2002.

[17] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, On a functional differential equation, *J. Inst. Math. Appl.*, 8:271–307, 1971.

[18] K. Gopalsamy, *Stability And Oscillation In Delay Differential Equations Of Population Dynamics,* Kluwer Academic, Dordrecht, 1992.

[19] K. Gopalsamy, and B. G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.*, 139:110–122, 1989.

[20] I. Gyori, and G. Ladas, *Oscillation Theory Of Delay Differential Equations*, Clarendon Press, Oxford, 1991.

[21] J.K. Hale, *Theory Of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[22] H. Huo, W. Li, and X. Liu, Existence and global attractivity of positive periodic solution of an impulsive delay differential equation, *Applicable Analysis,* 83:1279–1290, 2004.

[23] T. Krisztin, A note on the convergence of the solutions of a linear functional differential equation, *J. Math. Anal. Appl.*, 145:17–25, 1990.

[24] Y. Kuang, *Delay Differential Equations With Applications in Population Dynamics,* Academic Press, Boston, 1993.

[25] X. Liu, and G. Ballinger, Existence and continuability of solutions for differential equations with delays and state-dependent impulses, *Nonlinear Anal. TMA* , 51:633–647, 2002.

[26] X. Liu, and G. Ballinger, Boundedness for impulsive delay differential equations and applications to population growth models, *Nonlinear Analysis,* 53:1041-1062, 2003.

[27] X. Liu, and G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations, *Computers and Mathematics with Applications,* 41:903–915, 2001.

[28] K. Mahler, On a special functional equation, *J. London Math. Soc.* 15, 115–123, 1940.

[29] G. Makay, and J. Terjeki, On the asymptotic bahavior of the pantograph equations, *E. J. Qualitative Theory of Diff. Equ.,* no. 2, 1–12, 1998.

[30] K. Murakami, Asymptotic constancy for systems of delay differential equations, *Nonlinear Analysis, Theory, Methods and Applications*, 30: 4595–4606, 1997.

[31] A.M. Samoilenko, and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, 1995.

[32] X. Tang, Z. He, and J. Yu, Stability theorems for delay differential equations with impulses, *Appl. Math. Comput.,* 131:373–381, 2002

[33] J. Yan, Oscillation properties of a second order impulsive delay differential equation, *Comp. and Math. with Appl.,* 47:253–258, 2004.

[34] J. Yan, A. Zhao, and L. Peng, Oscillation of impulsive delay differential equations and applications to population dynamics, *ANZIAM J. (Series B),* 46:545–554, 2005.

[35] J.S. Yu, and B. G. Zhang, Stability theorem for delay differential equations with impulses, *J. Math. Anal. Appl.,* 199:162–175, 1996.

[36] D. Yujun, Periodic boundary value problems for functional differential equations with impulses, *J. Math. Anal. Appl.*, 210:170–181, 1997.

[37] F. Zhang, Z. Ma, and J. Yan, Boundary value problems for first order impulsive delay differential equations with a parameter, *J. Math. Anal. Appl.*, 290:213–223, 2004.

[38] Y. Zhang, A. Zhao, and J. Yan, Oscillation criteria for impulsive delay differ ential equations, *J. Math. Anal. Appl.*, 205:461–470, 1997.

[39] A. Zhao, and J. Yan, Asymptotic behavior of solutions of impulsive delay differential equations, *J. Math. Anal. Appl.*, 201:943–954, 1996.