Dynamic Systems and Applications 17 (2008) 109-120

BLOW-UP FOR THE EULER-BERNOULLI BEAM PROBLEM WITH A FRACTIONAL BOUNDARY DISSIPATION

SORAYA LABIDI AND NASSER-EDDINE TATAR

Laboratoire de Mathématiques Appliquées, Université de Versailles, 45, Avenue des Etats-Unis 78035, Versailles Cedex France (labidis1@yahoo.fr)
King Fahd University of Petroleum and Minerals, Mathematical Sciences Department, Dhahran 31261, Saudi Arabia (tatarn@kfupm.edu.sa)

ABSTRACT. We consider a beam problem with a polynomial source and a boundary damping of order between 0 and 1. Sufficient conditions on the initial data are established to have blow up of solutions in finite time.

Key words and phrases: Blow up, fractional derivative, boundary feedback, Euler-Bernoulli beam

AMS (MOS) Subject Classification: 45K05, 35B40, 35B45, 35B37, 93B52, 26A33

1. INTRODUCTION

In this paper we investigate the collapse in finite time of solutions to the following initial boundary value problem known as the Euler-Bernoulli beam problem

$$(1)\qquad \begin{cases} u_{tt}+\Delta^2 u = a\left|u\right|^{p-1}u,\ x\in\Omega,\ t>0 \\ u=\frac{\partial u}{\partial \upsilon}=0,\ x\in\Gamma_1,\ t>0 \\ \Delta u=0,\ x\in\Gamma_0,\ t>0 \\ \frac{\partial \Delta u}{\partial \upsilon}=\frac{b}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u_s(s)ds,\ x\in\Gamma_0,\ t>0 \\ u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right),\ x\in\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ such that $\Gamma=\Gamma_0\cup\Gamma_1$ and $\Gamma_0\cap\Gamma_1=$. The constants a and b are positive. The exponent p is greater than 1. The initial data $u_0\left(x\right)$ and $u_1\left(x\right)$ are given functions, $\partial/\partial\upsilon$ denotes the outward normal derivative and $\Gamma(\beta)$ is the usual Euler gamma function. The power β in the integral term is such that $0<\beta<1$.

This problem describes the damping of transversal vibrations of a beam. Controls are forces or torques applied on a portion of the boundary of the beam. This feedback is intended to reduce the effect of reflected waves.

Received July 09, 2006

Note that the integral term in (1) is the fractional derivative (in the sense of Caputo) of order $1-\beta$ (see [16]) defined by

$$\partial_t^{1-\beta} w\left(t\right) = I^{\beta} \frac{d}{dt} w\left(t\right), \quad 0<\beta<1$$

where I^{γ}, $\gamma>0$ denotes the Riemann-Liouville fractional integral

$$I^{\gamma} w\left(t\right) = \frac{1}{\Gamma(\gamma)} \int_0^t \left(t-s\right)^{\gamma-1} w\left(s\right) ds.$$

The reader is referred to the books [15–17] for more on derivatives and integrals of fractional order.

In [8], the present authors have proved an exponential growth result for a similar problem to (1) where the polynomial source is at the boundary and competes with the fractional damping in the same portion of the boundary. Namely, the following problem has been investigated

$$\begin{cases} u_{tt} - h\Delta u_{tt} + \Delta^2 u = 0, \ x\in\Omega, \ t>0 \\ u = \dfrac{\partial u}{\partial \upsilon} = 0, \ x\in\Gamma_1, \ t>0 \\ \Delta u = u_t, \ x\in\Gamma_0, \ t>0 \\ \dfrac{\partial \Delta u}{\partial \upsilon} - h\dfrac{\partial u_{tt}}{\partial \upsilon} + a\left|u\right|^{p-1} u = \dfrac{b}{\Gamma(\beta)} \int\limits_0^t \left(t-s\right)^{\beta-1} u_t ds + \dfrac{\partial u_t}{\partial \upsilon}, \ x\in\Gamma_0, \ t>0 \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right) \ x\in\Omega. \end{cases}$$

It is clear that here the situation is somehow favorable. Indeed, as both the source and the damping are acting on the same location, one does not have the difficulties of passing from one part of the boundary or the domain to another. In the present work we have to find a way to go from part of the boundary to the whole domain or vice-versa. In addition, in [8] an exponential growth result is proved in an infinite time interval. That is when time goes to infinity, whereas in the present work we prove a stronger result. We prove that solutions blow up in finite time.

For the wave equation without the source term (that is $a=0$) a similar problem has been studied by Mbodje and Montseny [12] and recently by Mbodje [11]. In both papers the problem is first converted into a coupled wave-diffusion system called an augmented system which may be put in the operator theoretical form $\dot{X}(t) = AX(t)$. Then, they apply the well-known techniques in the semigroup theory to prove existence and uniqueness and LaSalle's invariance principle to derive an asymptotic decay of solutions. In [11], the same problem but with an exponentially modified kernel $t^{\beta-1}e^{-\delta t}/\Gamma(\beta)$, $0<\beta<1$, $\delta>0$ (see [17]) has been reconsidered. In addition to the well-posedness, the asymptotic convergence towards zero and a decay rate of convergence, the author proves that solutions cannot decay uniformly exponentially to zero. An interesting problem is also studied in [14].

A closely related problem to ours is when the damping acts in the whole domain rather than on the boundary or part of the boundary. This problem has been investigated by Matignon *et al.* [10] using the same method mentioned above. An asymptotic decay result has been proven but without a precise decay rate. The problem with a strong fractional derivative (and $a = 0$) has been studied by the second author (Tatar) in [19]. Solutions are proved to be asymptotically convergent to the stationary solution zero in an exponential manner. The case $a \neq 0$ is considered in [13]. The same problem as in [10] (with an internal fractional damping and a polynomial source) has been treated in a series of papers [6, 18–20, 1]. In these papers the authors established several results on exponential growth and blow up in finite time. Finally, we mention here that the case $\Omega = \mathbf{R}^n$ and with $h(t,x)\,|u|^p$ as nonlinear source has been discussed in [7].

In this work we prove that solutions of the problem (1) with fractional boundary feedback may blow up in finite time. In fact, it is shown that for any fixed $T > 0$ we can find initial data whose associated solution blows up in a finite time $T_* < T$. The standard existing methods and techniques do not apply to this problem because of the singularity present in the definition of the fractional derivative.

The remaining part of the paper is organized as follows: In the next section we prepare some material which will be needed in the sequel. Section 3 is devoted to the statement and proof of our result on blow up in finite time.

2. PRELIMINARIES

In this section we prepare some definitions and some lemmas which will be needed in the proof of our result. We assume, without loss of generality, that $a = 1$ (in fact we could do the same with b). Let us define the classical energy associated to problem (1) by

$$E(t) := \int_\Omega \left\{ \frac{1}{2}u_t^2 + \frac{1}{2}\,|\Delta u|^2 - \frac{1}{p+1}\,|u|^{p+1} \right\} dx.$$

It is easily seen that

$$\frac{dE(t)}{dt} = -\frac{b}{\Gamma(\beta)} \int_{\Gamma_0} u_t \int_0^t (t-s)^{\beta-1}\, u_t ds\, d\sigma.$$

This shows that there is no guarantee that the system is dissipative with respect to the classical energy. Replacing t by s and s by z then integrating from 0 to t, we get

$$E(t) - E(0) = -\frac{b}{\Gamma(\beta)} \int_0^t \int_{\Gamma_0} u_t \int_0^s (s-z)^{\beta-1}\, u_z(z) dz\, d\sigma\, ds.$$

Hence, $E(t) \leq E(0)$ for all $t \geq 0$, because $t^{\beta-1}$, $0 < \beta < 1$, is a positive definite function. We will also need the following inequality.

Lemma 2.1 (Young inequality, see [2])**.** *Let $f \in L^p(\mathbf{R})$ and $g \in L^q(\mathbf{R})$ with $1 \leq p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbf{R})$ and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Lemma 2.2 (See [9])**.** *Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary, then*

$$\|u\|_{H_0^{(1-\theta)s_1+\theta s_2}(\Omega)} \leq c(\theta, s_1, s_2) \|u\|_{H_0^{s_1}}^{1-\theta} \|u\|_{H_0^{s_2}}^{\theta}$$

for all $s_i > 0$, $0 < \theta < 1$. In particular (when the Poincaré inequality is valid), we have

$$\|u\|_{H_0^{\theta}(\Omega)} \leq c(\theta) \|u\|_2^{1-\theta} \|\nabla u\|_2^{\theta}$$

for $0 < \theta < 1$.

Lemma 2.3 (See [9])**.** *Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary Γ_0, then*

$$\|u\|_{l,\Gamma_0} \leq C(l, s, \Omega) \|u\|_{H^s(\Omega)}$$

for all $l \geq 2$ and $s = \frac{n}{2} - \frac{n-1}{l} > 0$.

Remark 2.4. Notice that by Hölder inequality and Lemma 2, we have $\|u\|_{m,\Gamma_0} \leq C(l, s, \Omega) \|u\|_{H^s(\Omega)}$ for $0 < s < 1$ and $s \geq \frac{n}{2} - \frac{n-1}{m}$. Moreover, a combination of the previous two lemmas yield $\|u\|_{m,\Gamma_0} \leq C \|u\|_2^{1-\theta} \|\nabla u\|_2^{\theta}$, for $0 < \theta < 1$.

3. BLOW UP IN FINITE TIME

This section contains the statement and proof of our result. To deal with the nonlinear source we need the embedding $H_0^1 \hookrightarrow L^p$ and therefore we assume that $2 \leq p < 2n/(n-2)$ if $n \geq 3$ and $p \geq 2$ if $n = 1, 2$. Concerning the existence and uniqueness of a local solution, $(u, u_t) \in H^2(\Omega) \times L^2(\Omega)$ and $u \in C^0((0, T_m), H_0^2(\Omega)) \cap C^1((0, T_m), L^2(\Omega))$ (where $H_0^2(\Omega) := \{v \in H^2(\Omega) : v = \partial v / \partial \upsilon = 0 \text{ on } \Gamma_1\}$), we refer the reader to the papers [11,12] and also to [3].

Theorem 3.1. *Let u be a solution of* (1) *with $0 < \beta < 1$ and $p > 3$. Then, for every fixed $T > 0$ there exist sufficiently large initial data u_0, u_1 and $0 < T^* < T$ for which u blows up at T^*.*

Proof. Let us define

$$H(t) = \int_0^t \int_\Omega \left\{ \frac{1}{p+1} |u|^{p+1} - \frac{1}{2} u_t^2 - \frac{1}{2} |\Delta u|^2 \right\} dx\, ds + (kt + l) \int_\Omega u_0^2 dx,$$

where k and l are positive constants to be determined later. Clearly,

$$\begin{aligned} H'(t) &= \int_\Omega \left\{ \tfrac{1}{p+1} |u|^{p+1} - \tfrac{1}{2} u_t^2 - \tfrac{1}{2} |\Delta u|^2 \right\} dx + k \int_\Omega u_0^2 dx \\ &= k \int_\Omega u_0^2 dx - E(t) \geq k \int_\Omega u_0^2 dx - E(0). \end{aligned}$$

We can already choose $k > E(0)/\int_\Omega u_0^2 dx$ so that

(2) $$k\int_\Omega u_0^2 dx - E(0) = H^{'}(0) > 0.$$

In this way $H^{'}(t) > 0$ and

(3) $$H^{'}(0) - H^{'}(t) = E(t) - E(0) = -\frac{b}{\Gamma(\beta)}\int_0^t\int_{\Gamma_0} u_s \int_0^s (s-z)^{\beta-1} u_z(z)dz\, d\sigma\, ds \leq 0.$$

The main idea of the proof is to come up with a functional $\Phi(t)$ which satisfies an inequality of the form $\Phi^{'}(t) \geq C\Phi^{\omega}(t)$ with $\omega > 1$. This implies blow up in finite time of the solution. This argument has been used in [3] for the wave equation with an internal dissipation of the form $|u_t|^{m-1} u_t$. Here we shall use a different functional which allows us to include a larger class of initial data. In doing so, several difficulties arouse when dealing with the nonlocal term given by the fractional damping and which contains a singular kernel. The functional we suggest is the following

(4) $$\Phi(t) = H^{1-\gamma}(t) + \frac{\varepsilon}{2}\left[\int_\Omega u^2 dx - \int_\Omega u_0^2 dx\right]$$

where $\varepsilon > 0$ and $0 < \gamma = \frac{p-1}{2(p+1)} < 1$. At $t = 0$, we have

(5) $$\Phi(0) = H^{1-\gamma}(0) = \left(l\int_\Omega u_0^2 dx\right)^{1-\gamma}$$

and the derivative of $\Phi(t)$ is

(6) $$\Phi^{'}(t) = (1-\gamma) H^{-\gamma}(t) H^{'}(t) + \varepsilon\int_\Omega u u_t dx.$$

We perform a differentiation and then an integration on (6) with respect to t to arrive at

(7) $$\Phi^{'}(t) = (1-\gamma) H^{-\gamma}(t) H^{'}(t) + \varepsilon\int_\Omega u_0 u_1\, dx + \varepsilon\int_0^t\int_\Omega u_s^2 dx\, ds + \varepsilon\int_0^t\int_\Omega u u_{ss} dx\, ds.$$

The objective behind this is to be able to use the properties of positive definite functions. We can have an idea on the last term in (7) by a multiplication of the equation in (1) by u. It appears that

(8) $$\int_0^t\int_\Omega u u_{tt} dx\, ds = -\int_0^t\int_\Omega |\Delta u|^2 dx\, ds - \frac{b}{\Gamma(\beta)}\int_0^t\int_{\Gamma_0} u\int_0^s (s-z)^{\beta-1} u_z(z)dz\, d\sigma\, ds + \int_0^t\int_\Omega |u|^{p+1} dx\, ds.$$

Let us denote $\frac{t^{\beta-1}}{\Gamma(\beta)}$ by $k_\beta(t)$, and for fixed t, we define the extensions by 0 as follows

$$Lw(\tau) = \begin{cases} w(\tau),\ if\ \ \tau \in [0,t] \\ 0,\ if\ \ \tau \in \mathbf{R}\backslash [0,t] \end{cases}$$

(this extension depends of course on t but this dependence is not mentioned for notational convenience), and

$$Lk_{\beta}(\tau) = \begin{cases} k_{\beta}(\tau), \; if \;\; \tau > 0 \\ 0, \; if \;\; \tau \leq 0. \end{cases}$$

It is easily seen that

$$\begin{array}{c} \frac{1}{\Gamma(\beta)} \int_0^t u(s) \int_0^s (s-z)^{\beta-1} u_z(z)\, dz\, ds \\ = \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\beta}(s-z)(Lu_z)(z)\, dz\, ds. \end{array}$$

Moreover, by Parseval theorem (see [2,5]), we get

$$\begin{array}{c} \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\beta}(s-z)(Lu_z)(z)\, dz\, ds \\ = \int_{-\infty}^{+\infty} F(Lu)(\sigma) \overline{F(Lk_{\beta} * Lu_z)}(\sigma) d\sigma, \end{array}$$

where $F(f)$ denotes the usual Fourier transform of f. Since the Fourier transform of the convolution gives the product of the Fourier transforms and the fact that (see [17])

$$k_{\gamma+\eta}(t) = (k_{\gamma} * k_{\eta})(t)$$

for all $0 < \gamma, \eta < 1$, we can split the right hand side of the previous identity into two terms and apply the Cauchy-Schwarz inequality and the Young inequality to obtain

$$\begin{array}{c} \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\beta}(s-z)\, Lu_t dz\, ds \\ \leq \left(\int_{-\infty}^{+\infty} \left| F(Lk_{\beta/2}) F(Lu) \right|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left| F(Lk_{\beta/2}) F(Lu_t) \right|^2 d\sigma \right)^{\frac{1}{2}} \\ \leq \frac{1}{4\delta} \int_{-\infty}^{+\infty} \left| F(Lk_{\beta/2}) F(Lu) \right|^2 d\sigma + \delta \int_{-\infty}^{+\infty} \left| F(Lk_{\beta/2}) F(Lu_t) \right|^2 d\sigma, \end{array}$$

for some $\delta > 0$. Further, by Theorem 16.5.1 in [4], we find

(9)
$$\begin{array}{c} \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\beta}(s-z)\, Lu_t dz\, ds \leq \frac{1}{\cos(\beta\pi/2)} \left[\delta \int_{-\infty}^{+\infty} Lu_t(s)(Lk_{\beta} * Lu_t)(s) ds \right. \\ \left. + \frac{1}{4\delta} \int_{-\infty}^{+\infty} Lu(s)(Lk_{\beta} * Lu)(s) ds \right]. \end{array}$$

Gathering (7), (8) and (9), we get

$$\begin{array}{c} \Phi'(t) \geq (1-\gamma) H^{-\gamma}(t) H'(t) + \varepsilon \int_{\Omega} u_0 u_1 dx + \varepsilon \int_0^t \int_{\Omega} u_s^2 dx\, ds \\ -\varepsilon \int_0^t \int_{\Omega} |\Delta u|^2 dx\, ds + \varepsilon \int_0^t \int_{\Omega} |u|^{p+1} dx\, ds - \frac{b\varepsilon}{\cos(\beta\pi/2)} \\ \times \left[\delta \int_{\Gamma_0} \int_{-\infty}^{+\infty} Lu_s(s)(Lk_{\beta} * Lu_z)(s) ds\, d\sigma + \frac{1}{4\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} Lu(s)(Lk_{\beta} * Lu)(s) ds\, d\sigma \right]. \end{array}$$

Thanks to (3), we can write that

(10)
$$\begin{array}{c} \Phi'(t) \geq \left[(1-\gamma) H^{-\gamma}(t) - \frac{\varepsilon\delta}{\cos\left(\frac{\beta\pi}{2}\right)} \right] H'(t) + \frac{\varepsilon\delta}{\cos(\beta\pi/2)} H'(0) + \varepsilon \int_{\Omega} u_0 u_1 dx \\ +\varepsilon \int_0^t \int_{\Omega} u_t^2 dx\, ds - \varepsilon \int_0^t \int_{\Omega} |\Delta u|^2 dx\, ds + \varepsilon \int_0^t \int_{\Omega} |u|^{p+1} dx\, ds \\ -\frac{b\varepsilon}{4\delta \cos(\beta\pi/2)} \int_{\Gamma_0} \int_{-\infty}^{+\infty} Lu(s)(Lk_{\beta} * Lu)(s) ds\, d\sigma. \end{array}$$

For the last term in the right hand side of (10), the Cauchy-Schwarz inequality yields

$$\begin{aligned} I &= \int_{\Gamma_0}\int_{-\infty}^{+\infty} Lu(s)(Lk_\beta * Lu)(s)ds\,d\sigma \\ &\leq \int_{\Gamma_0}\left(\int_{-\infty}^{+\infty}|Lu|^2\,ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(Lk_\beta * Lu)^2(s)ds\right)^{\frac{1}{2}} d\sigma. \end{aligned}$$

Furthermore, appealing to Lemma 1 (Young inequality), we see that

$$\begin{aligned} I = \int_{\Gamma_0}\int_{-\infty}^{+\infty} Lu(s)(Lk_\beta * Lu)(s)ds\,d\sigma &\leq \left(\int_0^t (t-s)^{\beta-1}ds\right)\left(\int_{\Gamma_0}\int_{-\infty}^{+\infty}|Lu|^2\,ds\,d\sigma\right) \\ &\leq \frac{t^\beta}{\beta}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|Lu|^2\,ds\,d\sigma \end{aligned}$$

Back to (10), we find

$$\begin{aligned} \Phi'(t) \geq &\left[(1-\gamma)H^{-\gamma}(t) - \frac{\varepsilon\delta}{\cos(\beta\pi/2)}\right]H'(t) + \frac{\varepsilon\delta}{\cos(\beta\pi/2)}H'(0) \\ &+\varepsilon\int_\Omega u_0u_1dx + \varepsilon\int_0^t\int_\Omega u_t^2dx\,ds - \varepsilon\int_0^t\int_\Omega|\Delta u|^2\,dx\,ds + \varepsilon\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds \\ &-\frac{b\varepsilon t^\beta}{4\delta\beta\cos(\beta\pi/2)}\int_{\Gamma_0}\int_0^t|u|^2\,ds\,d\sigma. \end{aligned}$$

As $H(t)\neq 0$ for all $t\geq 0$, then taking $\delta = M\cos(\beta\pi/2)\,H^{-\gamma}(t)$, we get

$$\begin{aligned} \Phi'(t) \geq &[(1-\gamma)-M\varepsilon]\,H^{-\gamma}(t)\,H'(t) + \varepsilon M H^{-\gamma}(t)\,H'(0) + \varepsilon\int_\Omega u_0u_1dx \\ &+\varepsilon\int_0^t\int_\Omega u_t^2dx\,ds - \varepsilon\int_0^t\int_\Omega|\Delta u|^2\,dx\,ds + \varepsilon\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds \\ &-\frac{b\varepsilon t^\beta}{4M\beta\cos^2(\beta\pi/2)}H^\gamma(t)\int_{\Gamma_0}\int_0^t|u|^2\,ds\,d\sigma. \end{aligned} \tag{11}$$

Now comes the crucial step of estimating the last term in (11) by existing terms in the inequality. By the definition of $H(t)$, we have

$$\begin{aligned} J &= H^\gamma(t)\int_{\Gamma_0}\int_0^t|u|^2\,ds\,d\sigma \\ &\leq \left[\frac{1}{p+1}\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds + (kt+l)\int_\Omega u_0^2dx\right]^\gamma\int_{\Gamma_0}\int_0^t|u|^2\,ds\,d\sigma. \end{aligned} \tag{12}$$

From Lemma 2 and Lemma 3 with $\theta = \frac{1}{2}$ (see also Remark 1) we derive

$$\int_{\Gamma_0}\int_0^t|u|^2\,ds\,d\sigma \leq C_1\left(\int_\Omega\int_0^t|u|^2\,ds\,dx\right)^{\frac{1}{2}}\left(\int_\Omega\int_0^t|\nabla u|^2\,ds\,dx\right)^{\frac{1}{2}}.$$

$$\begin{aligned} J \leq &\frac{C_2(p-1)}{2(p+1)}\left(\frac{1}{p+1}\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds + (kT+l)\int_\Omega u_0^2dx\right) \\ &+\frac{p+3}{2(p+1)}\left(\int_\Omega\int_0^t|u|^2\,ds\,dx\right)^{\frac{p+1}{p+3}}\left(\int_\Omega\int_0^t|\nabla u|^2\,ds\,dx\right)^{\frac{p+1}{p+3}} \end{aligned} \tag{13}$$

where $C_2 := C_1^{\frac{2(p+1)}{p-1}}$. The Hölder inequality allows us to write

$$\begin{aligned} \left(\int_\Omega\int_0^t|u|^2\,ds\,d\sigma\right)^{\frac{p+1}{p+3}} &\leq \left[|\Omega|^{\frac{p-1}{p+1}}T^{\frac{p-1}{p+1}}\left(\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds\right)^{\frac{2}{p+1}}\right]^{\frac{p+1}{p+3}} \\ &\leq A_1\left(\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds\right)^{\frac{2}{p+3}} \end{aligned}$$

with $A_1 = (|\Omega|\,T)^{\frac{p-1}{p+3}}$. Then (13) becomes

$$\begin{aligned} J \leq &\frac{(p-1)C_2}{2(p+1)}\left[\frac{1}{p+1}\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds + (kT+l)\int_\Omega u_0^2dx\right] \\ &+\frac{(p+3)A_1}{2(p+1)}\left(\int_0^t\int_\Omega|u|^{p+1}\,dx\,ds\right)^{\frac{2}{p+3}}\left(\int_\Omega\int_0^t|\nabla u|^2\,ds\,dx\right)^{\frac{p+1}{p+3}}. \end{aligned}$$

Notice now that $\frac{p+3}{2}$ and $\frac{p+3}{p+1}$ are conjugate exponents. Therefore, applying once again the Young inequality, we obtain

$$\begin{array}{c} J \leq \frac{(p-1)C_2}{2(p+1)} \left[\frac{1}{p+1} \int_0^t \int_\Omega |u|^{p+1}\, dx\, ds + (kT+l) \int_\Omega u_0^2 dx \right] \\ + \frac{(p+3)A_1}{2(p+1)} \left[\frac{2}{p+3} \int_0^t \int_\Omega |u|^{p+1}\, dx\, ds + \frac{p+1}{p+3} \int_\Omega \int_0^t |\nabla u|^2\, ds\, dx \right]. \end{array}$$

With the help of Poincaré inequality, we deduce that

$$J \leq A_2 \int_0^t \int_\Omega |u|^{p+1}\, dx\, ds + A_3 \int_0^t \int_\Omega |\Delta u|^2\, dx\, ds + A_4 \tag{14}$$

where $A_2 = \frac{(p-1)C_2}{2(p+1)^2} + \frac{A_1}{p+1}$, $A_3 = \frac{C_p A_1}{2}$ (C_p is the Poincaré constant) and $A_4 = \frac{(p-1)C_2}{2(p+1)} (kT+l) \int_\Omega u_0^2 dx$. From the relations (14), (11) and the fact that $H(t)$ and $H'(0)$ are positive, we infer that

$$\begin{array}{c} \Phi'(t) \geq [(1-\gamma) - M\varepsilon] H^{-\gamma}(t) H'(t) + \varepsilon \int_\Omega u_0 u_1 dx \\ + \varepsilon \int_0^t \int_\Omega u_t^2 dx\, ds - \varepsilon \int_0^t \int_\Omega |\Delta u|^2\, dx\, ds + \varepsilon \int_0^t \int_\Omega |u|^{p+1}\, dx\, ds \\ - \frac{\varepsilon B_1}{M} \int_\Omega \int_0^t |u|^{p+1}\, ds\, dx - \frac{\varepsilon B_2}{M} \int_\Omega \int_0^t |\Delta u|^2\, ds\, dx - \frac{\varepsilon B_3}{M} \end{array} \tag{15}$$

where $B_1 = \frac{bT^\beta A_2}{4\beta \cos^2(\beta\pi/2)}$, $B_2 = \frac{bT^\beta A_3}{4\beta \cos^2(\beta\pi/2)}$ and $B_3 = \frac{bT^\beta A_4}{4\beta \cos^2(\beta\pi/2)}$. Choosing ε such that $0 < \varepsilon \leq \frac{1-\gamma}{M}$, (15) implies that

$$\begin{array}{c} \Phi'(t) \geq \varepsilon \int_\Omega u_0 u_1 dx + \varepsilon \int_0^t \int_\Omega u_t^2 dx\, ds - \varepsilon \left(1 + \frac{B_2}{M}\right) \int_0^t \int_\Omega |\Delta u|^2\, dx\, ds \\ + \varepsilon \left(1 - \frac{B_1}{M}\right) \int_\Omega \int_0^t |u|^{p+1}\, ds\, dx - \frac{\varepsilon B_3}{M}. \end{array} \tag{16}$$

Let us add and substract $4\varepsilon H(t)$ to the right hand side of (16), we find

$$\begin{array}{c} \Phi'(t) \geq 4\varepsilon H(t) + \varepsilon \int_\Omega u_0 u_1 dx + 3\varepsilon \int_0^t \int_\Omega u_t^2 dx\, ds \\ + \left(1 - \frac{B_1}{M} - \frac{4}{p+1}\right) \varepsilon \int_\Omega \int_0^t |u|^{p+1}\, ds\, dx \\ + \left(1 - \frac{B_2}{M}\right) \varepsilon \int_0^t \int_\Omega |\Delta u|^2\, dx\, ds - 4\varepsilon (kT+l) \int_\Omega u_0^2 dx - \frac{\varepsilon B_3}{M}. \end{array}$$

Select the initial data u_0 and u_1 such that

$$\begin{cases} \int_\Omega u_0 u_1 dx - 4(kT+l) \int_\Omega u_0^2 dx - \frac{B_3}{M} \geq 0 \\ \frac{p-3}{p+1} - \frac{B_1}{M} \geq \tilde{b} > 0 \\ 1 - \frac{B_2}{M} \geq 0 \end{cases} \tag{17}$$

for some $\tilde{b} > 0$. It suffices to select first u_0 and u_1 such that

$$\int_\Omega u_0 u_1 dx - 4(kT+l) \int_\Omega u_0^2 dx > 0 \tag{18}$$

and then choose M large enough so that

$$\int_\Omega u_0 u_1 dx - 4(kT+l) \int_\Omega u_0^2 dx \geq \frac{B_3}{M} > 0$$

and also the second and third conditions in (17) are satisfied.

Consequently,

$$\Phi'(t) \geq 4\varepsilon H(t) + 3\varepsilon \int_0^t \int_\Omega u_t^2 dx\, ds + \varepsilon b_1 \int_\Omega \int_0^t |u|^{p+1}\, ds\, dx. \tag{19}$$

On the other hand, we have

$$\Phi^{\frac{1}{1-\gamma}}(t) \le 2^{\frac{1}{1-\gamma}} \left[H(t) + \varepsilon^{\frac{1}{1-\gamma}} \left(\int_0^t \int_\Omega u u_t dx\, ds \right)^{\frac{1}{1-\gamma}} \right]. \tag{20}$$

It is easy to see that, by the Cauchy-Schwarz inequality and Hölder inequality, we have

$$\begin{aligned} \int_0^t \int_\Omega u u_t dx\, ds &\le \int_0^t \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}} ds \\ &\le C_3 \int_0^t \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{1}{p+1}} \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}} ds \\ &\le C_3 \left(\int_0^t \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}} ds\right)^{1/2} \left(\int_0^t \int_\Omega |u_t|^2 dx\, ds\right)^{1/2}. \end{aligned}$$

Moreover, because $\frac{2}{(p+1)(1-2\gamma)} = 1$, it follows that

$$\begin{aligned} \left(\int_0^t \int_\Omega u u_t dx\, ds\right)^{\frac{1}{1-\gamma}} &\le C_4 \left\{ \int_0^t \int_\Omega |u_t|^2 dx\, ds + \left(\int_0^t \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}} ds \right)^{\frac{1}{1-2\gamma}} \right\} \\ &\le C_4 \left\{ \int_0^t \int_\Omega |u_t|^2 dx\, ds + T^\mu \left(\int_0^t \int_\Omega |u|^{p+1} dx\, ds \right)^{\frac{2}{(p+1)(1-2\gamma)}} \right\}, \end{aligned}$$

where $\mu = \frac{p-1}{(p+1)(1-2\gamma)}$. This estimate, when used in (20), yields

$$\Phi^{\frac{1}{1-\gamma}}(t) \le 2^{\frac{1}{1-\gamma}} H(t) + 2^{\frac{1}{1-\gamma}} \varepsilon^{\frac{1}{1-\gamma}} C_4 \left\{ \int_0^t \int_\Omega |u_t|^2 dx\, ds + T^\mu \int_0^t \int_\Omega |u|^{p+1} dx\, ds \right\}$$

and therefore with the help of (19) we see that

$$\Phi^{\frac{1}{1-\gamma}}(t) \le K \Phi'(t) \tag{21}$$

for some sufficiently large K (depending on T). Integrating (21) over $(0, t)$, we find

$$\Phi^{\frac{\gamma}{1-\gamma}}(t) \ge \frac{1}{\Phi^{-\frac{\gamma}{1-\gamma}}(0) - \frac{\gamma}{K(1-\gamma)} t}$$

Consequently, $\Phi(t)$ blows up at some time $T^* \le \frac{K(1-\gamma)}{\gamma} \Phi^{-\frac{\gamma}{1-\gamma}}(0) < T$. The last inequality holds if $\Phi(0)$ is chosen so that $\Phi(0)^{\frac{\gamma}{1-\gamma}} > \frac{K(1-\gamma)}{\gamma T}$ that is if l is chosen so that $l^\gamma > K(1-\gamma)/\gamma T \left(\int_\Omega u_0^2 dx\right)^\gamma$. □

proposition 3.2. The set of initial data u_0 and u_1 satisfying (2) and (18) is not empty.

Proof. First, we show that we can find u_0 such that

$$16(kT + l) \int_\Omega u_0^2 dx + \frac{1}{2} \int_\Omega |\Delta u_0|^2 dx < k \int_\Omega u_0^2 dx + \frac{1}{p+1} \int_\Omega |u_0|^{p+1} dx. \tag{22}$$

Suppose for contradiction that we always have

$$k \int_\Omega u_0^2 dx + \frac{1}{p+1} \int_\Omega |u_0|^{p+1} dx \le 16(kT + l) \int_\Omega u_0^2 dx + \frac{1}{2} \int_\Omega |\Delta u_0|^2 dx$$

Let $u_0 = \delta v_0$ for an arbitrary $\delta > 0$, then

$$k\delta^2 \int_\Omega v_0^2 dx + \frac{\delta^{p+1}}{p+1} \int_\Omega |v_0|^{p+1} dx \le 16\delta^2 (kT + l) \int_\Omega v_0^2 dx + \frac{\delta^2}{2} \int_\Omega |\Delta v_0|^2 dx.$$

Simplifying by δ^2, we get

$$\frac{\delta^{p-1}}{p+1} \int_\Omega |v_0|^{p+1} dx \le \frac{1}{2} \int_\Omega |\Delta v_0|^2 dx + 16 (kT + l) \int_\Omega v_0^2 dx.$$

Observe that the right hand side is independent on δ. This is impossible and hence there exists u_0 such that (22) holds. Select now $u_1 > 4\sqrt{2} (kT + l) u_0$ such that

$$16 (kT + l) \int_\Omega u_0^2 dx < \frac{1}{2} \int_\Omega u_1^2 dx < k \int_\Omega u_0^2 dx + \frac{1}{p+1} \int_\Omega |u_0|^{p+1} dx - \frac{1}{2} \int_\Omega |\Delta u_0|^2 dx.$$

The second inequality means that (2) is satisfied and

$$\int_\Omega u_0 u_1 dx > 4\sqrt{2} (kT + l) \int_\Omega u_0^2 dx > 4 (kT + l) \int_\Omega u_0^2 dx$$

implies that (18) holds. □

Acknowledgments: The second author is very grateful to King Fahd University of Petroleum and Minerals for its continuous support.

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