DIFFERENTIABLE PERTURBATIONS OF
ORNSTEIN-UHLENBECK OPERATORS

L. MANCA
Dipartimento di Matematica P. e A., Università di Padova,
Via Trieste 63, 35121 Padova, Italy
manca@math.unipd.it

ABSTRACT. We prove an extension theorem for a small perturbation of the Ornstein-Uhlenbeck operator \((L, D(L))\) in the space of all uniformly continuous and bounded functions \(f : H \rightarrow \mathbb{R}\), where \(H\) is a separable Hilbert space. We consider a perturbation of the form \(N_0 \varphi = L \varphi + (D \varphi, F)\) where \(F : H \rightarrow H\) is bounded and Fréchet differentiable with uniformly continuous and bounded differential. Hence, we prove that \(N_0\) is essentially \(m\)-dissipative and its closure in \(C_b(H)\) coincides with the infinitesimal generator of a diffusion semigroup associated to a stochastic differential equation in \(H\).

AMS (MOS) Subject Classification: 47B38, 47A55.

1. INTRODUCTION AND SETTING OF THE PROBLEM

Let \(H\) be a separable Hilbert space endowed with scalar product \(\langle \cdot, \cdot \rangle\) and norm \(|\cdot|\). We shall always identify \(H\) with its topological dual space \(H^*\). \(\mathcal{L}(H)\) is the Banach space of all the linear and continuous maps in \(H\), endowed with the usual norm \(||\cdot||_{\mathcal{L}(H)}\). With \(C_b(H)\) (resp. \(C_b(H; H)\)) we denote the Banach space of all uniformly continuous and bounded functions \(f : H \rightarrow \mathbb{R}\) (resp. \(f : H \rightarrow H\)), endowed with the supremum norm \(||\cdot||\) (resp. \(||\cdot||_0\)). We also denote by \(C^1_b(H)\) (resp. \(C^1_b(H; H)\)) the space of all \(f \in C_b(H)\) (resp. \(C_b(H; H)\)) that are Fréchet differentiable with differential in \(C_b(H; H)\) (resp. with uniformly continuous and bounded differential \(Df : H \rightarrow \mathcal{L}(H)\)). We assume the following

Hypothesis 1.1. (i) \(A : D(A) \subset H \rightarrow H\) is the infinitesimal generator of a strongly continuous semigroup \((e^{tA})_{t \geq 0}\) of type \(G(1, \omega)\), i.e. there exists \(\omega \in \mathbb{R}\) such that

\[
\| e^{tA} \|_{\mathcal{L}(H)} \leq e^{\omega t}, \quad t \geq 0;
\]

(ii) \(Q \in \mathcal{L}(H)\) is self adjoint and positive;
(iii) For any \(t > 0\) the linear operator \(Q_t\), defined as

\[
Q_t x = \int_0^t e^{sA} Q e^{sA^*} x \, ds, \quad x \in H, \quad t \geq 0,
\]
is of trace class.

(iv) \( F \in C_b^1(H; H) \), and \( K = \sup_{x \in H} \| DF(x) \|_{\mathcal{L}(H)}. \)

It is well known (see, for instance, [4]) that thanks to conditions (i)-(iii) it is possible to define the so called Ornstein-Uhlenbeck (OU) semigroup \((R_t)_{t \geq 0}\) in \( C_b(H) \) by the formula
\[
R_t \varphi(x) = \int_0^t \varphi(e^{tA}x + y)N_Q_t(dy), \quad x \in H,
\]
where \( N_Q_t \) is the Gaussian measure on \( H \) of mean 0 and covariance operator \( Q_t \) (see [4]). It turns out that the OU semigroup in \( C_b(H) \) is not a strongly continuous semigroup with respect to the supremum norm, but it is strongly continuous with respect to weaker topologies (See [1], [5], [6], [9]). However, it is possible to define its infinitesimal generator by its resolvent or, in an equivalent way, by means of the approach of the \( \pi \)-semigroups introduced in [9]

\[
\begin{cases}
D(L) = \left\{ \varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t \to 0^+} \frac{R_t \varphi(x) - \varphi(x)}{t} = g(x), \\
x \in H, \sup_{t \in (0,1)} \left\| \frac{R_t \varphi - \varphi}{t} \right\| < \infty \right\}
\end{cases}
\]

\[
L \varphi(x) = \lim_{t \to 0^+} \frac{R_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(L), x \in H.
\]

We are interested in the operator \((N_0, D(N_0))\) defined by

\[
N_0 \varphi = L \varphi + \mathcal{F} \varphi, \quad \varphi \in D(N_0) = D(L) \cap C_b^1(H),
\]

where

\[
\mathcal{F} \varphi(x) = \langle D \varphi(x), F(x) \rangle.
\]

Now let us consider the stochastic differential equation in \( H \)
\[
\begin{cases}
dX(t) = (AX(t) + F(X(t)))dt + Q^{1/2}dW(t) & t > 0, \\
X(0) = x & x \in H,
\end{cases}
\]

where \((W(t))_{t \geq 0}\) is a cylindrical Wiener process defined on a stochastic basis \((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, \mathbb{P})\). Since \( F \in C_b^1(H; H) \), problem (1.5) has a unique mild solution \((X(t, x))_{t \geq 0, x \in H}\) (see [4]), that is for any \( x \in H \) the process \( \{X(\cdot, x), t \geq 0\} \) is adapted to the filtration \((\mathcal{G}_t)_{t \geq 0}\) and it is continuous in mean square, i.e.

\[
\lim_{t \to s} \mathbb{E}[|X(t, x) - X(s, x)|^2] = 0, \quad \forall s \geq 0.
\]

This allows us to define a transition semigroup \((P_t)_{t \geq 0}\) in \( C_b(H) \), by setting

\[
P_t \varphi(x) = \mathbb{E} [\varphi(X(t, x))], \quad t \geq 0, \varphi \in C_b(H), x \in H.
\]
The semigroup \((P_t)_{t \geq 0}\) is not strongly continuous in \(C_b(H)\). However, it is a \(\pi\)-semigroup, and we can define its infinitesimal generator \((N, D(N))\) in the same way as for the OU semigroup

\[
D(N) = \left\{ \varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = g(x), \quad x \in H, \sup_{t \in (0,1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\| < \infty \right\}
\]

\[
N \varphi(x) = \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(N), \ x \in H.
\]

The main result of this paper is the following

**Theorem 1.2.** Let us assume that Hypothesis 1.1 holds. Then, the operator \((N_0, D(N_0))\), defined by \(D(N_0) = D(L) \cap C^1_b(H)\) and \(N_0 \varphi = L \varphi + F \varphi, \ \forall \varphi \in D(N_0)\), is \(m\)-dissipative in \(C_b(H)\) and its closure is the operator \((N, D(N))\).

In [2], it is proved that Theorem 1.2 holds with \(F \in C^{1,1}_b(H; H)\), that is \(F\) is Fréchet differentiable and its differential \(DF : H \to \mathcal{L}(H)\) is Lipschitz continuous.

Perturbations of OU operators as been the object of several papers (see, for instance, [2, 3, 5–7, 10]). Frequently, additional assumptions are taken on the OU operator in order to have \(D(L) \subset C^1_b(H)\), see [5], [6].

In order to prove Theorem 1.2 we develop a technique introduced in [2]. The idea is the following: since \(F \in C^1_b(H; H)\), there exists a unique solution \(\eta(\cdot, x)\) of the abstract Cauchy problem

\[
\begin{cases}
\frac{d}{d\varepsilon} \eta(\varepsilon, x) = F(\eta(\varepsilon, x)), & \varepsilon > 0, \\
\eta(0, x) = x, & x \in H.
\end{cases}
\]

Then, for any \(\varepsilon > 0\) we define the operators \(\mathcal{F}_\varepsilon : C_b(H) \to C_b(H)\) and \(N_\varepsilon : D(N_\varepsilon) \subset C_b(H) \to C_b(H)\) by setting

\[
\mathcal{F}_\varepsilon \varphi(x) = \frac{1}{\varepsilon} \left( \varphi(\eta(\varepsilon, x)) - \varphi(x) \right),
\]

\[
\begin{cases}
D(N_\varepsilon) = D(L) \cap C^1_b(H), \\
N_\varepsilon \varphi = L \varphi + \mathcal{F}_\varepsilon \varphi, & \varphi \in D(N_\varepsilon).
\end{cases}
\]

By an approximation argument, we are able to prove that the operator \((N_0, D(N_0))\) is \(m\)-dissipative in \(C_b(H)\). Then, by the Lumer-Phillips theorem, it will follow that the closure of \((N_0, D(N_0))\) coincides with the operator \((N, D(N))\).
1.1. Properties of $\mathcal{F}_\varepsilon$. The following lemma collects some useful properties of $\eta$.

**Lemma 1.3.** The following estimates hold

\begin{align}
(1.6) & \quad |\eta(t, x)| \leq e^{\|F\|_0 t} |x|; \\
(1.7) & \quad |\eta(t, x) - \eta(t, y)| \leq e^{Kt} |x - y|; \\
(1.8) & \quad |\eta(t, x) - x| \leq c\|F\|_0 t \\
(1.9) & \quad \|\eta_x(t, x)\|_{L(H)} \leq e^{Kt} \\
(1.10) & \quad \|\eta_x(t, x) - \eta_x(t, y)\|_{L(H)} \leq e^{Kt} \theta_{DF}(e^{Kt} |x - y|),
\end{align}

where $\theta_{DF} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of $DF$.

**Proof.** (1.6), (1.8), (1.9) have been proved in [2, Lemma 2.1].

(1.7). We have

\[|\eta(t, x) - \eta(t, y)| \leq |x - y| + \int_0^t |F(\eta(s, x)) - F(\eta(s, y))| ds \leq K \int_0^t |\eta(s, x) - \eta(s, y)| ds.\]

Then (1.7) follows by Gronwall’s Lemma.

(1.10). Let $x, y, h \in H$ and set

\[r^h(t) = \eta_x(t, x) \cdot h - \eta_x(t, y) \cdot h = p^h(t, x) - p^h(t, y),\]

where $P^h(t, x) = \eta_x(t, x) \cdot h$ and $p^h(t, y) = \eta_x(t, y) \cdot h$. Then $r^h(t)$ fulfills the following equation

\[
\begin{aligned}
\frac{d}{dt} r^h(t) &= DF(\eta(t, x)) r^h(t) + [DF(\eta(t, x)) - DF(\eta(t, y))] p^h(t, x), \quad t > 0 \\
r^h(0) &= 0.
\end{aligned}
\]

Since $|DF(\eta(t, x)) r^h(t)| \leq K |r^h(t)|$ it follows that $r^h(t)$ is bounded by

\[|r^h(t)| \leq \int_0^t e^{K(t-s)} \|DF(\eta(s, x)) - DF(\eta(s, y))\|_{L(H)} |p^h(s, x)| ds.\]

By taking into account that $DF : H \to L(H; H)$ is uniformly continuous and bounded, we denote by $\theta_{DF}$ the modulus of continuity of $DF$. Hence, by (1.7), (1.9) we have

\[|r^h(t)| \leq \int_0^t e^{Ks} \theta_{DF}(|\eta(s, x) - \eta(s, y)|) ds |h| \leq e^{Kt} \theta_{DF}(e^{Kt} |x - y|) |h| \]

\[\square\]

**Proposition 1.4.** For any $\varphi \in C_b^1(H)$ we have

\begin{align}
(1.11) & \quad \lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon \varphi = \mathcal{F} \varphi \quad \text{in } C_b(H). \\
(1.12) & \quad \|\mathcal{F}_\varepsilon \varphi\| \leq \|D \varphi\|_0 \|F\|_0.
\end{align}
Proof. For all \( \varphi \in C^1_b(H) \) we have
\[
\mathcal{F}_\varepsilon \varphi(x) - \mathcal{F} \varphi(x) = \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(\eta(s,x)) - D\varphi(x), F(\eta(s,x)) \rangle \, ds \\
+ \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(x), F(\eta(s,x)) - F(x) \rangle \, ds.
\]
Then by (1.8) we have
\[
|\mathcal{F}_\varepsilon \varphi(x) - \mathcal{F} \varphi(x)| \leq \frac{1}{\varepsilon} \int_0^\varepsilon (|\theta_{D\varphi}(\eta(s,x) - x)| + |\theta_{D\varphi}||\eta(s,x) - x|) \, ds \\
\leq \frac{1}{\varepsilon} \int_0^\varepsilon (|F||0s|)|\varphi||F||0s| + |\theta_{D\varphi}||K||\varphi||F||0s| \, ds \\
\leq (|\theta_{D\varphi}||F||0\varepsilon|) + |\theta_{D\varphi}||K||\varphi||F||0
\]
where \( \theta_{D\varphi} \) is the modulus of continuity of \( D\varphi \). This yields (1.11). Moreover, we have
\[
\mathcal{F}_\varepsilon \varphi(x) = \frac{1}{\varepsilon} \int_0^\varepsilon \langle D\varphi(\eta(s,x)), F(\eta(s,x)) \rangle \, ds
\]
that implies (1.12).

1.2. \( m \)-dissipativity of \( N \). Given \( \varepsilon > 0 \) we introduce the following approximating operator
\[
N_\varepsilon = L + \mathcal{F}_\varepsilon, \quad D(N_\varepsilon) = D(L) \cap C^1_b(H).
\]
We have
\begin{proposition}
\( N_\varepsilon \) is an essentially \( m \)-dissipative operator in \( C_b(H) \) for any \( \varepsilon > 0 \). Moreover, for any \( f \in C^1_b(H) \) and any \( \lambda > \omega + (e^{\varepsilon K} - 1)/\varepsilon \) the operator
\[
R(\lambda, N_\varepsilon) = (1 - T_\lambda)^{-1}R \left( \lambda + \frac{1}{\varepsilon}, L \right),
\]
where \( T_\lambda : C_b(H) \to C_b(H) \) is defined by
\begin{equation}
T_\lambda \psi(x) = R \left( \lambda + \frac{1}{\varepsilon}, L \right) \left[ \frac{1}{\varepsilon} \psi(\eta(\varepsilon,x)) \right], \quad x \in H, \, \psi \in C_b(H)
\end{equation}
maps \( C^1_b(H) \) into \( D(L) \cap C^1_b(H) \) and
\begin{equation}
\|DR(\lambda, N_\varepsilon)f\|_0 \leq \frac{1}{\lambda - \omega - e^{\varepsilon K} - 1} \|Df\|_0.
\end{equation}
\end{proposition}

Proof. Let \( \varepsilon > 0, \lambda > 0, \, f \in C_b(H) \). The equation
\[
\lambda \varphi_\varepsilon - L \varphi_\varepsilon - \mathcal{F}(\varphi_\varepsilon) = f
\]
is equivalent to
\[
\left( \lambda + \frac{1}{\varepsilon} \right) \varphi_\varepsilon - L \varphi_\varepsilon - \mathcal{F}(\varphi_\varepsilon) = f + \frac{1}{\varepsilon} \varphi_\varepsilon(\eta(\varepsilon,\cdot))
\]
and to
\[ (1.15) \quad \varphi_\varepsilon = R\left(\lambda + \frac{1}{\varepsilon}, L\right) f + T_\lambda \varphi_\varepsilon. \]

Since, as we can easily see, for any \( \lambda > 0 \)
\[ (1.16) \quad \|T_\lambda \psi\| \leq \frac{1}{1 + \lambda \varepsilon} \|\psi\|, \quad \forall \psi \in C_b(H), \]
the operator \( T_\lambda \) is a contraction in \( C_b(H) \) and so equation (1.15) has a unique solution \( \varphi_\varepsilon \in C_b(H) \) done by \( \varphi_\varepsilon = R(\lambda, N_\varepsilon) f \). Moreover, by (1.13), (1.16) it holds
\[ \|\varphi_\varepsilon\| \leq \frac{1}{\lambda + \varepsilon} \left[ \|f\| + \frac{1}{\varepsilon} \|\varphi_\varepsilon\| \right]. \]

Consequently,
\[ \|\varphi_\varepsilon\| \leq \frac{1}{\lambda} \|f\|. \]

Then, \( N_\varepsilon \) is \( m \)-dissipative. Now let \( f \in C_b^1(H) \). We recall that for any \( \lambda > 0 \), \( \psi \in C_b(H) \)
\[ (1.17) \quad R(\lambda, L)\psi(x) = \int_0^\infty e^{-\lambda t} R_t \psi(x) dt \]
and that
\[ DR_t \psi(x) = \int_H e^{tA^*} D\psi(e^{tA} x + y) N_{Q_1} (dy). \]

Hence, for any \( \lambda > \omega \)
\[ (1.18) \quad DR(\lambda, L)\psi(x) = \int_0^\infty \int_H e^{-\lambda t} e^{tA^*} D\psi(e^{tA} x + y) N_{Q_1} (dy) dt \]
and so
\[ (1.19) \quad \|DR(\lambda, L)\psi\|_0 \leq \frac{1}{\lambda - \omega} \|D\psi\|_0 \]

Moreover, as it can be easily seen by (1.18), \( DR(\lambda, L)\psi \) is uniformly continuous. Then \( R(\lambda, L) : C_b^1(H) \to C_b^1(H) \). Now, in order to prove that \( T_\lambda : C_b^1(H) \to C_b^1(H) \) it is sufficient to show that \( \psi(\eta(\varepsilon, x)) \in C_b^1(H) \), for any \( \psi \in C_b^1(H) \). Indeed, by a standard computation, we have
\[ D\psi(\eta(\varepsilon, \cdot))(x) = \eta^*_x(\varepsilon, x) D\psi(\eta(\varepsilon, x)), \quad x \in H. \]

Consequently, by (1.7), (1.10) we have
\[ |D\psi(\eta(\varepsilon, \cdot))(x) - D\psi(\eta(\varepsilon, \cdot))(\overline{x})| \leq \|\eta^*_x(\varepsilon, x) - \eta^*_x(\varepsilon, \overline{x})\|_{\mathcal{L}(H)}|D\psi(\eta(\varepsilon, x))| \]
\[ + \|\eta^*_x(\varepsilon, \overline{x})\|_{\mathcal{L}(H)}|D\psi(\eta(\varepsilon, x)) - D\psi(\eta(\varepsilon, \overline{x}))| \]
\[ \leq e^{K} \theta_{DF}(e^{K} |x - \overline{x}|) \|D\psi\|_0 + e^{K} \theta_{D\psi}(|\eta(\varepsilon, x) - \eta(\varepsilon, \overline{x})|) \]
\[ \leq e^{K} \theta_{DF}(e^{K} |x - \overline{x}|) \|D\psi\|_0 + e^{K} \theta_{D\psi}(e^{K} |x - \overline{x}|), \]
for any \(x, \eta \in H\). So, \(DT_\lambda \psi(\cdot)\) is uniformly continuous. Now we prove that \(T_\lambda\) is a contraction in \(C_b^1(H)\). By (1.13), (1.17) we have

\[
T_\lambda \psi(x) = \frac{1}{\varepsilon} \int_0^\infty e^{-(\lambda + \frac{1}{\varepsilon})t} R_t \psi(\eta(\varepsilon, \cdot))(x) dt
\]

Then

\[
DT_\lambda \psi(x) = \frac{1}{\varepsilon} \int_0^\infty \int_H e^{-(\lambda + \frac{1}{\varepsilon})t_1} e^{t_1A} \eta(\varepsilon, e^{t_1A} x + y)) D\psi(\eta(\varepsilon, e^{t_1A} x + y)) N_{Q_\varepsilon}(dy) dt
\]

By (1.9) it follows

\[
|DT_\lambda \psi(x)| \leq \frac{1}{\varepsilon} \int_0^\infty e^{-(\lambda + \frac{1}{\varepsilon} - \omega)t} e^{\varepsilon K} \|D\psi\|_0 dt = \frac{e^{\varepsilon K}}{1 + \varepsilon(\lambda - \omega)} \|D\psi\|_0.
\]

Therefore, for any \(\lambda > \omega + (e^{\varepsilon K} - 1)/\varepsilon\) the linear operator \(T_\lambda\) is a contraction in \(C_b^1(H)\) and its resolvent satisfies

\[
(1 - T_\lambda)^{-1}(C_b^1(H)) \subset C_b^1(H),
\]

(1.20)

\[
\|D(1 - T_\lambda)^{-1}\|_0 \leq \frac{1}{1 - \frac{e^{\varepsilon K}}{1 + \varepsilon(\lambda - \omega)}} \|D\psi\|_0.
\]

This implies

\[
R(\lambda, N_\varepsilon)(C_b^1(H)) = (1 - T_\lambda)^{-1} R\left(\lambda + \frac{1}{\varepsilon}, L\right) C_b^1(H) \subset C_b^1(H).
\]

Then, since \(C_b^1(H)\) is dense in \(C_b(H)\), it follows that \(N_\varepsilon\) is essentially \(m\)-dissipative. Finally, (1.14) follows by (1.19) and (1.20).

**Lemma 1.6.** The operator \(N_0\) is dissipative in \(C_b(H)\).

**Proof.** We have to prove that \(\|\lambda \varphi - N_0 \varphi\| \geq \lambda \|\varphi\|\) for any \(\varphi \in D(N_0), \lambda > 0\). So, if \(\varphi \in D(L) \cap C_b^1(H)\) and \(\lambda > 0\) we set

\[
\lambda \varphi - L \varphi - \mathcal{F} \varphi = f.
\]

then for any \(\varepsilon > 0\) we have

\[
\lambda \varphi - N_\varepsilon \varphi = f + \mathcal{F} \varphi - \mathcal{F}_\varepsilon \varphi.
\]

It follows

\[
\varphi = R(\lambda, N_\varepsilon)(f + \mathcal{F} \varphi - \mathcal{F}_\varepsilon \varphi)
\]

and

\[
\|\varphi\| \leq \frac{1}{\lambda}(\|f\| + \|\mathcal{F} \varphi - \mathcal{F}_\varepsilon \varphi\|)
\]

Then by (1.11) it follows

\[
\|\varphi\| \leq \frac{1}{\lambda}\|f\|.
\]
Since \( N_0 \) is dissipative, its closure \( \overline{N}_0 \) is still dissipative (maybe it is multivalued).

By the following theorem follows Theorem 1.2.

**Theorem 1.7.** \( N_0 \) is essentially m-dissipative.

**Proof.** Let \( f \in C^1_b(H), \varepsilon \in (0,1) \) and \( \lambda > \omega + e^K - 1 \). We denote by \( \varphi_\varepsilon \) the solution of

\[
\lambda \varphi_\varepsilon - N_0 \varphi_\varepsilon = f.
\]

By Proposition (1.5) we have \( \varphi_\varepsilon \in D(L) \cap C^1_b(H) = D(N_0) \), then \( \varphi_\varepsilon \) is solution of

\[
\lambda \varphi_\varepsilon - N_0 \varphi_\varepsilon = f + \mathcal{F}_e \varphi_\varepsilon - \mathcal{F} \varphi_\varepsilon.
\]

We claim that \( \mathcal{F}_e \varphi_\varepsilon - \mathcal{F} \varphi_\varepsilon \to 0 \) in \( C_b(H) \) as \( \varepsilon \to 0^+ \). Indeed it holds

\[
\mathcal{F}_e \varphi_\varepsilon(x) - \mathcal{F} \varphi_\varepsilon(x) = \frac{1}{\varepsilon} \int_0^\varepsilon (\langle D\varphi_\varepsilon(\eta(s,x)), F(\eta(s,x)) \rangle + \langle D\varphi_\varepsilon(x), F(x) \rangle) \, ds
\]

Hence

\[
|\mathcal{F}_e \varphi_\varepsilon(x) - \mathcal{F} \varphi_\varepsilon(x)| \leq \frac{1}{\varepsilon} \int_0^\varepsilon |\langle D\varphi_\varepsilon(\eta(s,x)), D\varphi_\varepsilon(x) \rangle| |F||_0 + |\langle D\varphi_\varepsilon||_0 F(\eta(s,x)) - F(x) \rangle| \, ds
\]

By (1.8) we have

\[
|F(\eta(s,x)) - F(x)| \leq K|\eta(s,x) - x| \leq K||F||_0 s \leq K||F||_0 \varepsilon.
\]

Notice now that since \( \varphi_\varepsilon = R(\lambda, N_\varepsilon) f \) and \( \varepsilon \in (0,1) \), by (1.14) it follows

\[
|D\varphi_\varepsilon||_0 \leq c_1 |Df||_0,
\]

for all \( \varepsilon \in (0,1) \), where \( c_1 = (\lambda - \omega - K e^K)^{-1} \). This also implies

\[
|D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x)||_0 \leq c_1 |Df(\eta(s,x) + \cdot) - Df(x + \cdot)||_0 \leq c_1 |\theta_{Df}(\eta(s,x) - x) - c_1 \theta_{Df}(F||_0 \varepsilon),
\]

where \( \theta_{Df} : \mathbb{R}^+ \to \mathbb{R}^+ \) is the modulus of continuity of \( Df \). So we find

\[
|\mathcal{F}_e \varphi_\varepsilon(x) - \mathcal{F} \varphi_\varepsilon(x)| \leq c_1 ||F||_0 \theta_{Df}(F||_0 \varepsilon) + c_1 |Df||_0 K||F||_0 \varepsilon.
\]

Then \( \mathcal{F}_e \varphi_\varepsilon - \mathcal{F} \varphi_\varepsilon \to 0 \) in \( C_b(H) \), as \( \varepsilon \to 0^+ \). Finally, we have obtained

\[
\lim_{\varepsilon \to 0^+} [\lambda \varphi_\varepsilon - N_0 \varphi_\varepsilon] = f
\]

in \( C_b(H) \). Therefore the closure of the range of \( \lambda - N_0 \) includes \( C^1_b(H) \), which is dense in \( C_b(H) \). So, since \( N_0 \) is dissipative, by the Lumer-Phillips theorem the closure \( \overline{N}_0 \) of \( N_0 \) is m-dissipative. \( \square \)
1.3. **Proof of Theorem 1.2.** By Theorem 1.7 the operator $N_0$ is $m$-dissipative in $C_b(H)$. It is also known that if $\varphi \in D(L) \cap C^1_b(H)$, then $N\varphi = L\varphi + F\varphi$ (see, for instance, [8]) and therefore $(N, D(N))$ is an extension of $(N_0, D(N_0))$. Finally, since the operator $(N, D(N))$ is closed (see Proposition 3.4 in [9]), by the Lumer-Phillips theorem it follows that the closure of $(N_0, D(N_0))$ in $C_b(H)$ coincides with $(N, D(N))$.

**REFERENCES**


