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ON SOLUTIONS OF A QUADRATIC URYSOHN INTEGRAL EQUATION ON AN UNBOUNDED INTERVAL

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ABSTRACT. We investigate the existence of solutions of a quadratic Urysohn integral equation on unbounded interval. The method used in our considerations depends on suitable conjunction of the technique of measures of noncompactness with the classical Schauder fixed point principle. Such an approach permits us to obtain our existence results under rather general assumptions.

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1. INTRODUCTION

The theory of integral equations creates an important branch of nonlinear functional analysis. Integral equations are applicable in numerous areas of science such as mathematical physics, mechanics, kinetic theory of gases, transport theory, engineering, economics, biology and so on (cf. [1, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14] and references therein). A lot of real world problems can be described and analysed with help of both linear and nonlinear integral equations [9, 10, 14].

Among nonlinear integral equations the equations of Urysohn type belong to the most general and applicable ones.

The purpose of this paper is to investigate the solvability of the Urysohn integral equation on unbounded interval. To ensure the generality of our investigations we will consider the quadratic Urysohn integral equation having the form

$$x(t) = a(t) + f(t, x(t)) \int_0^\infty u(t, s, x(s))ds \ , \ t \in \mathbb{R}_+ = [0, \infty) \ . \tag{1.1}$$

Integral equations of this type contain as special cases a lot of functional and integral equations considered in nonlinear analysis. For example, the Urysohn integral equation on bounded interval of the form

$$x(t) = a(t) + \int_0^T u(t, s, x(s))ds$$

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and the Urysohn integral equation on unbounded interval

$$x(t) = a(t) + \int_0^\infty u(t, s, x(s))ds$$

are special cases of the equation (1.1), among others.

In the investigations of this paper we show that under rather general assumptions the integral equation (1.1) has a continuous and bounded solution on the interval $\mathbb{R}_+$ which vanishes at infinity.

Our existence results concerning (1.1) will be proved with help of the suitable combination of the technique of measures of noncompactness and the classical Schauder fixed point principle. Such method of proving was applied in the paper [2]. It enables us to overcome some difficulties appearing in the proof of existence results when we apply classical approach.

The results obtained in this paper generalize several ones obtained up to now. We illustrate our results by suitable examples showing the applicability of the method developed in this paper.

2. NOTATION AND AUXILIARY FACTS

In this section we collect a few auxiliary facts concerning mainly measures of noncompactness (cf. [3]). Let $(E, ||\cdot||)$ be a real Banach space with the zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. We will write B_r to denote the ball $B(\theta, r)$.

If X is a subset of E then symbols $\overline{X}$, Conv$X$ stand for the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E will be denoted by $\mathfrak{M}_E$ while its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$.

Following [3] we accept the following definition of a measure of noncompactness.

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness if it satisfies the following conditions:

1^o The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$
2^o $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$
3^o $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$
4^o $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$
5^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ defined in 1^o is called the kernel of the measure of noncompactness μ.

Remark 2.2. Let us notice that the intersection set X_∞ from 5^o is a member of the kernel of the measure of noncompactness μ. In fact, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for any $n = 1, 2, ...$ we have that $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. This property of the intersection set X_∞ will be very important in our further considerations.

For further facts concerning measures of noncompactness we refer to [3].

Our considerations concerning the integral equation (1.1) will be placed in the Banach space $BC(\mathbb{R}_+)$ consisting of real functions defined, continuous and bounded on $\mathbb{R}_+$. This space is endowed by the norm

$$||x|| = \sup\{|x(t)| : t \geq 0\} \ .$$

Now we recollect the definition of the measure of noncompactness in the space $BC(\mathbb{R}_+)$ which will be used in further considerations [3].

Let X be a nonempty bounded subset of the space $BC(\mathbb{R}_+)$. Fix a positive number T. For $x \in X$ and $\varepsilon > 0$ let us denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \varepsilon\} \ .$$

Further, let us put:

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\} \ ,$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon) \ ,$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X) \ .$$

Moreover, we put

$$\beta(X) = \lim_{T \to \infty} \left\{ \sup_{x \in X} \{\sup\{|x(t)| : t \geq T\}\} \right\} \ .$$

Finally, let us define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \beta(X) \ .$$

It may be shown [3] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and tend to zero at infinity uniformly with respect to the set X, i.e. for each $\varepsilon > 0$ there exists $T > 0$ with the property that $|x(t)| \leq \varepsilon$ for $t \geq T$ and $x \in X$. This property of $\ker \mu$ will be crucial in our further study.

3. MAIN RESULT

In this section we will study the existence of solutions of the quadratic Urysohn integral equation (1.1). Our considerations are situated in the Banach space $BC(\mathbb{R}_+)$ described in the previous section.

The integral equation (1.1) will be investigated under the following assumptions:

(i) $a \in BC(\mathbb{R}_+)$ and $a(t) \to 0$ as $t \to \infty$,

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t, 0) \to 0$ as $t \to \infty$,

(iii) the function f satisfies the Lipschitz condition with respect to the second variable i.e. there exists $k > 0$ such that

$$|f(t,x) - f(t,y)| \leq k|x - y|$$

for $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$,

(iv) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a contiunuous and nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t,s,x)| \leq g(t,s)h(|x|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

(v) for every $t \geq 0$ the function $s \to g(t,s)$ is integrable on $\mathbb{R}_+$ and the function $t \to \int\limits_0^\infty g(t,s)ds$ is bounded on $\mathbb{R}_+$

(vi)

$$\lim_{T\to\infty} \left\{ \sup \left\{ \int\limits_T^\infty g(t,s)ds : t \in [0,T] \right\} \right\} = 0 \ .$$

Observe that based on assumptions (ii) and (v) we may define the following finite constants:

$$F = \sup \{|f(t,0)| : \ t \geq 0\} \ ,$$

$$G = \sup \left\{ \int\limits_0^\infty g(t,s)ds : \ t \geq 0 \right\} \ .$$

Now we formulate our last assumption:

(vii) the inequality

$$||a|| + FGh(r) + kGrh(r) \leq r$$

has a positive solution r_0 such that $kGh(r_0) < 1$.

Remark 3.1. Notice that if r_0 satisfies the inequality from (vii) then we obtain

$$kGh(r_0) \leq 1 - \frac{||a||}{r_0} - \frac{FGh(r_0)}{r_0} \ .$$

Thus the condition $kGh(r_0) < 1$ is satisfied provided the functions $a(t)$ or $t \to f(t,0)$ do not vanish on $\mathbb{R}_+$.

Now we can formulate our main result.

Theorem 3.2. *Under assumptions (i)-(vii) equation* (1.1) *has at least one solution* $x = x(t)$ *in the space* $BC(\mathbb{R}_+)$ *such that* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Consider the operator U defined on the space $BC(\mathbb{R}_+)$ by the formula

$$(Ux)(t) = a(t) + f(t, x(t)) \int_0^\infty u(t, s, x(s))ds, \ t \geq 0 \ .$$

At first we show that the function Ux is continuous on $\mathbb{R}_+$.

To do this fix arbitrarily $T > 0$ and $\varepsilon > 0$. Next, take arbitrary numbers $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Then, keeping in mind our assumptions, we obtain:

$$|(Ux)(t) - (Ux)(s)| \leq |a(t) - a(s)| +$$

$$+ \left| f(t, x(t)) \int_0^\infty u(t, \tau, x(\tau))d\tau - f(s, x(s)) \int_0^\infty u(t, \tau, x(\tau))d\tau \right| +$$

$$+ \left| f(s, x(s)) \int_0^\infty u(t, \tau, x(\tau))d\tau - f(s, x(s)) \int_0^\infty u(s, \tau, x(\tau))d\tau \right| \leq$$

$$\leq \omega^T(a, \varepsilon) + |f(t, x(t)) - f(s, x(s))| \int_0^\infty |u(t, \tau, x(\tau))|d\tau +$$

$$+ |f(s, x(s))| \int_0^\infty |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))|d\tau \leq$$

$$\leq \omega^T(a, \varepsilon) + [|f(t, x(t)) - f(t, x(s))| + |f(t, x(s)) - f(s, x(s))|] \int_0^\infty g(t, \tau)h(|x(\tau)|)d\tau +$$

$$+ [|f(s, x(s)) - f(s, 0)| + |f(s, 0)|] \int_0^\infty |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))|d\tau \leq$$

$$\leq \omega^T(a, \varepsilon) + \left[k|x(t) - x(s)| + \omega^T_{||x||}(f, \varepsilon)\right] h(||x||) \int_0^\infty g(t, \tau)d\tau +$$

$$+ [k|x(s)| + |f(s, 0)|] \int_0^\infty |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))|d\tau \ , \tag{3.1}$$

where we denoted

$$\omega^T_d(f, \varepsilon) = \sup\{|f(t, y) - f(s, y)| : \ t, s \in [0, T], \ y \in [-d, d], \ |t - s| \leq \varepsilon\} \ .$$

Obviously in the above conducted calculations we should put $||x||$ instead of d.

Now, form estimate (3.1) we get:

$$|(Ux)(t)-(Ux)(s)| \leq \omega^T(a,\varepsilon)+kGh(||x||)\omega^T(x,\varepsilon)+$$

$$+Gh(||x||)\omega^T_{||x||}(f,\varepsilon)+(k||x||+F)\int_0^\infty |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau \leq$$

$$\leq \omega^T(a,\varepsilon)+kGh(||x||)\omega^T(x,\varepsilon)+Gh(||x||)\omega^T_{||x||}(f,\varepsilon)+$$

$$+(k||x||+F)\left\{\int_0^T |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau \; + \right.$$

$$\left. + \int_T^\infty [|u(t,\tau,x(\tau))|+|u(s,\tau,x(\tau))|]d\tau\right\} \leq$$

$$\leq \omega^T(a,\varepsilon)+kGh(||x||)\omega^T(x,\varepsilon)+Gh(||x||)\omega^T_{||x||}(f,\varepsilon)+$$

$$+(k||x||+F)\left\{\int_0^T \omega^T_{||x||}(u,\varepsilon)d\tau+\int_T^\infty (g(t,\tau)+g(s,\tau))h(||x||)d\tau\right\},$$

where, similarly as above, we denoted

$$\omega^T_d(u,\varepsilon)=\sup\{|u(t,\tau,y)-u(s,\tau,y)|:\ t,s,\tau\in[0,T],\ |t-s|\leq\varepsilon,\ y\in[-d,d]\}.$$

Let us notice that $\omega^T_{||x||}(f,\varepsilon)\to 0$ and $\omega^T_{||x||}(u,\varepsilon)\to 0$ as $\varepsilon\to 0$, which is a consequence of the uniform continuity of the function f on the set $[0,T]\times[-||x||,||x||]$ and the function u on the set $[0,T]\times[0,T]\times[-||x||,||x||]$, respectively.

In what follows from the last estimate we derive:

$$|(Ux)(t)-(Ux)(s)| \leq \omega^T(a,\varepsilon)+kGh(||x||)\omega^T(x,\varepsilon)+$$

$$+Gh(||x||)\omega^T_{||x||}(f,\varepsilon)+(k||x||+F)T\omega^T_{||x||}(u,\varepsilon)+$$

$$+(k||x||+F)\cdot 2\sup\left\{\int_T^\infty g(t,\tau)d\tau:\ t\in[0,T]\right\}\cdot h(||x||). \tag{3.2}$$

Further observe that in virtue of assumption (vi) we can choose a number T so big that the last term of the estimate (3.2) is sufficiently small. Hence, taking into account the facts established above we infer that the function Ux is continuous on the interval $[0,T]$ for any $T>0$ big enough. This implies that Ux is continuous on the whole interval $\mathbb{R}_+$.

Now we show that the function Ux is bounded on $\mathbb{R}_+$. In fact, using our assumptions for arbitrarily fixed $t\in\mathbb{R}_+$ we have:

$$|(Ux)(x)| \leq |a(t)|+|f(t,x(t))|\int_0^\infty |u(t,s,x(s))|ds \leq$$

$$\leq |a(t)| + [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \int_0^\infty g(t,s)h(|x(s)|)ds \leq$$

$$\leq |a(t)| + kGh(||x||)|x(t)| + Gh(||x||)|f(t,0)| \; . \tag{3.3}$$

Hence we get

$$|(Ux)(t)| \leq ||a|| + kG||x||h(||x||) + FGh(||x||) \; , \tag{3.4}$$

which means that the function Ux is bounded on $\mathbb{R}_+$.

Linking this fact with the continuity of the function Ux on $\mathbb{R}_+$ we conclude that the operator U transforms the space $BC(\mathbb{R}_+)$ into itself.

Further, from (3.4) we obtain

$$||Ux|| \leq ||a|| + kG||x||h(||x||) + FGh(||x||) \; .$$

Combining this inequality with assumption (vii) we infer that U is a self-mapping of the ball B_{r_0}, where $r_0 > 0$ is a number existing on the base of the mentioned assumption.

In what follows let us take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon > 0$ and $T > 0$ and take an arbitrary function $x \in X$. Then, using the estimate (3.2) we obtain:

$$\omega^T(Ux,\varepsilon) \leq \omega^T(a,\varepsilon) + kGh(r_0)\omega^T(x,\varepsilon) + Gh(r_0)\omega^T_{r_0}(f,\varepsilon) +$$

$$+ (kr_0 + F)T\omega^T_{r_0}(u,\varepsilon) + 2(kr_0+F)h(r_0)\sup\left\{\int_T^\infty g(t,s)ds : \; t \in [0,T]\right\} \; .$$

Hence we get

$$\omega^T(UX,\varepsilon) \leq \omega^T(a,\varepsilon) + kGh(r_0)\omega^T(X,\varepsilon) + Gh(r_0)\omega^T_{r_0}(f,\varepsilon) +$$

$$+ (kr_0 + F)T\omega^T_{r_0}(u,\varepsilon) + 2(kr_0+F)h(r_0)\sup\left\{\int_T^\infty g(t,s)ds : \; t \in [0,T]\right\} \; .$$

Now, taking into account the properties of the components involved in the above inequality, we have:

$$\omega^T_0(UX) \leq kGh(r_0)\omega^T_0(X) +$$

$$+ 2(kr_0+F)h(r_0)\cdot\sup\left\{\int_T^\infty g(t,s)ds : \; t \in [0,T]\right\} \; .$$

Combining this inequality with assumption (vi), we derive the estimate

$$\omega_0(UX) \leq kGh(r_0)\omega_0(X) \; . \tag{3.5}$$

Further, taking $x \in X$ and choosing arbitrarily $T > 0$, in view of the estimate (3.3) we obtain

$$\sup\{|(Ux)(t)| : t \geq T\} \leq \sup\{|a(t)| : t \geq T\} +$$

$$+ kGh(r_0)\sup\{|x(t)| : t \geq T\} + Gh(r_0)\sup\{|f(t,0)| : t \geq T\} .$$

Hence, in view of the assumptions (i) and (ii), we get

$$\beta(UX) \leq kGh(r_0)\beta(X) . \tag{3.6}$$

Now, linking (3.5) and (3.6) we obtain

$$\mu(UX) \leq kGh(r_0)\mu(X) , \tag{3.7}$$

where μ is the measure of noncompactness defined in Section 2

Next, let us consider the sequence of sets $(B^n_{r_0})$, where $B^1_{r_0} = \text{Conv}U(B_{r_0})$, $B^2_{r_0} = \text{Conv}U(B^1_{r_0})$ and so on. Observe that all sets of this sequence are nonempty, bounded, closed and convex. Moreover, $B^{n+1}_{r_0} \subset B^n_{r_0}$ for $n = 1, 2, ...$ Further, keeping in mind (3.7) we get

$$\mu(B^n_{r_0}) \leq q^n\mu(B_{r_0}) , \tag{3.8}$$

where we put $q = kGh(r_0)$. Obviously, in view of (vii) we have that $q < 1$. Apart from this we can calculate that $\mu(B_{r_0}) = 3r_0$. In virtue of (3.8) this implies that $\lim_{n\to\infty} \mu(B^n_{r_0}) = 0$. Thus, from the condition 5^o of Definition 2.1 we infer that the set $Y = \bigcap_{n=1}^{\infty} B^n_{r_0}$ is nonempty, bounded, closed and convex. Moreover, by Remark 2.2 we deduce that $Y \in \ker\mu$. It should be also noted that the operator U maps the set Y into itself.

Now we show that U is continuous on the set Y. To do this fix $\varepsilon > 0$ and take functions $x, y \in Y$ such that $||x-y|| \leq \varepsilon$. Taking into account the fact that $Y \in \ker\mu$ and the description of sets belonging to $\ker\mu$ we can find a number $T > 0$ such that for each $z \in Y$ and $t \geq T$ the inequality $|z(t)| \leq \varepsilon$ is satisfied. Since $U : Y \to Y$ we have that $Ux, Uy \in Y$. Thus, for $t \geq T$ we obtain

$$|(Ux)(t) - (Uy)(t)| \leq |(Ux)(t)| + |(Uy)(t)| \leq 2\varepsilon \tag{3.9}$$

On the other hand, for $t \in [0, T]$ we get:

$$|(Ux)(t) - (Uy)(t)| \leq |f(t, x(t)) - f(t, y(t))| \int_0^\infty |u(t, s, x(s))|ds +$$

$$+ |f(t, y(t))| \int_0^\infty |u(t, s, x(s)) - u(t, s, y(s))|ds \leq \varepsilon k \int_0^\infty g(t, s)h(|x(s)|)ds +$$

$$+ (k|y(t)| + |f(t,0)|) \int_0^\infty |u(t,s,x(s)) - u(t,s,y(s))| ds \le \varepsilon k G h(r_0) +$$

$$+ (k|y(t)| + |f(t,0)|) \left\{ \int_0^T |u(t,s,x(s)) - u(t,s,y(s))| ds + \right.$$

$$\left. + \int_T^\infty (|u(t,s,x(s))| + |u(t,s,y(s))|) ds \right\} \le \varepsilon k G h(r_0) + (kr_0 + F) T \overline{\omega}_{r_0}^T(u,\varepsilon) +$$

$$+ (kr_0 + F) 2 \int_T^\infty g(t,s) h(r_0) ds \le \varepsilon k G h(r_0) + (kr_0 + F) T \overline{\omega}_{r_0}^T(u,\varepsilon) +$$

$$+ 2(kr_0 + F) h(r_0) \sup \left\{ \int_T^\infty g(t,s) ds : \ t \in [0,T] \right\} , \tag{3.10}$$

where we denoted

$$\overline{\omega}_{r_0}^T(u,\varepsilon) = \sup \{ |u(t,s,x) - u(t,s,y)| : \ t,s \in [0,T], \ x,y \in [-r_0,r_0], \ |x-y| \le \varepsilon \} \ .$$

Observe that $\overline{\omega}_{r_0}^T(u,\varepsilon) \to 0$ as $\varepsilon \to 0$ which is an easy consequnce of the uniform continuity of the function $u(t,s,x)$ on the set $[0,T] \times [0,T] \times [-r_0,r_0]$. Moreover, we can choose T in such a way (cf. the assumption (vi)) that the last term in estimate (3.10) is small enough. Taking into account the above facts and the estimates (3.9) and (3.10) we conclude that the operator U is continuous on the set Y.

Finally, linking all above established properties of the set Y and the operator $U : Y \to Y$ and using the Schauder fixed point principle we infer that the operator U has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of the integral equation (1.1). Moreover, keeping in mind that $Y \in \ker \mu$ we obtain that $x(t) \to 0$ as $t \to \infty$.

This completes the proof. □

Now, let us pay attention to the fact that the existence result contained in Theorem 3.2 does not cover some important cases of equation (1.1). It is caused by the requirement that $f(t,0) \to 0$ as $t \to \infty$ imposed in the assumption (ii). In fact, observe that in the case of the classical Urysohn equation

$$x(t) = a(t) + \int_0^\infty u(t,s,x(s)) ds \tag{3.11}$$

we have that $f(t,x) \equiv 1$ so $f(t,0) \equiv 1$ and the condition in question is not satisfied. Thus Theorem 3.2 fails to work in the case of the above equation.

It turns out that we can formulate an existence result which covers also the Urysohn intgeral equation of the type (3.11), among others. Namely, we have to replace the requirement $f(t,0) \to 0$ as $t \to \infty$ by other one.

Indeed, we have the following result.

Theorem 3.3. *Suppose there are satisfied assumptions (i), (iii)-(vii) of Theorem 3.2 and the assumption (ii) has the form:*

(ii') $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is continuous and the function* $t \to f(t,0)$ *is bounded on* $\mathbb{R}_+$.

Moreover, we assume the following hypothesis:

$$(viii) \qquad \lim_{t\to\infty} \int_0^\infty g(t,s)ds = 0 \ .$$

Then integral equation (1.1) *has a solution* $x = x(t)$ *in the space* $BC(\mathbb{R}_+)$ *such that* $x(t) \to 0$ *as* $t \to \infty$.

The proof of the above theorem is similar to that of Theorem 3.2 and we omit it.

In the next section we show that the assumption (viii) admits a lot of natural realizations.

4. REMARKS AND EXAMPLES

In this section we will discuss some details concerning the assumptions imposed in our existence results contained in Theorems 3.2 and 3.3.

At the beginning let us pay attention to the fact that in the theory of improper Riemann integral with a parameter there is used the notion of uniform convergence of the improper integral with respect to a parameter [11]. We recall this definition adopting it to our situation.

Thus, let us assume that the function $g(t,s) = g: \ \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is such that the integral

$$(4.1) \qquad \int_0^\infty g(t,s)ds$$

exists for any fixed $t \in \mathbb{R}_+$.

Definition 4.1. We say that the integral (4.1) is uniformly convergent with respect to $t \in \mathbb{R}_+$ if

$$\lim_{t\to\infty} \int_0^T g(t,s)ds = \int_0^\infty g(t,s)ds$$

uniformly with respect to $t \in \mathbb{R}_+$.

Equivalently: The integral (4.1) is uniformly convergent with respect to $t \in \mathbb{R}_+$ if

$$\lim_{T\to\infty} \left\{ \sup_{t\in\mathbb{R}_+} \int_T^\infty g(t,s)ds \right\} = 0 \ . \tag{4.2}$$

Let us notice that if the integral (4.1) is uniformly convergent with respect to $t \in \mathbb{R}_+$ then it is satisfied the equality from the assumption (vi). In fact, this implication is a simple consequence of the inequality

$$\sup_{t\in[0,T]} \int_T^\infty g(t,s)ds \le \sup_{t\in\mathbb{R}_+} \int_T^\infty g(t,s)ds$$

being valid for any $T > 0$.

Further we show that the converse implication is not true.

Example 4.2. Consider the function $g(t,s) = g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined in the following way

$$g(t,s) = \begin{cases} 1/t \text{ for } t > s+1 \text{ and } s \ge 0 \\ h(t,s) \text{ for } s \le t \le s+1 \text{ and } s \ge 0 \\ 0 \text{ for } t < s \text{ and } s \ge 0 \ , \end{cases}$$

where $h(t,s)$ is chosen in such a way that the function $g(t,s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Now, fix arbitrarily $T > 0$. Then, for $t \in [0,T]$ we have that $\int_T^\infty g(t,s)ds = 0$ since $s \ge T \ge t$. This implies that there is satisfied the condition from (vi).

On the other hand, taking $t > T+1$ we get

$$\int_T^\infty g(t,s)ds \ge \int_T^{t-1} \frac{1}{t}ds = \frac{1}{t}(t-1-T) = 1 - \frac{1}{t} - \frac{T}{t} \ .$$

Hence we obtain

$$\sup_{t\in\mathbb{R}} \int_T^\infty g(t,s)ds \ge \sup_{t\ge T} \int_T^\infty g(t,s)ds \ge 1$$

for any $T > 0$. Thus the integral $\int_0^\infty g(t,s)ds$ is not uniformly convergent with respect to $t \in \mathbb{R}$ and there is not satisfied the condition from assumption (viii).

Further we provide some other examples explaining the relations among assumptions (vi), (viii) and the uniform convergence with respect to a parameter i.e. the condition (4.2).

Example 4.3. Let $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by the formula

$$g(t,s) = \begin{cases} \sin\left(s - \frac{t}{1-t}\right) \text{ for } \frac{t}{1-t} \le s \le \frac{t}{1-t} + \pi \text{ and for } 0 \le t < 1 \\ 0 \text{ otherwise.} \end{cases}$$

Obviously for this function there is satisfied the condition from assumption (viii).

On the other hand, fix $T > 0$ and take t such that $\frac{t}{1-t} > T$. Then we have:

$$\int_T^\infty g(t,s)ds \geq \int_{\frac{t}{1-t}}^{\frac{t}{1-t}+\pi} \sin\left(s - \frac{t}{1-t}\right) ds = 2 \ .$$

This implies that there is not satisfied the condition (4.2).

Apart from this notice that in our situation we have $\frac{T}{1+T} < t$ thus taking $\frac{T}{1+T} < t < T$ we infer that there is not satisfied assumption (vi).

Example 4.4. Consider the function $g = g(t,s)$ defined by the formula

$$g(t,s) = e^{-se^t}$$

for $t \geq 0$ and $s \geq 0$.

Then, standard calculation yields

$$\int_0^\infty g(t,s)ds = e^{-t} \ .$$

which implies that

$$\lim_{t\to\infty} \int_0^\infty g(t,s)ds = 0 \ .$$

This means that there is satisfied the condition from assumption (viii).

Moreover,it is easily seen that the function $g(t,s)$ satisfies also the condition (4.2). Consequently, the condition from (vi) is also satisfied.

Example 4.5. Let us take the function given by the formula

$$g(t,s) = \frac{\arctan ts}{s^2+1}$$

for $t \geq 0$ and $s \geq 0$.

Observe that

$$\int_0^\infty \frac{ds}{s^2+1} = \frac{\pi}{2}$$

and the function $f(t,s) = \arctan ts$ is monotonic with respect to s and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$. Thus the integral $\int_0^\infty g(t,s)ds$ is uniformly convergent with respect to $t \in \mathbb{R}_+$ (cf. [11]). On the other hand from the elementary inequality $\arctan x \geq \frac{x}{1+x}$ being valid for $x \geq 0$, we obtain that

$$\int_0^\infty g(t,s)ds = \int_0^\infty \frac{\arctan ts}{1+s^2} ds \geq \int_0^\infty \frac{ts}{(s^2+1)(ts+1)} ds \ .$$

By the standard calculation we obtain

$$\int_0^\infty \frac{ts}{(s^2+1)(ts+1)} ds = \frac{t^2}{t^2+1}\left(\frac{\pi}{2} - \frac{lnt}{t}\right) .$$

This yields

$$\lim_{t\to\infty} \int_0^\infty \frac{\arctan ts}{s^2+1} ds \geq \frac{\pi}{2}$$

which means that the function $g(t,s)$ does not satisfy assumption (viii).

In the sequel we give two examples of quadratic Urysohn integral equations of the form (1.1) which satisfy assumptions imposed in Theorem 3.2 or 3.3.

Example 4.6. Let us take into account the following quadratic integral equation

$$x(t) = te^{-4t^2} + \arctan(t + x(t)) \int_0^\infty e^{-s(t+1)} x^2(s) ds . \tag{4.3}$$

This equation is a special case of the equation (1.1), if we put $a(t) = te^{-4t^2}$, $f(t,x) = \arctan(t+x)$, $u(t,s,x) = e^{-s(t+1)}x^2$.

It is easy to verify that there is satisfied assumption (i) and $||a|| = \frac{1}{8}e^{-1/16}$. Next observe that $f(t,x)$ satisfies assumption (ii') of Theorem 3.3 with $f(t,0) = \arctan t$ being bounded and $F = \pi/2$ (cf. Section 2).

Obviously the function $f(t,x)$ satisfies the Lipschitz condition with respect to x with the constant $k = 1$, so the hypothesis (iii) holds.

Further, we have that (iv) is satisfied with $g(t,s) = e^{-s(t+1)}$ and $h(r) = r^2$. It can be calculated that

$$\int_0^\infty g(t,s) ds = \frac{1}{t+1} .$$

So it is satisfied assumption (viii). Hence we infer also that assumption (v) holds and $G = 1$.

Moreover, for an arbitrarily fixed $T > 0$ we have

$$\int_T^\infty g(t,s) ds = \frac{1}{t+1} e^{-T(t+1)} .$$

This implies that

$$\sup_{t\in[0,T]} \int_T^\infty g(t,s) ds = e^{-T} ,$$

so there is satisfied assumption (vi).

Now we conclude that the inequality from (vii) has the form

$$\frac{1}{8}e^{-1/16} + \frac{\pi}{2}r^2 + r^3 \le r \ .$$

It is easily seen that this inequality has a positive solution r_0. For example, $r_0 = 1/4$. Apart from this we have that $kGh(r_0) = 1/16 < 1$.

Finally we see that there are satisfied assumptions of Theorem 3.3. This implies that the equation (4.3) has a solution $x = x(t)$ belonging to the space $BC(\mathbb{R}_+)$ which vanishes at infinity.

Example 4.7. Consider the quadratic Urysohn integral equation having the form

$$x(t) = \frac{t}{t^2+16} + x(t)\int_0^\infty \ln\left(1 + \sqrt{|x(t)|}e^{-s(t^2+2)/(t^2+1)}\right) ds \ . \tag{4.4}$$

We show that this equation satisfies assumptions of Theorem 3.2.

Indeed, put $a(t) = t/(t^2+16)$, $f(t,x) = x$, $u(t,s,x) = \ln\left(1 + \sqrt{|x|}e^{-s(t^2+2)/(t^2+1)}\right)$. Then we see that there are satisfied assumptions (i), (ii), (iii) with $||a|| = 1/8$, $F = 0$ and $k = 1$.

Further, we get

$$|u(t,s,x)| \le e^{-s(t^2+2)/(t^2+1)}\sqrt{|x|} \ ,$$

so there is satisfied assumption (iv), where $h(r) = \sqrt{r}$ and $g(t,s) = e^{-s(t^2+2)/(t^2+1)}$. Moreover, if we write the function $g(t,s)$ in the form

$$g(t,s) = e^{-s/(t^2+1)}e^{-s} \ ,$$

we have that $\int_0^\infty e^{-s}ds = 1$ and the function $f(t,s) = e^{-s/(t^2+1)}$ is bounded and monotonic with respect to s on $\mathbb{R}_+$. Thus applying again the classical result from [11] we deduce that the integral $\int_0^\infty g(t,s)ds$ is uniformly convergent with respect to $t \in \mathbb{R}_+$, so there are satisfied assumptions (v) and (vi). Moreover, it is easy to calculate that

$$\int_0^\infty g(t,s)ds = \frac{t^2+1}{t^2+2}$$

which implies that $G = 1$.

Further we have that the inequality from (vii) has the form $\frac{1}{8} + r\sqrt{r} \le r$. Obviously $r_0 = 4/9$ is a positive solution of this inequality for which $kGh(r_0) = 2/3 < 1$.

Thus we showed that there are satisfied all assumptions of Theorem 3.2. This yields that the equation (4.4) has a solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ vanishing at infinity.

REFERENCES

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, Positive Solutions of Differential, Difference and Integral equations, *Kluwer Academic Publishers*, Dordrecht, 1999.

[2] J. Banaś , Existence results for Volterra-Stieltjes quadratic integral equations on an unbounded interval, *Math. Scand.* **98** (2006), 143–160.

[3] J. Banaś and K. Goebel, Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Appl. Math., vol. 60, *Marcel Dekker*, New York and Basel, 1980.

[4] T. A. Burton, Volterra Integral and Differential Equations, *Academic Press,* New York, 1983.

[5] L. W. Busbridge, The Mathematics of Radiative Transfer, *Cambridge Univ. Press*, Cambridge, 1960.

[6] B. Cahlon and B. Eskin, Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation, *J. Math. Anal. Appl.* **83** (1981), 159–171.

[7] K. M. Case and P. F. Zweigel, Linear Transport Theory, *Addison-Wesley,* Reading, MA, 1967.

[8] S. Chandrasekhar, Radiative Transfer, *Oxford Univ. Press*, London, 1950.

[9] C. Corduneanu, Integral Equations and Applications, *Cambridge Univ. Press*, Cambridge, 1991.

[10] K. Deimling, Nonlinear Functional Analysis, *Springer Verlag*, Berlin, 1985.

[11] G. M. Fichtenholz, Differential and Integral Calculus, vol. II, *PWN*, Warsaw, 1980 (in Polish).

[12] S. Hu, M. Khavanin and W. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.* **34** (1989), 261–266.

[13] M. Väth, Volterra and Integral Equations of Vector Functions, Pure and Applied Math., *Marcel Dekker*, New York, 2000.

[14] P. P. Zabrejko, A. I. Koshlev, M. A. Krasnosel'skii, S. G. Mikhlin, L. S. Rakovschik and V. J. Stetsenko, Integral Equations, *Nordhoff*, Leyden, 1975.

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