

## EXISTENCE OF POSITIVE SOLUTIONS FOR PERIODIC BOUNDARY VALUE PROBLEMS WITH IMPULSE EFFECTS

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**ABSTRACT.** This paper is devoted to study the existence of multiple positive solutions for the second order periodic boundary value problems with impulse effects. By imposing different conditions on nonlinearity, we establish various of existence results. Besides, some results generalize Jiang [5] for ordinary differential equations. In particular, nonlinearity involving the first derivative of  $x$  is considered.

**Key words:** Periodic boundary value problem; Impulse effects; Multiple positive solutions.

**AMS (MOS) Subject Classifications:** 34B15; 34B18; 34B37.

### 1. INTRODUCTION

In this paper, we shall study the existence of multiple positive solutions for the periodic boundary value problems with impulse effects

$$(1.1) \quad \begin{cases} x'' + Mx = f(t, x), & t \neq t_k, \quad t \in J, \\ -\Delta x|_{t=t_k} = I_k(x(t_k)), & \Delta x'|_{t=t_k} = J_k(x(t_k)), \quad k = 1, 2, \dots, l, \\ x(0) = x(2\pi), & x'(0) = x'(2\pi). \end{cases}$$

and

$$(1.2) \quad \begin{cases} x'' + Mx = g(t, x, x'), & t \neq t_k, \quad t \in J, \\ -\Delta x|_{t=t_k} = I_k(x(t_k)), & \Delta x'|_{t=t_k} = J_k(x(t_k)), \quad k = 1, 2, \dots, l, \\ x(0) = x(2\pi), & x'(0) = x'(2\pi). \end{cases}$$

Here,  $J = [0, 2\pi]$ ,  $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = 2\pi$ ,  $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ ,  $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ , where  $x^i(t_k^+)$  (respectively  $x^i(t_k^-)$ ) denotes the right limit (respectively left limit) of  $x^i(t)$  at  $t = t_k$ ,  $i = 0, 1$ . Throughout this paper, assume

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that  $0 < M < \frac{1}{4}$ ,  $f \in C(J \times R^+, R^+)$ ,  $g \in C(J \times R^+ \times R, R^+)$ ,  $I_k \in C(R^+, R)$ ,  $J_k \in C(R^+, R^+)$ ,  $R^+ = [0, \infty)$  with  $|I_k(x)| \leq \frac{\sin 2m\pi}{2m(1-\cos 2m\pi)} J_k(x)$ ,  $x \in R^+$ ,  $m = \sqrt{M}$ .

In recent years, impulsive and periodic boundary value problems have been studied extensively in the literature, please see [3–14].

In [3, 5, 6, 13], periodic boundary value problems were studied extensively. Jiang [5] has applied the Krasnosel'skii fixed point theorem to establish the existence of positive solution to problem

$$(1.3) \quad \begin{cases} x'' + Mx = f(t, x), & t \in [0, 2\pi], \\ x(0) = x(2\pi), & x'(0) = x'(2\pi). \end{cases}$$

He proved that there exists at least one positive solution provided one of the following conditions holds:

- (i)  $\lim_{x \rightarrow 0} \max_{t \in [0, 2\pi]} \frac{f(t, x)}{x} = 0$  and  $\lim_{x \rightarrow \infty} \min_{t \in [0, 2\pi]} \frac{f(t, x)}{x} = +\infty$ ;
- (ii)  $\lim_{x \rightarrow 0} \min_{t \in [0, 2\pi]} \frac{f(t, x)}{x} = +\infty$  and  $\lim_{x \rightarrow \infty} \max_{t \in [0, 2\pi]} \frac{f(t, x)}{x} = 0$ .

On the other hand, impulsive differential equations were studied extensively. In [7, 8, 10, 11, 12], authors used the method of lower and upper solutions with the monotone iterative technique to study impulsive differential equations. In [1, 9], authors used the Krasnosel'skii fixed point theorem in a cone to impulsive differential equations and obtained the existence of positive solutions.

However, in [1, 7, 9, 10, 11, 12], the authors made the condition that the nonlinearity  $f$  depends only on  $x$  and not on the first derivative of  $x$ . In [8], the equation

$$x'' = f(t, x, x'), \quad t \in [a, b], \quad t \neq t_i, i = 1, 2, \dots, p$$

with impulsive effects and nonlinear boundary conditions was studied by upper and lower solutions methods. The existence of at least one solution was obtained. By so far, very few multiplicity positive solutions were established for impulsive boundary value problem with nonlinear terms depending on the first derivative. By using fixed point theory [2], we establish the existence of positive solutions. This is the first time to apply fixed point theory [2] to impulsive boundary value problem.

For the case of  $I_k = J_k = 0$ ,  $k = 1, 2, \dots, l$ , (1.1) is reduced to (1.3). Our some results extend the corresponding results in [5]. Besides, impulsive effect occur at both  $x(t_i)$  and  $\Delta x(t_i)$ , which extend those in [9].

For convenience, we always use the notations:

$$\begin{aligned} f_0 &= \liminf_{x \rightarrow 0^+} \min_{t \in [0, 2\pi]} \frac{f(t, x)}{x}, & J_0(k) &= \liminf_{x \rightarrow 0^+} \frac{J_k(x)}{x}, \\ f^\infty &= \limsup_{x \rightarrow +\infty} \max_{t \in [0, 2\pi]} \frac{f(t, x)}{x}, & J^\infty(k) &= \limsup_{x \rightarrow +\infty} \frac{J_k(x)}{x}, \\ f^0 &= \limsup_{x \rightarrow 0^+} \max_{t \in [0, 2\pi]} \frac{f(t, x)}{x}, & J^0(k) &= \limsup_{x \rightarrow 0^+} \frac{J_k(x)}{x}, \\ f_\infty &= \liminf_{x \rightarrow +\infty} \min_{t \in [0, 2\pi]} \frac{f(t, x)}{x}, & J_\infty(k) &= \liminf_{x \rightarrow +\infty} \frac{J_k(x)}{x}. \end{aligned}$$

In this paper, some of the following hypotheses are satisfied:

(H<sub>1</sub>) :

$$\left[ 2\pi f_0 + \sum_{i=1}^l J_0(i) \right] \sigma > 2\pi M, \quad \left[ 2\pi f_\infty + \sum_{i=1}^l J_\infty(i) \right] \sigma > 2\pi M,$$

where  $\sigma = \frac{G(0)}{2G(\pi)+G(0)}$ ,  $G(0) = \frac{\sin 2m\pi}{2m(1-\cos 2m\pi)}$ ,  $G(\pi) = \frac{\sin m\pi}{2m(1-\cos 2m\pi)}$ ,  $m = \sqrt{M}$ .

(H<sub>2</sub>) :

$$2\pi f^0 + \sum_{i=1}^l J^0(i) < 2\pi\sigma M, \quad 2\pi f^\infty + \sum_{i=1}^l J^\infty(i) < 2\pi\sigma M.$$

(H<sub>3</sub>) : There is a  $p > 0$  such that  $0 \leq x \leq p$  and  $0 \leq t \leq 2\pi$  imply

$$f(t, x) \leq \eta p, \quad J_k(x) \leq \eta_k p,$$

where  $\eta, \eta_k \geq 0$  satisfy  $\eta + \sum_{k=1}^l \eta_k > 0$ ,  $2\pi\eta G(\pi) + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l \eta_i < 1$ .

(H<sub>4</sub>) : There is a  $p > 0$  such that  $\sigma p \leq x \leq p$  and  $0 \leq t \leq 2\pi$  imply

$$f(t, x) \geq \lambda p, \quad J_k(x) \geq \lambda_k p,$$

where  $\lambda, \lambda_k \geq 0$  satisfy  $\lambda + \sum_{k=1}^l \lambda_k > 0$  and  $2\pi G(0)\lambda + \frac{G(0)}{2} \sum_{i=1}^l \lambda_i > 1$ .

For the remainder of this section, we present some results which will be needed in Section 3 and Section 4.

Let  $E$  be a Banach space and  $K \subset E$  be a cone in  $E$ . Assume that  $\Omega$  is a bounded open subset of  $E$  and let  $\partial\Omega$  be its boundary. Let  $\Phi : K \cap \bar{\Omega} \rightarrow K$  be a continuous and completely continuous mapping. If  $\Phi u \neq u$  for every  $u \in K \cap \partial\Omega$ , then the fixed point index  $i(\Phi, K \cap \Omega, K)$  is defined. If  $i(\Phi, K \cap \Omega, K) \neq 0$ , then  $\Phi$  has a fixed point in  $K \cap \Omega$ .

For  $r > 0$ , let  $K_r = \{u \in K : \|u\|_{PC} < r\}$  and  $\partial K_r = \{u \in K : \|u\|_{PC} = r\}$ , which is the relative boundary of  $K_r$  in  $K$ . The following three Lemmas are needed in our argument.

**Lemma 1.1** ([11]). *Let  $\Phi : K \rightarrow K$  be a continuous and completely continuous mapping and  $\Phi u \neq u$  for  $u \in \partial K_r$ . Thus one has the following conclusions:*

- (i) *If  $\|u\| \leq \|\Phi u\|$  for  $u \in \partial K_r$ , then  $i(\Phi, K_r, K) = 0$ ;*
- (ii) *If  $\|u\| \geq \|\Phi u\|$  for  $u \in \partial K_r$ , then  $i(\Phi, K_r, K) = 1$ .*

**Lemma 1.2** ([11]). *Let  $\Phi : K \rightarrow K$  be a continuous and completely continuous mapping with  $\mu\Phi u \neq u$  for every  $u \in \partial K_r$  and  $0 < \mu \leq 1$ . Then  $i(\Phi, K_r, K) = 1$ .*

**Lemma 1.3** ([11]). *Let  $\Phi : K \rightarrow K$  be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:*

- (i)  *$\inf_{u \in \partial K_r} \|\Phi u\| > 0$ ;*
- (ii)  *$\mu\Phi u \neq u$  for every  $u \in \partial K_r$  and  $\mu \geq 1$ .*

*Then,  $i(\Phi, K_r, K) = 0$ .*

**Definition 1.1.** The map  $\psi$  is said to be a nonnegative continuous concave functional on cone  $P$  provided that  $\psi : P \rightarrow [0, \infty)$  is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ . Similarly, we say the map  $\alpha$  is a nonnegative continuous convex functional on  $P$  provided that:  $\alpha : P \rightarrow [0, \infty)$  is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ .

Let  $r > a > 0$ ,  $L > 0$  be constants,  $\psi$  is a nonnegative continuous concave functional and  $\alpha, \beta$  nonnegative continuous convex functionals on the cone  $P$ . Define convex sets

$$P(\alpha, r; \beta, L) = \{y \in P | \alpha(y) < r, \beta(y) < L\},$$

$$\overline{P}(\alpha, r; \beta, L) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L\},$$

$$P(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) < r, \beta(y) < L, \psi(y) > a\},$$

$$\overline{P}(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L, \psi(y) \geq a\}.$$

The following assumptions about the nonnegative continuous convex functionals  $\alpha, \beta$  will be used:

(A1) there exists  $M > 0$  such that  $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$ , for all  $x \in P$ ;

(A2)  $P(\alpha, r; \beta, L) \neq \emptyset$  for all  $r > 0, L > 0$ .

**Lemma 1.4** (Bai and Ge [2]). *Let  $E$  be a Banach space,  $P \subset E$  a cone and  $r_2 \geq d > b > r_1 > 0, L_2 \geq L_1 > 0$ . Assume that  $\alpha, \beta$  are nonnegative continuous convex functionals satisfying (A1) and (A2),  $\psi$  is a nonnegative continuous concave functional on  $P$  such that  $\psi(y) \leq \alpha(y)$  for all  $y \in \overline{P}(\alpha, r_2; \beta, L_2)$ , and  $T : \overline{P}(\alpha, r_2; \beta, L_2) \rightarrow \overline{P}(\alpha, r_2; \beta, L_2)$  is a completely continuous operator. Suppose*

(B1)  $\{y \in P(\alpha, d; \beta, L_2; \psi, b) | \psi(y) > b\} \neq \emptyset, \psi(Ty) > b$  for  $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$ ;

(B2)  $\alpha(Ty) < r_1, \beta(Ty) < L_1$  for all  $y \in \overline{P}(\alpha, r_1; \beta, L_1)$ ;

(B3)  $\psi(Ty) > b$  for all  $y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$  with  $\alpha(Ty) > d$ .

*Then  $T$  has at least three fixed points  $y_1, y_2$  and  $y_3$  in  $\overline{P}(\alpha, r_2; \beta, L_2)$  with*

$$y_1 \in P(\alpha, r_1; \beta, L_1), \quad y_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

*and*

$$y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

The paper is organized as follows: In Section 2, we give some important properties for the Green's function and some fundamental results for later use. In Section 3, we establish the multiple existence results for (1.1). In Section 4, we establish the existence results for (1.2). In Section 5, some example are present to illustrate our main results.

## 2. PRELIMINARIES

In order to define the solution of (1.1) and (1.2) we shall consider the following spaces.

Let  $J' = J \setminus \{t_1, t_2, \dots, t_l\}$ ,

$$PC(J, R) = \{x : J \mapsto R : x|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), x(t_k^-) = x(t_k), \\ \exists x(t_k^+), k = 1, 2, \dots, l\}$$

is a Banach space with norm  $\|x\|_{PC} = \sup_{t \in [0, 2\pi]} |x(t)|$ . Let

$$PC^1(J, R) = \{x : J \mapsto R : x|_{(t_k, t_{k+1})}, x'|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), x(t_k^-) = x(t_k), \\ x'(t_k^-) = x'(t_k), \exists x(t_k^+), x'(t_k^+), k = 1, 2, \dots, l\}$$

with the norm  $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$ , then  $PC^1(J, R)$  is also a Banach space.

**Definition 2.1.** A function  $x \in PC^1(J, R) \cap C^2(J', R)$  is called a solution of (1.1) (or (1.2)) if it satisfies the differential equation

$$x'' + Mx = f(t, x), t \in J' \quad (\text{or } x'' + Mx = f(t, x, x'), t \in J')$$

and the function  $x$  satisfies the conditions  $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k))$ ,  $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-) = -J_k(x(t_k))$ , and the periodic boundary conditions  $x(0) = x(2\pi)$ ,  $x'(0) = x'(2\pi)$ .

**Lemma 2.1** ([14]). *A function  $x \in PC^1(J) \cap C^2(J')$  is a solution of problem (1.1) if and only if  $x \in PC(J)$  is a solution of the equation*

$$(2.1) \quad x(t) = \int_0^{2\pi} G(t, s)f(s, x(s))ds + \sum_{k=1}^l G(t, t_k)J_k(x(t_k)) \\ + \sum_{k=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_k} I_k(x(t_k)),$$

where  $G(t, s)$  is the Green's function to the periodic boundary value problem  $x'' + Mx = 0$ ,  $x(0) = x(2\pi)$ ,  $x'(0) = x'(2\pi)$ , and

$$G(t, s) := \frac{1}{\Gamma} \begin{cases} \sin m(t - s) + \sin m(2\pi - t + s), & 0 \leq s \leq t \leq 2\pi, \\ \sin m(s - t) + \sin m(2\pi - s + t), & 0 \leq t \leq s \leq 2\pi, \end{cases}$$

here  $\Gamma = 2m(1 - \cos 2m\pi)$ .

**Lemma 2.2.** *The Green's function  $G(t, s)$  is defined in Lemma 2.1, then the following inequalities holds*

- (a)  $\frac{\sin 2m\pi}{2m(1 - \cos 2m\pi)} = G(0) \leq G(t, s) \leq G(\pi) = \frac{\sin m\pi}{2m(1 - \cos 2m\pi)}$ ;
- (b)  $\left| \frac{\partial G(t, s)}{\partial s} \right| \leq \frac{1}{2}, t, s \in [0, 2\pi], \quad \left| \frac{\partial G(t, s)}{\partial t} \right| \leq \frac{1}{2}, t, s \in [0, 2\pi]$ ;
- (c)  $m^2 G(0) \leq \frac{\partial}{\partial t} \left( \frac{\partial G(t, s)}{\partial s} \right) \leq 2m^2 G(\pi), t, s \in [0, 2\pi]$ .

*Proof.* It is clear that (a) holds. Now we shall show that (b) holds. By computing,

$$\frac{\partial G(t, s)}{\partial s} = \frac{1}{\Gamma} \begin{cases} -m \cos m(t-s) + m \cos m(2\pi - t + s), & 0 \leq s \leq t \leq 2\pi, \\ m \cos m(s-t) - m \cos m(2\pi - s + t), & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

Let  $h(\theta) = \frac{1}{\Gamma} [-m \cos m\theta + m \cos m(2\pi - \theta)]$ ,  $\theta \in [0, 2\pi]$ . Then

$$\begin{aligned} h'(\theta) &= \frac{1}{\Gamma} [m^2 \sin m\theta + m^2 \sin m(2\pi - \theta)] \\ &= \frac{2m^2}{\Gamma} \sin m\pi \cos m(\theta - \pi) \geq 0. \end{aligned}$$

So  $-\frac{1}{2} = h(0) \leq h(\theta) \leq h(2\pi) = \frac{1}{2}$ , i.e.  $|h(\theta)| \leq \frac{1}{2}$ . So  $\left| \frac{\partial G(t, s)}{\partial s} \right| \leq \frac{1}{2}$ ,  $t, s \in [0, 2\pi]$ . On the other hand,

$$\frac{\partial G(t, s)}{\partial t} = \frac{1}{\Gamma} \begin{cases} m \cos m(t-s) - m \cos m(2\pi - t + s), & 0 \leq s \leq t \leq 2\pi, \\ -m \cos m(s-t) + m \cos m(2\pi - s + t), & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

Let  $l(\theta) = \frac{1}{\Gamma} [m \cos m\theta - m \cos m(2\pi - \theta)]$ ,  $\theta \in [0, 2\pi]$ . Then

$$\begin{aligned} l'(\theta) &= \frac{1}{\Gamma} [-m^2 \sin m\theta - m^2 \sin m(2\pi - \theta)] \\ &= -\frac{2m^2}{\Gamma} \sin m\pi \cos m(\theta - \pi) \leq 0. \end{aligned}$$

So  $-\frac{1}{2} = l(2\pi) \leq l(\theta) \leq l(0) = \frac{1}{2}$ , i.e.  $|l(\theta)| \leq \frac{1}{2}$ . So  $\left| \frac{\partial G(t, s)}{\partial t} \right| \leq \frac{1}{2}$ ,  $t, s \in [0, 2\pi]$ .

At last, we show that (c) holds. By computing,

$$\left| \frac{\partial}{\partial t} \left( \frac{\partial G(t, s)}{\partial s} \right) \right| = \frac{1}{\Gamma} \begin{cases} m^2 \sin m(t-s) + m^2 \sin m(2\pi - t + s), & 0 \leq s \leq t \leq 2\pi, \\ m^2 \sin m(s-t) + m^2 \sin m(2\pi - s + t), & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

Let  $p(\theta) = \frac{1}{\Gamma} [m^2 \sin m\theta + m^2 \sin m(2\pi - \theta)]$ , Then

$$\begin{aligned} p'(\theta) &= \frac{1}{\Gamma} [m^3 \cos m\theta - m^3 \cos m(2\pi - \theta)] \\ &= \frac{1}{\Gamma} [-2m^3 \sin m\pi \sin m(\theta - \pi)]. \end{aligned}$$

For  $\theta \in [0, \pi]$ ,  $p'(\theta) \geq 0$ , thus  $m^2 G(0) = h(0) \leq h(\theta) \leq h(\pi) = 2m^2 G(\pi)$ ;

For  $\theta \in [\pi, 2\pi]$ ,  $p'(\theta) \leq 0$ , thus  $m^2 G(0) = h(2\pi) \leq h(\theta) \leq h(\pi) = 2m^2 G(\pi)$ .

So (c) holds.  $\square$

For every positive solution of problem (1.1), one has

$$\|x\|_{PC} = \sup_{t \in [0, 2\pi]} |x(t)|.$$

Without loss of generality, we assume  $\lim_{t \rightarrow \xi} |x(t)| = \|x\|_{PC}$ ,  $\xi \in [t_k, t_{k+1}]$ ,  $k \in \{0, 1, \dots, l\}$ , then by Lemma 2.2(a),

$$(2.2) \quad \|x\|_{PC} \leq G(\pi) \int_0^{2\pi} f(s, x(s)) ds$$

$$\begin{aligned}
 & + \lim_{t \rightarrow \xi} \left\{ \left| \sum_{i=1}^l G(t, t_i) J_i(x(t_i)) + \sum_{i=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_i} I_i(x(t_i)) \right| \right\} \\
 & = G(\pi) \int_0^{2\pi} f(s, x(s)) ds + \sum_{i=1}^l |G(\xi, t_i) J_i(x(t_i))| \\
 & \quad + \sum_{i=1}^l \left| \frac{\partial G(\xi, s)}{\partial s} \Big|_{s=t_i} I_i(x(t_i)) \right| \\
 & \leq G(\pi) \int_0^{2\pi} f(s, x(s)) ds + G(\pi) \sum_{i=1}^l J_i(x(t_i)) + \frac{1}{2} \sum_{i=1}^l |I_i(x(t_i))| \\
 & \leq G(\pi) \int_0^{2\pi} f(s, x(s)) ds + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l J_i(x(t_i)).
 \end{aligned}$$

For any  $t \in [0, 2\pi]$ , without loss of generality, we assume that  $t \in [t_k, t_{k+1})$ , then

$$\begin{aligned}
 (2.3) \quad x(t) & \geq G(0) \int_0^{2\pi} f(s, x(s)) ds + \sum_{i=1}^l G(t, t_i) J_i(x(t_i)) \\
 & \quad + \sum_{i=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_i} I_i(x(t_i)) \\
 & \geq G(0) \int_0^{2\pi} f(s, x(s)) ds + G(0) \sum_{i=1}^l J_i(x(t_i)) - \frac{1}{2} \sum_{i=1}^l |I_i(x(t_i))| \\
 & \geq G(0) \int_0^{2\pi} f(s, x(s)) ds + \frac{G(0)}{2} \sum_{i=1}^l |J_i(x(t_i))| \\
 & = \frac{G(0)}{G(\pi)} G(\pi) \int_0^{2\pi} f(s, x(s)) ds \\
 & \quad + \frac{G(0)}{2G(\pi) + G(0)} \left[ G(\pi) + \frac{G(0)}{2} \right] \sum_{i=1}^l J_i(x(t_i)) \\
 & \geq \min \left\{ \frac{G(0)}{G(\pi)}, \frac{G(0)}{2G(\pi) + G(0)} \right\} \|x\|_{PC} \\
 & = \frac{G(0)}{2G(\pi) + G(0)} \|x\|_{PC} := \sigma \|x\|_{PC}.
 \end{aligned}$$

Let  $K$  be a cone in  $PC(J, R)$  which is defined as

$$K = \{x \in PC(J, R) : x(t) \geq \sigma \|x\|_{PC}, t \in J\}.$$

Define an operator  $\Phi : K \rightarrow K$  as follows

$$(\Phi x)(t) = \int_0^{2\pi} G(t, s) f(s, x(s)) ds + \sum_{k=1}^l G(t, t_k) J_k(x(t_k))$$

$$+ \sum_{k=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_k} I_k(x(t_k)).$$

Then we have the following Lemma.

**Lemma 2.3.**  $\Phi(K) \subset K$ .

*Proof.* For  $x \in K$ , we have the inequalities from (2.2) (2.3) that

$$\begin{aligned} \|\Phi x\|_{PC} &\leq G(\pi) \int_0^{2\pi} f(s, x(s)) ds + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l J_i(x(t_i)), \\ (\Phi x)(t) &\geq \frac{G(0)}{2G(\pi) + G(0)} \|\Phi x\|_{PC} := \sigma \|\Phi x\|_{PC}, t \in [0, 2\pi] \end{aligned}$$

Thus,  $\Phi(K) \subset K$ . □

It is clear that  $\Phi : K \rightarrow K$  is continuous and completely continuous.

### 3. MAIN RESULTS FOR (1.1)

The following theorems are our main results.

**Theorem 3.1.** *Assume that  $(H_1)$  and  $(H_3)$  are satisfied. Then problem (1.1) has at least two positive solutions  $x_1$  and  $x_2$  with*

$$0 < \|x_1\|_{PC} < p < \|x_2\|_{PC}.$$

**Corollary 3.2.** *The conclusion of Theorem 3.1 is valid if  $(H_1)$  are replaced by:*

$$(H_1^*) \quad f_0 = \infty \text{ or } \sum_{i=1}^l J_0(i) = \infty, \quad f_\infty = \infty \text{ or } \sum_{i=1}^l J_\infty(i) = \infty.$$

**Theorem 3.3.** *Assume that  $(H_2)$  and  $(H_4)$  are satisfied. Then problem (1.1) has at least two positive solutions  $x_1$  and  $x_2$  with*

$$0 < \|x_1\|_{PC} < p < \|x_2\|_{PC}.$$

**Corollary 3.4.** *The conclusions of Theorem 3.3 is valid if  $(H_2)$  is replaced by:*

$$(H_2^*) \quad f^0 = 0 \text{ and } J^0(i) = 0, \quad f^\infty = 0 \text{ and } J^\infty(i) = 0, \quad i = 1, 2, \dots, l.$$

**Theorem 3.5.** *Assume the following conditions are satisfied:*

$$\left[ 2\pi f_0 + \sum_{i=1}^l J_0(i) \right] \sigma > 2\pi M, \quad 2\pi f^\infty + \sum_{i=1}^l J^\infty(i) < 2\pi \sigma M.$$

*Then (1.1) has at least one positive solution.*

**Corollary 3.6.** *Assume the following conditions are satisfied:*

$$f_0 = \infty \text{ or } \sum_{i=1}^l J_0(i) = \infty, \quad f^\infty = 0 \text{ and } J^\infty(i) = 0, \quad i = 1, \dots, l.$$

*Then (1.1) has at least one positive solution.*

**Theorem 3.7.** *Assume the following conditions are satisfied:*

$$2\pi f^0 + \sum_{i=1}^l J^0(i) < 2\pi\sigma M, \quad \left[ 2\pi f_\infty + \sum_{i=1}^l J_\infty(i) \right] \sigma > 2\pi M.$$

Then (1.1) has at least one positive solution.

**Corollary 3.8.** *Assume that*

$$f^0 = 0 \text{ and } J^0(i) = 0, \ i = 1, \dots, l, \ f_\infty = \infty \text{ or } \sum_{i=1}^l J_\infty(i) = \infty.$$

Then (1.1) has at least one positive solution.

*Remark 3.1.* Corollaries 3.6, 3.8 are the generalization of Theorem 1 of [5].

In order to prove the main results, we need the following two Lemmas.

**Lemma 3.9.** *If (H<sub>3</sub>) is satisfied, then  $i(\Phi, K_p, K) = 1$ .*

*Proof.* Let  $x \in K$  with  $\|x\|_{PC} = p$ . It follows from (H<sub>3</sub>) and (2.2) that

$$\begin{aligned} \|\Phi x\|_{PC} &\leq \int_0^{2\pi} G(\pi) f(s, x(s)) ds + \left[ G(\pi) + \frac{G(0)}{2} \right] \sum_{i=1}^l J_i(x(t_i)) \\ &\leq p \left[ 2\pi\eta G(\pi) + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l \eta_i \right] < p = \|x\|_{PC}. \end{aligned}$$

This shows that

$$\|\Phi x\|_{PC} < \|x\|_{PC}, \ \forall x \in \partial K_p.$$

It is obvious that  $\Phi x \neq x$  for  $x \in \partial K_p$ . Therefore,  $i(\Phi, K_p, K) = 1$  follows from Lemma 1.1(ii). □

**Lemma 3.10.** *If (H<sub>4</sub>) is satisfied, then  $i(\Phi, K_p, K) = 0$ .*

*Proof.* Let  $x \in K$  with  $\|x\|_{PC} = p$ . Then by (H<sub>4</sub>) (2.3), we have

$$\begin{aligned} (\Phi x)(t) &\geq G(0) \int_0^{2\pi} f(s, x(s)) ds + \frac{G(0)}{2} \sum_{i=1}^l J_i(x(t_i)) \\ &\geq p \left[ 2\pi G(0)\lambda + \frac{G(0)}{2} \sum_{i=1}^l \lambda_i \right] > p = \|x\|_{PC}. \end{aligned}$$

This shows that

$$\|\Phi x\|_{PC} > \|x\|_{PC}, \ \forall x \in \partial K_p.$$

Also clearly  $\Phi x \neq x$  for  $x \in \partial K_p$ . Therefore,  $i(\Phi, K_p, K) = 0$  follows from Lemma 1.1(i). □

**Proof of Theorem 3.1.** According to Lemma 3.9, we have that

$$(3.1) \quad i(\Phi, K_p, K) = 1.$$

Suppose that  $(H_1)$  holds. There exists  $0 < \varepsilon < 1$  sufficiently small such that

$$(3.2) \quad (1 - \varepsilon)\sigma \left[ 2\pi f_0 + \sum_{i=1}^l J_0(i) \right] > 2\pi M, \quad (1 - \varepsilon)\sigma \left[ 2\pi f_\infty + \sum_{i=1}^l J_\infty(i) \right] > 2\pi M.$$

By the notations  $f_0, J_0$ , one can find  $0 < r_0 < p$  such that

$$(3.3) \quad f(t, x) \geq f_0(1 - \varepsilon)x, \quad J_k(x) \geq J_0(k)(1 - \varepsilon)x, \quad \forall t \in [0, 2\pi], \quad 0 < x < r_0.$$

Let  $r \in (0, r_0)$ . Then for  $x \in \partial K_r$  we have

$$x(t) \geq \sigma \|x\|_{PC} = \sigma r \text{ for } t \in [0, 2\pi],$$

and so

$$\begin{aligned} (\Phi x)(t) &= \int_0^{2\pi} G(t, s) f(s, x(s)) ds + \sum_{k=1}^l G(t, t_k) J_k(x(t_k)) + \sum_{k=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_k} I_k(x(t_k)) \\ &\geq G(0) \int_0^{2\pi} f(s, x(s)) ds + \frac{G(0)}{2} \sum_{k=1}^l J_k(x(t_k)) \\ &\geq G(0) f_0(1 - \varepsilon) \int_0^{2\pi} x(s) ds + \frac{G(0)}{2} (1 - \varepsilon) \sum_{k=1}^l J_0(k) x(t_k) \\ &\geq (1 - \varepsilon) \sigma r \left[ 2\pi f_0 G(0) + \frac{G(0)}{2} \sum_{k=1}^l J_0(k) \right], \end{aligned}$$

from which we see that  $\inf_{x \in \partial K_r} \|\Phi x\|_{PC} > 0$ , namely, hypothesis (i) of Lemma 1.3 holds.

Next we show that  $\mu \Phi x \neq x$  for any  $x \in \partial K_r$  and  $\mu \geq 1$ . If this is not true, then there exist  $x_0 \in \partial K_r$  and  $\mu_0 \geq 1$  such that  $\mu_0 \Phi x_0 = x_0$ . Note that  $x_0(t)$  satisfies

$$(3.4) \quad \begin{cases} x_0''(t) + Mx_0(t) = \mu_0 f(t, x_0(t)), & t \in J', \\ -\Delta x_0|_{t=t_k} = \mu_0 I_k(x_0(t_k)), \quad \Delta x_0'|_{t=t_k} = \mu_0 J_k(x_0(t_k)), & k = 1, 2, \dots, l, \\ x_0(0) = x_0(2\pi), \quad x_0'(0) = x_0'(2\pi). \end{cases}$$

Integrating from 0 to  $2\pi$ , use integration by parts in the left side, without loss of generality we assume that  $t_k \leq \pi \leq t_{k+1}, k \in \{0, 1, \dots, l\}$ , then one has

$$(3.5) \quad \begin{aligned} \int_0^{2\pi} [x_0''(t) + Mx_0(t)] dt &= - \sum_{i=1}^l \Delta x_0'(t_i) + M \int_0^{2\pi} x_0(t) dt \\ &= -\mu_0 \sum_{i=1}^l J_i(x_0(t_i)) + M \int_0^{2\pi} x_0(t) dt. \end{aligned}$$

Thus

$$(3.6) \quad \begin{aligned} M \int_0^{2\pi} x_0(t) dt &= \mu_0 \left[ \int_0^{2\pi} f(t, x_0(t)) dt + \sum_{i=1}^l J_i(x_0(t_i)) \right] \\ &\geq (1 - \varepsilon) \left[ 2\pi f_0 + \sum_{i=1}^l J_0(i) \right] \sigma r. \end{aligned}$$

So we obtain

$$(3.7) \quad 2\pi Mr \geq (1 - \varepsilon) \left[ 2\pi f_0 + \sum_{i=1}^l J_0(i) \right] \sigma r,$$

which contradicts with (3.2).

Hence  $\Phi$  satisfies the hypothesis of Lemma 1.3 in  $K_r$ , we have

$$(3.8) \quad i(\Phi, K_r, K) = 0.$$

On the other hand, from  $(H_1)$ , there exists  $H > p$  such that

$$(3.9) \quad f(t, x) \geq f_\infty(1 - \varepsilon)x, \quad J_k(x) \geq J_\infty(k)(1 - \varepsilon)x, \quad \forall t \in [0, 2\pi], x \geq H.$$

Choose  $R > R_0 := \max\{\frac{H}{\sigma}, p\}$ . Let  $x \in \partial K_R$ . Since  $x(t) \geq \sigma \|x\|_{PC} = \sigma R > H$  for  $t \in [0, 2\pi]$ , from (3.9) we see that

$$(3.10) \quad \begin{aligned} f(t, x(t)) &\geq f_\infty(1 - \varepsilon)x(t), \quad \forall t \in [0, 2\pi], \\ J_k(x(t_k)) &\geq J_\infty(k)(1 - \varepsilon)x(t_k), \quad k = 1, 2, \dots, l. \end{aligned}$$

Essentially the same reasoning as above yields  $\inf_{x \in \partial K_R} \|\Phi x\|_{PC} > 0$ . Next we show that if  $R$  is large enough, then  $\mu \Phi x \neq x$  for any  $x \in \partial K_R$  and  $\mu \geq 1$ . In fact, if there exist  $x_0 \in \partial K_R$  and  $\mu_0 \geq 1$  such that  $\mu_0 \Phi x_0 = x_0$ , then  $x_0(t)$  satisfies equation (3.4).

Integrating from 0 to  $2\pi$ , using integration by parts in the left side to obtain (3.5). By (3.10),

$$\begin{aligned} M \int_0^{2\pi} x_0(t) dt &= \mu_0 \left[ \int_0^{2\pi} f(t, x_0(t)) dt + \sum_{i=1}^l J_i(x_0(t_i)) \right] \\ &\geq (1 - \varepsilon) \left[ 2\pi f_\infty + \sum_{i=1}^l J_\infty(i) \right] \sigma r. \end{aligned}$$

So we obtain

$$(3.11) \quad 2\pi Mr \geq (1 - \varepsilon) \left[ 2\pi f_\infty + \sum_{i=1}^l J_\infty(i) \right] \sigma r,$$

which contradicts with (3.2).

Hence hypothesis (ii) of Lemma 1.3 is satisfied and

$$(3.12) \quad i(\Phi, K_R, K) = 0.$$

In view of (3.1), (3.8) and (3.12), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = -1, \quad i(\Phi, K_p \setminus \bar{K}_r, K) = 1.$$

Thus,  $\Phi$  has fixed points  $x_1$  and  $x_2$  in  $K_p \setminus \bar{K}_r$  and  $K_R \setminus \bar{K}_p$ , respectively, which means  $x_1(t)$  and  $x_2(t)$  are positive solution of the problem (1.1) and  $0 < \|x_1\|_{PC} < p < \|x_2\|_{PC}$ .

**Proof of Theorem 3.3.** According to Lemma 3.10, we have that

$$(3.13) \quad i(\Phi, K_p, K) = 0.$$

Suppose that  $(H_2)$  holds, there exists  $0 < \varepsilon < \min\{\lambda_1 - f^0, \lambda_1 - f^\infty\}$  such that

$$(3.14) \quad 2\pi\sigma M > 2\pi(f^0 + \varepsilon) + \sum_{i=1}^l (J^0(i) + \varepsilon), \quad 2\pi\sigma M > 2\pi(f^\infty + \varepsilon) + \sum_{i=1}^l (J^\infty(i) + \varepsilon).$$

One can find  $0 < r_0 < p$  such that

$$(3.15) \quad f(t, x) \leq (f^0 + \varepsilon)x, \quad J_k(x) \leq (J^0(k) + \varepsilon)x, \quad \forall t \in [0, 2\pi], 0 \leq x \leq r_0.$$

Let  $r \in (0, r_0)$ . We now prove that  $\mu\Phi x \neq x$  for any  $x \in \partial K_r$  and  $0 < \mu \leq 1$ . If this is not true, then there exist  $x_0 \in \partial K_r$  and  $0 < \mu_0 \leq 1$  such that  $\mu_0\Phi x_0 = x_0$ . Then  $x_0(t)$  satisfies equation (3.4). Integrating from 0 to  $2\pi$  ( use (3.5) (3.15)) to obtain

$$(3.16) \quad \begin{aligned} M \int_0^{2\pi} x_0(t) dt &= \mu_0 \left[ \int_0^{2\pi} f(t, x_0(t)) dt + \sum_{i=1}^l J_i(x_0(t_i)) \right] \\ &\leq (f^0 + \varepsilon) \int_0^{2\pi} x_0(t) dt + \sum_{i=1}^l (J^0(i) + \varepsilon) x_0(t_i) \\ &\leq 2\pi r (f^0 + \varepsilon) + \sum_{i=1}^l (J^0(i) + \varepsilon) r. \end{aligned}$$

So

$$2\pi\sigma M r \leq \left[ 2\pi(f^0 + \varepsilon) + \sum_{i=1}^l (J^0(i) + \varepsilon) \right] r,$$

which is a contradiction with (3.14). By Lemma 1.2, we have

$$(3.17) \quad i(\Phi, K_r, K) = 1.$$

On the other hand, from  $(H_2)$ , there exists  $H > p$  such that

$$(3.18) \quad f(t, x) \leq (f^\infty + \varepsilon)x, \quad J_k(x) \leq (J^\infty(k) + \varepsilon)x \quad \forall t \in [0, 2\pi], x \geq H.$$

Choose  $R > R_0 = \max\{\frac{H}{\sigma}, p\}$ . Let  $x \in \partial K_R$ , then (3.18) holds since  $x(t) \geq \sigma\|x\|_{PC} = \sigma R > H$  for  $t \in [0, 2\pi]$ . Now we will show that  $\mu\Phi x \neq x$  for any  $x \in \partial K_R$  and  $0 < \mu \leq 1$ . In fact, if there exist  $x_0 \in \partial K_R$  and  $0 < \mu_0 \leq 1$  such that  $\mu_0\Phi x_0 = x_0$ , then  $x_0(t)$  satisfies equation (3.4).

Integrating from 0 to  $2\pi$  ( use (3.5) (3.18)) to obtain

$$\begin{aligned} M \int_0^{2\pi} x_0(t) dt &= \mu_0 \left[ \int_0^{2\pi} f(t, x_0(t)) dt - \sum_{i=1}^l J_i(x_0(t_i)) \right] \\ &\leq (f^\infty + \varepsilon) \int_0^{2\pi} x_0(t) dt + \sum_{i=1}^l (J^\infty(i) + \varepsilon) x_0(t_i) \\ &\leq 2\pi r (f^\infty + \varepsilon) + r \sum_{i=1}^l (J^\infty(i) + \varepsilon) \\ &= r \left[ 2\pi(f^\infty + \varepsilon) + \sum_{i=1}^l (J^\infty(i) + \varepsilon) \right]. \end{aligned}$$

So

$$2\pi\sigma M r \leq r \left[ 2\pi(f^\infty + \varepsilon) + \sum_{i=1}^l (J^\infty(i) + \varepsilon) \right],$$

which is a contradiction with (3.14).

Let  $R > \max\{p, \frac{H}{\sigma}\}$ , then for any  $x \in \partial K_R$  and  $0 < \mu \leq 1$ , we have  $\mu\Phi x \neq x$ . Hence hypothesis of Lemma 1.2 also holds. By Lemma 1.2,

$$(3.19) \quad i(\Phi, K_R, K) = 1.$$

In view of (3.13), (3.17) and (3.19), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = 1, \quad i(\Phi, K_p \setminus \bar{K}_r, K) = -1.$$

Thus,  $\Phi$  has fixed points  $x_1$  and  $x_2$  in  $K_p \setminus \bar{K}_r$  and  $K_R \setminus \bar{K}_p$ , respectively, which means  $x_1(t)$  and  $x_2(t)$  are positive solution of the problem (1.1) and  $0 < \|x_1\|_{PC} < p < \|x_2\|_{PC}$ .

**Proof of Theorems 3 and 4.** The proof follows the ideas in the proof of Theorems 1 and 2.

#### 4. MAIN RESULTS FOR (1.2)

Let  $P$  be a cone in  $PC^1(J, R)$  which is defined as

$$P = \{x \in PC^1(J, R) : x(t) \geq \sigma \|x\|_{PC}, t \in J\}.$$

Define an operator  $\Phi : P \rightarrow P$  as follows

$$\begin{aligned} (Tx)(t) = & \int_0^{2\pi} G(t, s)g(s, x(s), x'(s))ds + \sum_{k=1}^l G(t, t_k)J_k(x(t_k)) \\ & + \sum_{k=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_k} I_k(x(t_k)). \end{aligned}$$

A function  $x \in PC^1(J) \cap C^2(J')$  is a solution of (1.2) if and only if  $x \in PC^1(J, R)$  is a fixed point of the operator  $T$ . Define the functionals

$$\alpha(x) = \sup_{t \in J} |x(t)|, \quad \beta(x) = \sup_{t \in J} |x'(t)|, \quad \psi(x) = \inf_{t \in J} |x(t)|.$$

Then  $\alpha, \beta : P \rightarrow [0, \infty)$  are nonnegative continuous convex functionals satisfying (A1), (A2);  $\psi$  is a nonnegative continuous concave functional with  $\psi(x) \leq \alpha(x)$  for all  $x \in P$ .

**Theorem 4.1.** *Suppose that there exists  $r_2 \geq \frac{b}{\sigma} > b > r_1 > 0, L_2 > L_1 > 0$  satisfying*

$$2 \left[ 1 + \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2 G(0)G(\pi) \right\} \right] \frac{b}{G(0)} < \min \left\{ \frac{r_2}{G(\pi)}, 2L_2 \right\}.$$

*If the following assumptions hold:*

$$(C1) \quad \max_{(t,x,y) \in J \times [0,r_1] \times [-L_1,L_1]} g(t, x, y) + \left( 1 + \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2 G(0)G(\pi) \right\} \right) \max_{x \in [0,r_1]} \sum_{i=1}^l J_i(x)$$

$$< \min \left\{ \frac{r_1}{G(\pi)}, 2L_1 \right\};$$

$$(C2) \quad \min_{(t,x,y) \in J \times [b, \frac{b}{\sigma}] \times [-L_2, L_2]} g(t, x, y) + \frac{1}{2} \min_{x \in [b, \frac{b}{\sigma}]} \sum_{i=1}^l J_i(x) > \frac{b}{G(0)};$$

$$(C3) \quad \max_{(t,x,y) \in J \times [0, r_2] \times [-L_2, L_2]} g(t, x, y) + \left( 1 + \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2 G(0)G(\pi) \right\} \right) \max_{x \in [0, r_2]} \sum_{i=1}^l J_i(x) < \min \left\{ \frac{r_2}{G(\pi)}, 2L_2 \right\}.$$

Then problem (1.2) has at least three positive solutions  $x_1, x_2, x_3$  with

$$x_1 \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in \{\bar{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$x_3 \in \bar{P}(\alpha, r_2; \beta, L_2) \setminus (\bar{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \bar{P}(\alpha, r_1; \beta, L_1)).$$

*Proof.* We will apply Lemma 1.4 to verify the existence of fixed points of the operator  $T$ . It is clear that  $T : P \rightarrow P$  is completely continuous. Now we will verify that all the conditions of Lemma 1.4 are satisfied. First we show  $T : \bar{P}(\alpha, r_2; \beta, L_2) \rightarrow \bar{P}(\alpha, r_2; \beta, L_2)$ . If  $x \in \bar{P}(\alpha, r_2; \beta, L_2)$ , then  $\alpha(x) \leq r_2, \beta(x) \leq L_2$ . By (2.2), (B3), Lemma 2.2 (b)(c), one has

$$\begin{aligned} \alpha(Tx) &= \sup \left\{ \left| \int_0^{2\pi} G(t, s) g(s, x(s), x'(s)) ds \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^l G(t, t_i) J_i(x(t_i)) + \sum_{i=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_i} I_i(x(t_i)) \right| \right\} \\ &\leq G(\pi) \int_0^{2\pi} g(s, x(s), x'(s)) ds + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l J_i(x(t_i)) \\ &\leq G(\pi) \max_{(t,x,y) \in J \times [0, r_2] \times [-L_2, L_2]} g(t, x, y) + \left( G(\pi) + \frac{G(0)}{2} \right) \max_{x \in [0, r_2]} \sum_{i=1}^l J_i(x) \\ &= G(\pi) \left\{ \max_{(t,x,y) \in J \times [0, r_2] \times [-L_2, L_2]} g(t, x, y) + \left( 1 + \frac{G(0)}{2G(\pi)} \right) \max_{x \in [0, r_2]} \sum_{i=1}^l J_i(x) \right\} \\ &\leq G(\pi) \left\{ \max_{(t,x,y) \in J \times [0, r_2] \times [-L_2, L_2]} g(t, x, y) \right. \\ &\quad \left. + \left( 1 + \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2 G(0)G(\pi) \right\} \right) \max_{x \in [0, r_2]} \sum_{i=1}^l J_i(x) \right\} \\ &< r_2. \end{aligned}$$

$$\beta(Tx) = \sup_{t \in J} \left\{ \left| \int_0^{2\pi} \frac{\partial G(t, s)}{\partial t} g(s, x(s), x'(s)) ds \right. \right.$$

$$\begin{aligned}
 & \left. + \sum_{i=1}^l \frac{\partial G(t, t_i)}{\partial t} J_i(x(t_i)) + \sum_{i=1}^l \frac{\partial}{\partial t} \left( \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_i} \right) I_i(x(t_i)) \right\} \\
 \leq & \frac{1}{2} \max_{(t,x,y) \in J \times [0,r_2] \times [-L_2,L_2]} g(t, x, y) + \frac{1}{2} \max_{x \in [0,r_2]} \sum_{i=1}^l J_i(x) \\
 & + 2m^2 G(\pi) \max_{x \in [0,r_2]} \sum_{i=1}^l |I_i(x)| \\
 \leq & \frac{1}{2} \left\{ \max_{(t,x,y) \in J \times [0,r_2] \times [-L_2,L_2]} g(t, x, y) + (1 + 4m^2 G(\pi) G(0)) \max_{x \in [0,r_2]} \sum_{i=1}^l J_i(x) \right\} \\
 \leq & \frac{1}{2} \max_{(t,x,y) \in J \times [0,r_2] \times [-L_2,L_2]} g(t, x, y) \\
 & + \left( 1 + \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2 G(0) G(\pi) \right\} \right) \max_{x \in [0,r_2]} \sum_{i=1}^l J_i(x) \\
 < & L_2.
 \end{aligned}$$

So  $T : \bar{P}(\alpha, r_2; \beta, L_2) \rightarrow \bar{P}(\alpha, r_2; \beta, L_2)$ . In the same way we can show  $T : \bar{P}(\alpha, r_1; \beta, L_1) \rightarrow \bar{P}(\alpha, r_1; \beta, L_1)$ , so the condition (B2) is satisfied. To check the condition (B1) in Lemma 1.4, we choose  $x(t) = \frac{b}{\sigma}$ ,  $t \in J$ . It is easy to see that  $x(t) = \frac{b}{\sigma} \in \bar{P}(\alpha, \frac{b}{\sigma}; \beta, L_2; \psi, b)$ ,  $\psi(x) = \frac{b}{\sigma} > b$ , and consequently,  $\{x \in \bar{P}(\alpha, \frac{b}{\sigma}; \beta, L_2; \psi, b) : \psi(x) > b\} \neq \emptyset$ . For  $x \in \bar{P}(\alpha, \frac{b}{\sigma}; \beta, L_2; \psi, b)$ , then  $\|x\|_{PC} \leq \frac{b}{\sigma}$ ,  $\|x'\|_{PC} \leq L_2$ ,  $x(t) \geq b$ ,  $t \in J$ . Now we show  $\psi(Tx) > b$ . By (C2)

$$\begin{aligned}
 \psi(Tx) &= \inf_{t \in J} \left\{ \int_0^{2\pi} G(t, s) g(s, x(s), x'(s)) ds + \sum_{i=1}^l G(t, t_i) J_i(x(t_i)) \right. \\
 & \quad \left. + \sum_{i=1}^l \frac{\partial G(t, s)}{\partial s} \Big|_{s=t_i} I_i(x(t_i)) \right\} \\
 &\geq G(0) \int_0^{2\pi} g(s, x(s), x'(s)) ds + \frac{G(0)}{2} \sum_{i=1}^l J_i(x(t_i)) \\
 &\geq G(0) \min_{(t,x,y) \in J \times [b, \frac{b}{\sigma}] \times [-L_2, L_2]} g(t, x, y) + \frac{G(0)}{2} \min_{x \in [b, \frac{b}{\sigma}]} \sum_{i=1}^l J_i(x) \\
 &> b.
 \end{aligned}$$

Finally, we verify that the condition (B3) in Lemma 1.4 holds. For  $x \in \bar{P}(\alpha, r_2; \beta, L_2; \psi, b)$  with  $\alpha(Tx) > \frac{b}{\sigma}$ , then by the definition  $\psi$  and (2.3) we have

$$\begin{aligned}
 \psi(Tx) &= \min_{t \in [l_1, l_2]} (Tx)(t) = \min_{t \in [l_1, l_2]} \int_0^\infty G(t, s) g(s, x(s), x'(s)) ds \\
 &\geq \sigma \sup_{t \in J} \int_0^{2\pi} G(t, s) g(s, x(s), x'(s)) ds \\
 &\geq \sigma \alpha(Tx) > b.
 \end{aligned}$$

Therefore, the operator  $T$  has three fixed points  $x_i \in \overline{P}(\alpha, r_2; \beta, L_2)$ ,  $i = 1, 2, 3$ , with

$$x_1 \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

□

## 5. EXAMPLE

**Example 5.1.** Consider the following impulsive boundary value problem

$$(5.1) \quad \begin{cases} x''(t) + \frac{1}{16}x(t) = x^\alpha + x^\beta, & t \in J', \quad 0 < \alpha < 1 < \beta, \\ -\Delta x|_{t=t_k} = c_k x(t_k), & \Delta x'|_{t=t_k} = c_k x(t_k), \quad (c_k \geq 0) \\ x(0) = x(2\pi), & x'(0) = x'(2\pi). \end{cases}$$

Then problem (5.1) has at least two positive solutions  $x_1$  and  $x_2$  with

$$0 < M < \frac{1}{4}, \quad 0 < \|x_1\|_{PC} < 1 < \|x_2\|_{PC}$$

provided

$$(5.2) \quad \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l c_i < 1.$$

To see this we will apply Corollary 3.2.

By (5.2),  $\eta > 0$  is chosen such that

$$0 < \eta < \frac{1}{2\pi G(0)} \left[ 1 - \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l c_i \right].$$

Set

$$f(t, x) = x^\alpha + x^\beta.$$

Note

$$f_0 = \infty, \quad f_\infty = \infty,$$

so  $(H_1^*)$  holds.

Let  $\eta_k = c_k$  then  $\eta, \eta_k$  satisfy (5.2) and

$$2\pi\eta G(0) + \left( G(\pi) + \frac{G(0)}{2} \right) \sum_{i=1}^l \eta_i < 1.$$

Let  $p = 1$ , then for  $0 \leq x \leq p$ , we have

$$f(t, x) = x^\alpha + x^\beta \leq p^\alpha + p^\beta = 2 < \eta p,$$

and

$$J_k(x) \leq c_k = \eta_k = \eta_k p,$$

thus  $(H_3)$  holds. The result follows from Corollary 3.2.

**Example 5.2.** Consider the following periodic boundary value problem with impulse effects

$$(5.3) \quad \begin{cases} x''(t) + \frac{1}{16}x(t) = g(t, x(t), x'(t)), & t \in [0, 2\pi], t \neq t_i, i = 1, 2, \\ -\Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{cases}$$

where

$$g(t, x, y) = \begin{cases} x^2 + \frac{y}{10000} + 0.005, & (t, x, y) \in [0, 2\pi] \times [0, 3] \times [-50, 50], \\ \frac{x}{10} + 8.7 + \frac{y}{10000} + 0.005, & (t, x, y) \in [0, 2\pi] \times [3, \infty) \times [-100, 100], \\ \frac{x}{10} + 8.7 + \frac{\sqrt{y}}{10} + 0.005, & (t, x, y) \in [0, 2\pi] \times [3, \infty) \times [100, \infty), \\ \frac{x}{10} + 8.7 + \frac{\sqrt{-y}}{10} + 0.005, & (t, x, y) \in [0, 2\pi] \times [3, \infty) \times (-\infty, -100], \end{cases}$$

$$|I_i(x)| \leq 2J_i(x), \quad J_i(x) = \begin{cases} 0, & x \in [0, 1], \\ 2x - 2, & x \in [1, 2], \\ \frac{x}{10} + \frac{9}{5}, & x \in [2, +\infty), \end{cases} \quad x \in R^+, \quad i = 1, 2.$$

By computing  $G(0) = 2, G(\pi) = \sqrt{2}, \sigma = \sqrt{2} - 1, \max \left\{ \frac{G(0)}{2G(\pi)}, 4m^2G(0)G(\pi) \right\} = \frac{\sqrt{2}}{2}$ .

Let  $b = 1, r_1 = \frac{1}{4}, r_2 = 100, L_1 = 50, L_2 = 100$ . The conditions of Theorem 4.1 are satisfied. So problem (5.3) has at least three positive solutions.

## REFERENCES

- [1] R.P. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations. *Appl. Math. Comput.* 2000, **114**: 51–59.
- [2] Zhanbing Bai, Weigao Ge, Existence of three positive solutions for some second-order boundary value problems, *Comput. Math. Appl.*, 48 (2004), 699–707.
- [3] Fuzhong Cong, Periodic solutions for second order differential equations, *Appl. Math. Letters* 18 (2005) 957–961.
- [4] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
- [5] Daqing Jiang, On the existence of positive solutions to second order periodic BVPs, *Acta Math. Sci.*, 18(1998), 31–35.
- [6] Daqing Jiang, Junjie Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, *Nonlinear Anal.* 50 (2002) 885–898.
- [7] S. G. Hristova, D. D. Bainov, Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations. *J. Math. Anal. Appl.* 1996, **1997**: 1–13.
- [8] Irena Rachunková and Jan Tomeček, Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions, *J. Math. Anal. Appl.* 292 (2004) 525–539.
- [9] Xiaoning Lin, Daqing Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, *J. Math. Anal. Appl.* 321 (2006) 501–514.
- [10] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*. World Scientific, Singapore, (1989).

- [11] E. K. Lee , Y.H. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equation. *Appl. Math. Comput.* 2004, **158**: 745–759.
- [12] Jianli Li, Jianhua Shen, Periodic boundary value problems for delay differential equations with impulses, *J. Comput. Appl. Math.* 193 (2006) 563–573.
- [13] Zhongxin Zhang, Junyu Wang, On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations, *J. Math. Anal. Appl.* 281 (2003) 99–107.
- [14] Zhongli Wei, Periodic boundary value problem for second order impulsive integrodifferential equations of Mixed type in Banach space, *J. Math. Appl. Anal.* 195(1995), 214–229.