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NONLINEAR INTEGRAL EQUATIONS IN BANACH SPACES AND HENSTOCK-KURZWEIL-PETTIS INTEGRALS

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ABSTRACT. We prove an existence theorem for the nonlinear integral equation :

$$x(t) = f(t) + \int_0^\alpha k_1(t,s)x(s)ds + \int_0^\alpha k_2(t,s)g(s,x(s))ds, \quad t \in I_\alpha = [0,\alpha], \quad \alpha \in R_+,$$

with the Henstock-Kurzweil-Pettis integrals. This integral equation can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation. The assumptions about the function g are really-weak: scalar measurability and weak sequential continuity with respect to the second variable. Moreover, we suppose that the function g satisfies some conditions expressed in terms of the measure of weak noncompactness.

Key words: existence of solution, Henstock-Kurzweil integral, Pettis integral, Henstock-Kurzweil-Pettis integral, nonlinear Fredholm integral equation, measures of weak noncompactness

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1. INTRODUCTION

The Henstock-Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals [15, 19, 25]. A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0,1]$ and $F(0) = 0$ can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957-58 and has since proved useful in the study of ordinary differential equations [4, 8, 23, 24, 31]. In the paper [7] S. S. Cao defined the Henstock integral in a Banach space, which is a generalization of the Bochner integral. The Pettis integral is also a generalization of the Bochner integral [30]. This notion is strictly relative to weak topologies in Banach spaces.

In [10], we generalized both concepts of integral introducing the Henstock-Kurzweil-Pettis integral.

Let $(E, \|\cdot\|)$ be a Banach space, E^*- its dual space and $I_\alpha = [0,\alpha]$, $\alpha \in R_+$. Moreover, let $(C(I_\alpha, E), \omega)$ denote the space of all continuous functions from I_α to E endowed with the topology $\sigma(C(I_\alpha, E), C(I_\alpha, E)^*)$. In this paper we will prove an

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existence theorem for the integral equation:

$$x(t) = f(t) + \int_0^\alpha k_1(t,s)x(s)ds + \int_0^\alpha k_2(t,s)g(s,x(s))ds, \tag{1}$$

where $g : I_\alpha \times E \to E$, $f : I_\alpha \to E$, $x : I_\alpha \to E$ are functions with values in E, $k_1,\ k_2 : I_\alpha \times I_\alpha \to R_+$ and the integrals are taken in the sense of Henstock-Kurzweil-Pettis [11].

Note that the previous integral equation can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1) (see, for example, [1, 2, 3, 20, 21, 26, 28, 29]).

The main result presented in this paper generalizes the previous ones.

A Kubiaczyk fixed point theorem [22] and the techniques of the theory of measure of weak noncompactness are used to prove the existence of solution of problem (1). The assumptions about the function g are really-weak: scalar measurability and weak sequential continuity with respect to the second variable. By using these conditions, we define a completely continuous operator F over the Banach space $C([0,\alpha])$, whose fixed points are solutions of (1). The fixed point theorem of Kubiaczyk [22] is used to prove the existence of a fixed point of the operator F.

Let us recall, that a function $f : I_\alpha \to E$ is said to be *weakly continuous* if it is continuous from I_α to E endowed with its weak topology. A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly-weakly sequentially continuous* if for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we will write $x_n \xrightarrow{\omega} x_0$.

Our fundamental tool is the measure of weak noncompactness developed by De-Blasi [6].

Let A be a bounded nonempty subset of E. The *measure of weak noncompactness* $\mu(A)$ is defined by

$$\mu(A) = \inf\{t > 0 : \quad \text{there exists } C \in K^\omega \quad \text{such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

We will use the following properties of the measure of weak noncompactness μ (for bounded nonempty subsets A and B of E):

(i) if $A \subset B$, then $\mu(A) \le \mu(B)$;
(ii) $\mu(A) = \mu(\bar{A})$, where $\bar{A}$ denotes the closure of A;
(iii) $\mu(A) = 0$ if and only if A is relatively weakly compact;

(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$;

(v) $\mu(\lambda A) = |\lambda|\mu(A)$, $(\lambda \in R)$;

(vi) $\mu(A + B) \leq \mu(A) + \mu(B)$;

(vii) $\mu(convA) = \mu(A)$.

It is necessary to remark that if μ has these properties, then the following Lemma is true.

Lemma 1.1 [27]. *Let $H \subset C(I_\alpha, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_\alpha$, $H(t) = \{h(t) \in E,\ h \in H\}$. Then $\beta(H(I_\alpha)) = \sup_{t \in I_\alpha} \beta(H(t))$ and the function $t \mapsto \beta(H(t))$ is continuous.*

In the proof of the main result we will apply the following fixed point theorem.

Theorem 1.2 [22]. *Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{conv}\left(\{x\} \cup F(V)\right) \quad \Rightarrow \quad V \quad \text{is relatively weakly compact,} \tag{2}$$

holds for every subset V of D, where $\overline{conv}\left(\{x\} \cup F(V)\right)$denotes the closure of the convex of $(\{x\} \cup F(V))$, then F has a fixed point.

Let us introduce the following definitions:

Definition 1.3 [30]. Let $G : [a, b] \to E$ and let $A \subset [a, b]$. The function $g : A \to E$ is a *pseudoderivative of* G on A if for each x^* in E^* the real-valued function x^*G is differentiable almost everywhere on A and $(x^*G)' = x^*g$ almost everywhere on A.

Definition 1.4 [15, 25]. A family $\mathcal{F}$ of functions F is said to be *uniformly absolutely continuous* in the restricted sense on X or, in short, uniformly $AC_*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every F in $\mathcal{F}$ and for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in X$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$, where $\omega(F, [a_i, b_i])$ denotes the oscillation of F over $[a_i, b_i]$ (i.e. $\omega(F, [a_i, b_i]) = \sup\{|F(r) - F(s)| : r, s \in [a_i, b_i]\}$).

A family F of functions F is said to be *uniformly generalized absolutely continuous* in the restricted sense on $[a, b]$ or uniformly ACG_* on $[a, b]$ if $[a, b]$ is the union of a sequence of closed sets A_i such that on each A_i, the family F is uniformly $AC_*(A_i)$.

2. HENSTOCK-KURZWEIL-PETTIS INTEGRAL IN BANACH SPACES

In this part we present the Henstock-Kurzweil-Pettis integral and we give properties of this integral.

Definition 2.1 [15, 25]. Let δ be a positive function defined on the interval $[a, b]$. A tagged interval $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [c, d]$.

The tagged interval $(x, [c, d])$ is subordinate to δ if $[c, d] \subseteq (x - \delta(x), x + \delta(x))$.

Let $P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n,\ n \in N\}$ be such a collection in $[a, b]$. Then

(i) The points $\{s_i : 1 \leq i \leq n\}$ are called the tags of P.

(ii) The intervals $\{[c_i, d_i] : 1 \leq i \leq n\}$ are called the intervals of P.

(iii) If $\{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is subordinate to δ for each i, then we write P is sub δ.

(iv) If $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then P is called a tagged partition of $[a, b]$.

(v) If P is a tagged partition of $[a, b]$ and if P is sub δ, then we write P is sub δ on $[a, b]$.

(vi) If $f : [a,\ b] \to E$, then $f(P) = \sum_{i=1}^{n} f(s_i)(d_i - c_i)$.

(vii) If F is defined on the subintervals of $[a, b]$, then $F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} [F(d_i) - F(c_i)]$.

If $F : [a, b] \to E$, then F can be treated as a function of intervals by defining $F([c, d]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a, b]$.

Definition 2.2 [15, 25]. A function $f : [a, b] \to R$ is *Henstock-Kurzweil integrable on* $[a, b]$ if there exists a real number L with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|f(P) - L| < \varepsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to δ.

The function f is *Henstock-Kurzweil integrable on a measurable set* $A \subset [a, b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The number L in Definition 2.2 is called *the Henstock-Kurzweil integral of* f and we will denote it by $(HK) \int_a^b f(t)dt$.

Definition 2.3 [7]. A function $f : [a, b] \to E$ is *Henstock-Kurzweil integrable* on $[a, b]$ ($f \in HK([a, b], E)$) if there exists a vector $z \in E$ with the following property: for every $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is a tagged partition of $[a, b]$ sub δ. The function f is Henstock-Kurzweil integrable on a measurable set $A \subset [a, b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The vector z is *the Henstock-Kurzweil integral of* f.

We remark that this definition includes the generalized Riemann integral defined by Gordon [16]. In a special case, when δ is a constant function, we get the Riemann integral.

Definition 2.4 [7]. A function $f : [a, b] \to E$ is *HL integrable* on $[a, b]$ ($f \in HL([a, b], E)$) if there exists a function $F : [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that if $P = \{(s_i, [c_i, d_i] : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ sub δ, then

$$\sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon.$$

Remark 1. We note that by triangle inequality:

$$f \in HL([a,b],E) \quad \text{implies } f \in HK([a,b],E).$$

In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

Definition 2.5 [30]. The function $f : I_\alpha \to E$ is *Pettis integrable* (P integrable for short) if

(i) $\forall_{x^* \in E^*}$ $x^* f$ is Lebesgue integrable on I_α,
(ii) $\forall_{A \subset I_\alpha}$, A-measurable $\exists_{g \in E}$ $\forall_{x^* \in E^*}$ $x^* g = (L) \int\limits_A x^* f(s) ds$,

where $(L) \int\limits_A$ denotes the Lebesgue integral over A.

Now we present a definition of an integral which is a generalization for both: Pettis and Henstock-Kurzweil integrals.

Definition 2.6 [11]. The function $f : I_\alpha \to E$ is *Henstock-Kurzweil-Pettis integrable* (HKP integrable for short) if there exists a function $g : I_\alpha \to E$ with the following properties:

(i) $\forall_{x^* \in E^*}$ $x^* f$ is Henstock-Kurzweil integrable on I_α and
(ii) $\forall_{t \in I_\alpha} \forall_{x^* \in E^*}$ $x^* g(t) = (HK) \int\limits_0^t x^* f(s) ds.$

This function g will be called a *primitive of* f and by $g(\alpha) = \int\limits_0^\alpha f(t) dt$ we will denote the *Henstock-Kurzweil-Pettis integral* of f on the interval I_α.

Remark 2. Each function which is HL integrable is integrable in the sense of Henstock-Kurzweil-Pettis. Our notion of integral is essentially more general than the previous ones (in Banach spaces):

(i) Pettis integral: by the definition of the Pettis integral and since each Lebesgue integrable function is HK integrable, a P integrable function is clearly HKP integrable.
(ii) Bochner, Riemann, and Riemann-Pettis integrals [16].
(iii) MsShane integral [14] or [17].
(iv) Henstock-Kurzweil (HL) integral ([7]).

We present below an example of a function which is HKP integrable but neither HL integrable nor P integrable.

Example. Let $f : [0,1] \to (L^\infty[0,1], \|\cdot\|_\infty)$ be defined as $f(t) = \chi_{[0,t]} + A(t) \cdot F'(t)$**,** where

$$F(t) = t^2 \sin t^{-2}, \quad t \in (0,1],\ F(0) = 0, \quad \chi_{[0,t]}(\tau) = \begin{cases} 1, & \tau \in [0,t], \\ 0, & \tau \notin [0,t], \end{cases} \quad t, \tau \in [0,1],$$

$A(t)(\tau) = 1$ for $\tau, t \in [0,1]$.

Put $f_1(t) = \chi_{[0,t]}$, $\quad f_2(t) = A(t) \cdot F'(t)$.

We will show that the function $f(t) = f_1(t) + f_2(t)$ is integrable in the sense of Henstock-Kurzweil-Pettis.

Observe that

$$x^*(f(t)) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).$$

Moreover, the function $x^*(f_1(t))$ is Lebesgue integrable (in fact f_1 is Pettis integrable [13]), so it is Henstock-Kurzweil integrable, and the function $x^*(f_2(t))$ is Henstock-Kurzweil integrable by Definition 2.2.

For each $x^* \in E^*$ the function $x^* f$ is not Lebesgue integrable because $x^* f_2$ is not Lebesgue integrable. So f is not Pettis integrable. Moreover, the function f_1 is not strongly measurable ([13]) and the function f_2 is strongly measurable. So their sum f is not strongly measurable. Then by Theorem 9 from [7] f is not HL integrable.

In this sequel we present some properties of the HKP integral which are important in the next part of our paper.

Theorem 2.7 [11]. *Let $f : [a, b] \to E$ be HKP integrable on $[a, b]$ and let $F(x) = \int_a^x f(s)ds$, $x \in [a, b]$. Then*

(i) *for each x^* in E^* the function $x^* f$ is HK integrable on $[a, b]$ and $(HK) \int_a^x x^*(f(s))ds = x^*(F(x))$*

(ii) *the function F is weakly continuous on $[a, b]$ and f is a pseudoderivative of F on $[a, b]$.*

Theorem 2.8 [11]. *Let $f : [a, b] \to E$. If $f = 0$almost everywhere on $[a, b]$, then f is HKP integrable on $[a, b]$ and $\int_a^b f(t)dt = 0$.*

Theorem 2.9 [11] (Mean value theorem for the HKP integral). *If the function $f : I_\alpha \to E$ is HKP integrable, then*

$$\int_I f(t)dt \in |I| \cdot \overline{conv} f(I),$$

where $\overline{conv} f(I)$ is the closure of the convex of $f(I)$, I is an arbitrary subinterval of I_α and $|I|$ is the length of I.

Theorem 2.10 [9]. *Let $f : I_\alpha \to E$and assume that $f_n : I_\alpha \to E$, $n \in N$, are HKP integrable on I_α. For each $n \in N$, let F_n be a primitive of f_n. If we assume that:*

(i) *$\forall x^* \in E^*$ $x^*(f_n(t)) \to x^*(f(t))$ a.e. on I_α,*

(ii) *for each $x^* \in E^*$, the family $G = \{x^* F_n : n = 1, 2, \dots)\}$ is uniformly ACG_* on I_α (i.e. weakly uniformly ACG_* on I_α),*

(iii) *for each $x^* \in E^*$, the set G is equicontinuous on I_α,* *then f is HKP integrable on I_α and* $\int\limits_0^t f_n(s)ds$ *tends weakly in E to* $\int\limits_0^t f(s)ds$ *for each $t \in I_\alpha$.*

3. EXISTENCE OF A SOLUTION

Now we prove the existence theorem for problem (1) under the weakest assumptions on g, as it is known.

For $x \in C(I_\alpha, E)$, we define the norm of x by: $\|x\|_C = \sup\{\|x(t)\|,\ t \in I_\alpha\}$.

Put $B = \{x \in C(I_\alpha, E) : x(0) = f(0),\ \|x\| \leq \|f(\cdot)\| + M, \quad M > 0\}$.

We define the operator $F : C(I_\alpha, E) \to C(I_\alpha, E)$ by

$$F(x)(t) = f(t) + \int\limits_0^\alpha k_1(t,s)x(s)ds + \int\limits_0^\alpha k_2(t,s)g(s,x(s))ds,\ t \in I_\alpha,\ \alpha \in R_+,\ x \in B,$$

where integrals are taken in the sense of Henstock-Kurzweil-Pettis.

Moreover, let $\Gamma = \{F(x) \in C(I_\alpha, E) : x \in B\}$ and let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int\limits_0^\alpha [k_1(t,s) + k_2(t,s)]u(s)ds, \quad t \in I_\alpha, \quad u \in B.$$

Now we present the existence theorem for problem (1).

A continuous function $x : I_\alpha \to E$ is said to be *a solution of problem* (1) if it satisfies the equation (1) for every $t \in I_\alpha$.

Theorem 3.1 *Assume that for each continuous function $x : I_\alpha \to E, g(\cdot, x(\cdot))$ is HKP integrable, $g(s,\cdot)$ is weakly-weakly sequentially continuous and $k_1, k_2 : I_\alpha \times I_\alpha \to R_+$ are measurable functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous. Moreover, let $L > 0$ and*

(3) $$\mu(g(I,X)) \leq L\mu(X) \quad \text{for each bounded subset } X \subset E, I \subset I_\alpha.$$

Suppose that Γ is equicontinuous and uniformly ACG_* on I_α. Moreover, let $(1 + L)r(K) < 1$. Then there exists at least one solution of problem (1) *on I_β, for some $0 < \beta \leq \alpha$, with continuous initial function f.*

Proof. By equicontinuity of Γ there exists some number β $(0 < \beta \leq \alpha)$ such that $\left\| \int\limits_0^\beta [k_1(t,s)x(s) + k_2(t,s)g(s,x(s)]ds \right\| \leq M$ for fixed $M > 0$, $t \in I_\beta$ and $x \in B$.

By our assumptions, the operator F is well defined and maps B into B. We will show that the operator F is weakly sequentially continuous.

By Lemma 9 of [27], a sequence $x_n(\cdot)$ is weakly convergent in $C(I_\beta, E)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_\beta$. Because $g(s,\cdot)$ is weakly-weakly sequentially continuous, so if $x_n \xrightarrow{\omega} x$ in $(C(I_\beta, E), \omega)$ then $g(s, x_n(s)) \xrightarrow{\omega} g(s, x(s))$ in E for $t \in I_\beta$ and by Theorem 2.10 we have

$$\lim_{n\to\infty} \int_0^\beta [k_1(t,s)x_n(s) + k_2(t,s)g(s,x_n(s))]ds = \int_0^\beta [k_1(t,s)x(s) + k_2(t,s)g(s,x(s))]ds$$

weakly in E, for each $t \in I_\beta$. We see that $F(x_n)(t) \to F(x)(t)$ weakly in E for each $t \in I_\beta$ so $F(x_n) \to F(x)$ in $(C(I_\beta, E), \omega)$.

Suppose that $V \subset B$ satisfies the condition $\bar{V} = \overline{conv}(\{x\} \cup F(V))$, for some $x \in B$. We will prove that V is relatively weakly compact, thus (2) is satisfied.

Let, for $t \in I_\beta$, $V(t) = \{v(t) \in E,\ v \in V\}$.

From the definition of B and Lemma 1.1, it follows that the function $v : t \mapsto \mu(V(t))$ is continuous on I_β.

We divide the interval I_β: $0 = t_0 < t_1 < \cdots < t_m = \beta$, where $t_i = \frac{i\beta}{m}$, $i = 0, 1, \ldots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) \in E : u \in V,\ t_i \le s \le t_{i+1}\}$, $i = 0, 1, \ldots, m-1$. By Lemma 1.1 and the continuity of v there exists $s_i \in T_i = [t_i, t_{i+1}]$ such that

$$\mu(V([t_i, t_{i+1}])) = \sup\{\mu(V(s)) : t_i \le s \le t_{i+1}\} =: v(s_i). \tag{4}$$

On the other hand, by the definition of the operator F and Theorem 2.11 we have

$$F(u)(t) = f(t) + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} [k_1(t,s)u(s) + k_2(t,s)g(s,u(s))]ds$$

$$\in f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}[k_1(t,T_i)V([t_i,t_{i+1}]) + k_2(t,T_i)g(T_i, V([t_i,t_{i+1}]))]$$

for each $u \in V$.

Therefore

$$F(V(t)) \subset f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}[k_1(t,T_i)V([t_i,t_{i+1}]) + k_2(t,T_i)g(T_i, V([t_i,t_{i+1}]))].$$

Using (3), (4) and the properties of the measure of weak noncompactness μ we obtain

$$\begin{aligned}\mu(F(V(t))) &\le \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_1(t,T_i)\mu(V([t_i,t_{i+1}])) \\ &\quad + \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_2(t,T_i)\mu(g(T_i, V([t_i,t_{i+1}]))) \\ &\le \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_1(t,T_i)v(s_i) + k_2(t,T_i)Lv(s_i)] \\ &= \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_1(t,T_i)v(s_i) + L\sum_{i=0}^{m-1} (t_{i+1} - t_i)k_2(t,T_i)v(s_i)\end{aligned}$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\sup_{s\in T_i}k_1(t,s)v(s_i)+L\sum_{i=0}^{m-1}(t_{i+1}-t_i)\sup_{s\in T_i}k_2(t,s)v(s_i)$$
$$=\sum_{i=0}^{m-1}(t_{i+1}-t_i)k_1(t,p_i)v(s_i)+L\sum_{i=0}^{m-1}(t_{i+1}-t_i)k_2(t,q_i)v(s_i),$$

where $s_i, p_i, q_i \in T_i$, hence

$$\mu(F(V(t)))\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)k_1(t,p_i)v(p_i)+\sum_{i=0}^{m-1}(t_{i+1}-t_i)[k_1(t,p_i)(v(s_i)-v(p_i))]$$
$$+L\sum_{i=0}^{m-1}(t_{i+1}-t_i)k_2(t,q_i)v(q_i)$$
$$+L\sum_{i=0}^{m-1}(t_{i+1}-t_i)[k_2(t,q_i)(v(s_i)-v(q_i))]$$
$$=\sum_{i=0}^{m-1}(t_{i+1}-t_i)k_1(t,p_i)v(p_i)+\frac{\beta}{m}\sum_{i=0}^{m-1}[k_1(t,p_i)(v(s_i)-v(p_i))]$$
$$+L\sum_{i=0}^{m-1}(t_{i+1}-t_i)k_2(t,q_i)v(q_i)+\frac{L\beta}{m}\sum_{i=0}^{m-1}[k_2(t,q_i)(v(s_i)-v(q_i))].$$

By the continuity of v we have $v(s_i)-v(p_i)<\varepsilon_1$ and $\varepsilon_1\to 0$ as $m\to\infty$ and $v(s_i)-v(q_i)<\varepsilon_2$ and $\varepsilon_2\to 0$ as $m\to\infty$.

So

$$\mu(F(V(t)))<\int_0^\beta k_1(t,s)v(s)ds+\beta\sup_{p\in I_\beta}k_1(t,p)\varepsilon_1$$
$$+L\int_0^\beta k_2(t,s)v(s)ds+L\beta\sup_{q\in I_\beta}k_2(t,q)\varepsilon_2.$$

Therefore

$$\mu(F(V(t)))\leq (1+L)\int_0^\beta [k_1(t,s)+k_2(t,s)]v(s)ds,\quad \text{for } t\in I_\beta. \tag{5}$$

Since $V=\overline{conv}(\{u\}\cup F(V))$, by the property of the measure of weak noncompactness we have $\mu(V(t))\leq\mu(F(V(t)))$ and so in view of (5), it follows that $v(t)\leq(1+L)\int_0^\beta[k_1(t,s)+k_2(t,s)]v(s)ds$, for $t\in I_\beta$. Because this inequality holds for every t$\in I_\beta$ and $(1+L)r(K)<1$, so by applying Gronwall's inequality [18], we conclude that $\mu(V(t))=0$, for t$\in I_\beta$. Hence Arzela-Ascoli‘s theorem implies that the set V is relatively compact. Consequently, by Theorem 1.2, F has a fixed point which is a solution of the problem (1).

Remark 3. The condition (3) in our Theorem 3.1 can be also generalized to the Sadovskii conditions: $\mu(F(I \times X)) < \mu(X)$, whenever $\mu(X) > 0$, where μ can be replaced by some axiomatic measure of weak noncompactness.

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