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EXISTENCE AND QUASILINEARIZATION FOR A CLASS OF NONLINEAR ELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we discuss some existence results and the application of quasilinearization methods to the solution of second order nonlinear self adjoint elliptic partial differential equation in $\mathbb{R}^n$ with Dirichlet boundary conditions. Under fairly general assumptions on the data of the problem we show the existence of a solution that can be obtained as the limit of a quadratically convergent nondecreasing sequence of approximate solutions. If the assumptions are strengthened, we show that the solution can be quadratically bracketed between two monotone sequences of approximate solutions of certain related linear equations.

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1. INTRODUCTION

Recently, there has been a lot of activity related to the theory of upper and lower solutions and quasilinearization (QSL) for nonlinear ordinary differential equations. See, for example, [1, 6, 10, 15, 17, 25] and [3, 5, 12, 13, 16, 18, 19, 20, 21, 22, 23] and the references therein. The current research activity on the QSL method involves development and application. For a discussion of the method see the manuscript [15] or the paper [21]. For an account on recent developments the reader is referred to [7] and the references therein.

Encouraged by the positive indications concerning the applicability of the method to various fields of mathematics and applied sciences, we treat in this article the development of the QSL method to partial differential equations of the form

$$\ell u = Fu, \tag{1.1}$$

where

$$\ell u\left(x\right)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_{i}}\left(a_{ij}\left(x\right)\frac{\partial u\left(x\right)}{\partial x_{j}}\right)+a_{0}\left(x\right)u\left(x\right),\ x\in\Omega\subset\mathbb{R}^{n}$$

with Dirichlet boundary conditions and F is a nonlinear operator. In a sense, this article is an application of the techniques developed for the general Hilbert space

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settings [7, 9] to the case where the operator involved is a partial differential operator. We shall see that some of the main results here can be proved in exactly the same way as in [7], whereas some of the other results have to be proved directly.

This paper consists of four sections in addition to the introduction. In Section 2 we present some preliminary material. In Section 3 we prove a general existence theorem under a weak continuity assumption on F and a certain coercivity condition. The proof makes use of a Galerkin type argument. In Section 4 we combine the Galerkin method with lower and upper solutions techniques to prove a basic existence theorem when problem (1.1) has lower and upper solutions. In Section 5 we discuss the application of the QSL method to generate sequences of solutions of certain perturbed problems converging quadratically in an energy norm to a solution of the differential equation (1.1).

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary and let $H_0^1(\Omega)$, $L^2(\Omega)$, $H^{-1}(\Omega)$ and $L^\infty(\Omega)$ denote the usual Sobolev and Lebesgue spaces. In general, the notation $\langle \xi, u\rangle$ will mean the pairing between the elements $\xi \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$. If both $\xi, u \in L^2(\Omega)$, then the pairing reduces to the inner product in $L^2(\Omega)$.

Suppose $a_0,\ a_{ij} \in L^\infty(\Omega)$, $i, j = 1, 2, \ldots, n$. Define the formal operator ℓ by

$$\begin{aligned}\ell u(x) &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) + a_0(x)\,u(x).\\ &= -\operatorname{div}(A(x)\nabla u(x)) + a_0(x)\,u(x),\ x \in \Omega,\end{aligned}$$

where $A(x) = (a_{ij}(x))$. Now ℓ induces an operator $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$Lu = \ell u \quad \forall u \in H_0^1(\Omega).$$

Furthermore, define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \langle A\nabla u, \nabla v\rangle + \langle a_0 u, v\rangle.$$

Lemma 2.1. *For $f \in H^{-1}(\Omega)$, the following are equivalent*

(i) $u \in H_0^1(\Omega)$, $Lu = f$,

(ii) $u \in H_0^1(\Omega)$, $a(u, v) = \langle f, v\rangle$ *for all* $v \in C_0^\infty(\Omega)$.

Proof. See [2]. □

Corollary 2.2. *The operator* $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is bounded.*

Proof. Since L is defined on all of $H_0^1(\Omega)$, we only need to show that it is closed. For this assume that we have a sequence $\{u_n\} \subset H_0^1(\Omega)$ converging to a $u \in H_0^1(\Omega)$ such that $Lu_n \to f$ in $H^{-1}(\Omega)$. Then $\nabla u_n \to \nabla u$ in $L^2(\Omega)^n$ and $u_n \to u$ in $L^2(\Omega)$. For any $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}\langle f, v\rangle &= \lim \langle Lu_n, v\rangle = \lim \left(\left\langle \nabla u_n, A^t \nabla v\right\rangle + \langle a_0 u_n, v\rangle\right) \\ &= \left\langle \nabla u, A^t \nabla v\right\rangle + \langle a_0 u, v\rangle = a(u, v).\end{aligned}$$

It follows from Lemma 2.1 that $Lu = f$. □

Observe that the operator $L^{\#}$ defined on $H_0^1(\Omega)$ by

$$L^{\#} u = \ell^* u,$$

where $\ell^* u = -\mathrm{div}\left(A^t \nabla u\right) + a_0 u$ satisfies

$$\langle Lu, v\rangle = a(u, v) = \left\langle u, L^{\#} v\right\rangle$$

for all $u, v \in H_0^1(\Omega)$.

We consider the equation

$$Lu = Fu, \tag{2.1}$$

where we make the following basic assumption on F.

(A) The domain of definition D_F of F is convex, $H^1(\Omega) \subseteq D_F \subseteq L^2(\Omega)$ and $F : D_F \to H^{-1}(\Omega)$ is a nonlinear weakly continuous (or continuous) operator with respect to the norms of $L^2(\Omega)$ and $H^{-1}(\Omega)$.

The weak continuity here means that if $\{u_n\}$ is a sequence in D_F such that $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in L^2(\Omega)$ then $\langle Fu_n, v\rangle \to \langle Fu, v\rangle$ $\forall v \in H_0^1(\Omega)$. Observe also that our settings here allow the dependence of F on u to include the gradient ∇u.

3. EXISTENCE OF SOLUTIONS

In this section we prove an existence theorem under the Assumption (A) and a coercivity condition that is required to hold on the surface of a ball of positive radius in $H_0^1(\Omega)$ (see equation (3.1) below). No other assumptions are made on the operator L. For instance, it is not required here that L be bounded below.

Assume $\{w_i\}_{i=1}^\infty$is an orthonormal basis for $H_0^1(\Omega)$ (actually, $\{w_i\}_{i=1}^\infty$ can be selected in $C_0^\infty(\Omega)$).

Theorem 3.1. *Assume* (A) *holds. Assume further that there exists a* $\rho > 0$ *such that the following coercivity condition holds:*

$$\langle Lu - Fu, u\rangle > 0 \ (< 0) \tag{3.1}$$

for all $u \in span\{w_i\}_{i=1}^\infty$ *with* $\|u\|_{H_0^1(\Omega)} = \rho$. *Then* (2.1) *has at least one solution.*

Proof. We first consider the case when the inequality in (3.1) is greater than zero. Let

$$V_m = \operatorname{span}\{w_1, w_2, \cdots, w_m\}.$$

Denote by $B_m(0,\rho)$ the open ball centered at the origin and with radius ρ in $H_0^1(\Omega)$. Consider the problem: Find $u_m \in V_m$ such that

$$\text{(Pm)} \qquad \langle Lu_m, v\rangle = \langle Fu_m, v\rangle \ \ \forall v \in V_m.$$

We show that (Pm) has a solution $u_m \in V_m$. Define the operator $T_m : \mathbb{R}^m \to \mathbb{R}^m$ as follows. Associate with each element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$, the element $u \in V_m$, where $u = \sum_{i=1}^m \alpha_i w_i$. Then $|\alpha|^2 = \|u\|^2_{H_0^1(\Omega)}$ and the association is an isometry between $\mathbb{R}^m$ and V_m. Define

$$T_m\alpha = \left(\langle (L-F)\,u, w_j\rangle\right)_{j=1}^m.$$

The weak continuity (or continuity) of F and the finite dimensionality of $\mathbb{R}^m$ imply that T_m is continuous. Furthermore, for all $u \in \partial B_m(0,\rho)$,

$$\langle T_m\alpha, \alpha\rangle = \sum_{j=1}^m \alpha_j \left\langle (L-F)\,u, w_j'\right\rangle = \langle (L-F)\,u, u\rangle > 0.$$

A consequence of Brouwer's fixed point theorem (see, e.g. [11]) then gives us that T_m has a zero α_m such that the corresponding $u_m \in \overline{B}_m(0,\rho)$.

Hence, we have a uniformly bounded sequence $\{u_m\}_{m=1}^\infty$ (in $H_0^1(\Omega)$ and in $L^2(\Omega)$) such that

$$\langle Lu_m, w_i\rangle = \langle Fu_m, w_i\rangle \quad \forall i \le m.$$

Since $\{u_m\}_{m=1}^\infty$ is weakly compact, we get a subsequence (denoted $\{u_{m_k}\}_{k=1}^\infty$) and an element $u \in H_0^1(\Omega)$ such that $u_{m_k} \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_{m_k} \to u$ in $L^2(\Omega)$ (as $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$). Since L is continuous, $Lu_{m_k} \rightharpoonup Lu$. Since F is weakly continuous (continuous), $Fu_{m_k} \rightharpoonup Fu$. ($Fu_{m_k} \to Fu$) in $H^{-1}(\Omega)$. Fixing i and taking the limit as $k \to \infty$ gives

$$\text{(3.2)} \qquad \langle Lu, w_i\rangle = \langle Fu, w_i\rangle.$$

Since i is arbitrary, equation (3.2), holds also for all $v \in D = \operatorname{span}\{w_i\}_{i=1}^\infty$. Since D is dense,

$$\langle Lu, v\rangle = \langle Fu, v\rangle \quad \forall v \in H_0^1(\Omega).$$

Therefore,

$$Lu = Fu.$$

Finally note that if the inequality in (3.1) is less than zero then we let $\widetilde{F} = -F$ and $\widetilde{L} = -L$. In this case $\left\langle \widetilde{L}u - \widetilde{F}u, u\right\rangle = -\langle Lu - Fu, u\rangle > 0.$ □

4. EXISTENCE IN THE PRESENCE OF UPPER AND LOWER SOLUTIONS

In this section we show that, if problem (2.1) has lower and upper solutions (see [19] or the definitions below), then the coercivity condition (3.1) can be traded in for a boundedness condition on the operator L. This condition is explicitly stated as follows:

There exists a $\mu \in \mathbb{R}$ such that

$$a\left(u,u\right) \geq \mu \left\|u\right\|^2_{H^1_0(\Omega)}. \tag{4.1}$$

This assumption is satisfied if, e.g., the coefficients $a_{ij}, i, j = 1, 2, \ldots, n$ satisfy the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}\left(x\right)\xi_i\xi_j \geq c_0 \left|\xi\right|^2 \quad \forall \xi \in \mathbb{R}^n,\ \forall x \in \Omega$$

for some $c_0 \in \mathbb{R}$.

In the following definition, we denote by γ_0 the trace operator (see [2]) on $H^1\left(\Omega\right)$.

Definition 4.1. A function $\alpha \in H^1\left(\Omega\right)$ is called a lower solution of (2.1) if $\ell\alpha \leq F\alpha$ and $\gamma_0\alpha \leq 0$ almost everywhere on $\partial\Omega$. A function $\beta \in H^1\left(\Omega\right)$ is called an upper solution of (2.1) if $\ell\beta \geq F\beta$ and $\gamma_0\beta \geq 0$.

Define the operator T_0 on $C^\infty_0\left(\Omega\right)$ by

$$T_0\varphi\left(x\right) = \max\left\{\varphi\left(x\right), 0\right\}. \tag{4.2}$$

Lemma 4.2. *The operator T_0 defined by* (4.2) *extends to a continuous operator from $H^1_0\left(\Omega\right)$ into itself.*

Proof. First, we observe that, for $\varphi \in C^\infty_0\left(\Omega\right)$, $|\varphi| \in H^1_0\left(\Omega\right)$. To see this, let $\psi \in C^\infty_0\left(\Omega\right)$ and set

$$\Omega^+ = \left\{x \in \Omega : \varphi\left(x\right) > 0\right\},$$
$$\Omega^- = \left\{x \in \Omega : \varphi\left(x\right) < 0\right\}$$

and

$$\Omega^0 = \left\{x \in \Omega : \varphi\left(x\right) = 0\right\}.$$

Then, Ω^+, Ω^- are open, Ω^0 is closed and

$$\begin{aligned}\int_\Omega |\varphi| \frac{\partial\psi}{\partial x_i} &= \left|\int_{\Omega^+} \varphi\frac{\partial\psi}{\partial x_i} - \int_{\Omega^-} \varphi\frac{\partial\psi}{\partial x_i}\right| \\ &= -\int_{\Omega^+} \psi\frac{\partial\varphi}{\partial x_i} + \int_{\Omega^-} \psi\frac{\partial\varphi}{\partial x_i} \\ &= -\int_\Omega \psi g,\end{aligned}$$

where,

$$g(x)=\begin{cases}\frac{\partial\varphi(x)}{\partial x_i}, & x\in\Omega^+\\ -\frac{\partial\varphi(x)}{\partial x_i} & x\in\Omega^-\\ 0 & \text{otherwise.}\end{cases}$$

Hence, $|\varphi|$ has an $L^2(\Omega)$ weak derivative. Therefore, $|\varphi|\in H_0^1(\Omega)$. Now, since $T_0\varphi=(|\varphi|+\varphi)/2$, $T_0\varphi\in H_0^1(\Omega)$. Furthermore,

$$\begin{aligned}\|T_0\varphi_1-T_0\varphi_2\|_{H_0^1(\Omega)} &\le \frac{1}{2}\||\varphi_1-\varphi_2|+(\varphi_1-\varphi_2)\|_{H_0^1(\Omega)}\\ &\le \|\varphi_1-\varphi_2\|_{H_0^1(\Omega)}.\end{aligned}$$

Finally, since the inclusion $C_0^\infty(\Omega)\subset H_0^1(\Omega)$ is dense, T_0 extends to all of $H_0^1(\Omega)$. □

Observe that if, for $\varphi\in H_0^1(\Omega)$, we set $\psi=T_0\varphi$ and Ω^+, Ω^-, Ω_0 as in the proof of the above theorem, then

$$\nabla\psi(x)=\begin{cases}\nabla\varphi(x), & x\in\Omega^+\cup\Omega_0^0\\ 0, & x\in\Omega^-,\end{cases}$$

where Ω_0^0 is the interior of Ω_0. Since $\partial\Omega_0=\partial\Omega\cup\partial\Omega^+\cup\partial\Omega^-$, it has measure zero. Therefore,

$$\begin{aligned}\int_\Omega|\nabla\psi(x)|^2\,dx &= \int_{\Omega^+\cup\Omega^-\cup\Omega_0^0}|\nabla\psi(x)|^2\,dx\\ &= \int_{\Omega^+}|\nabla\psi(x)|^2\,dx=\int_{\Omega^+}|\nabla\varphi(x)|^2\,dx\\ &\le \int_\Omega|\nabla\varphi(x)|^2\,dx.\end{aligned}$$

Hence $\|T_0\varphi\|_{H_0^1(\Omega)}\le\|\varphi\|_{H_0^1(\Omega)}$.

Lemma 4.3. *For every $u\in H_0^1(\Omega)$, $\langle Lu,T_0u\rangle\ge\mu\|T_0u\|^2_{L^2(\Omega)}$.*

Proof. Let $\varphi\in C_0^\infty(\Omega)$. Let $\Omega^+=\{x\in\Omega:\varphi(x)>0\}$. Since $T_0\varphi\in H_0^1(\Omega)$,

$$\begin{aligned}\langle L\varphi,T_0\varphi\rangle &= \langle-\operatorname{div}(A\nabla\varphi)+a_0\varphi,T_0\varphi\rangle=\langle A\nabla\varphi,\nabla T_0\varphi\rangle+\langle a_0\varphi,T_0\varphi\rangle\\ &= \int_{\Omega^+}A\nabla\varphi\cdot\nabla\varphi+\int_{\Omega^+}a_0\varphi^2=a(T_0\varphi,T_0\varphi)\ge\mu\|T_0\varphi\|^2_{H_0^1(\Omega)}.\end{aligned}$$

Next, given $u\in H_0^1(\Omega)$, let $\{\varphi_n\}$ be a sequence in $C_0^\infty(\Omega)$ converging to u in $H_0^1(\Omega)$. Since L is bounded, $L\varphi_n\to Lu$. Then, since T_0 is also continuous

$$\langle Lu,T_0u\rangle=\lim\langle L\varphi_n,T_0\varphi_n\rangle\ge\mu\lim\|T_0\varphi_n\|^2_{H_0^1(\Omega)}=\mu\|T_0u\|^2_{H_0^1(\Omega)}.$$

□

If we start with the definition of T_0 on $C^\infty\left(\overline{\Omega}\right)$, we can, similarly, show that it extends to a continuous linear operator on $H^1(\Omega)$ into itself.

Given $v \in H^1(\Omega)$, we can then define the operators T_v and T^v on $H^1(\Omega)$ by

$$\begin{aligned} T_v\varphi &= v + T_0(\varphi - v), \\ T^v\varphi &= v - T_0(v - \varphi). \end{aligned}$$

For $\alpha, \beta \in H^1(\Omega)$ such that $\alpha \le \beta$ we let $[\alpha, \beta] = \{u \in L^2(\Omega) : \alpha \le u \le \beta\}$ and define the operator $Q : H^1(\Omega) \to H^1(\Omega)$ by

$$Qu(x) := T^\beta T_\alpha u(x) = \begin{cases} \beta(x) & \text{if } u(x) > \beta(x), \\ u(x) & \text{if } \alpha(x) \le u(x) \le \beta(x), \\ \alpha(x) & \text{if } u(x) < \alpha(x). \end{cases}$$

Then, by Lemma 4.2 Q is continuous on $H^1(\Omega)$. However, it is not weakly continuous (see [26]). For this reason we need to strenghten Assumption (A) to:

(A′) The domain of definition D_F of F is convex, $H^1(\Omega) \subseteq D_F \subseteq L^2(\Omega)$ and $F : D_F \to H^{-1}(\Omega)$ is a nonlinear weakly continuous and continuous operator with respect to the norms of $L^2(\Omega)$ and $H^{-1}(\Omega)$.

Theorem 4.4. *Suppose* (A′) *holds and that problem* (2.1) *has a lower solution α and an upper solution β such that $\alpha \le \beta$. Assume further that inequality* (4.1) *holds with $\mu \ge 0$. Then* (2.1) *has a solution u such that $\alpha \le u \le \beta$.*

Proof. Choose $\lambda > \mu$ and consider the modified problem

$$Lu + \lambda u = F^* u, \tag{4.3}$$

where $F^* := (F + \lambda) Q$. Note that $Q(H^1(\Omega))$ is weakly pre-compact in $L^2(\Omega)$ since it is bounded. Since the operator $F + \lambda$ is weakly continuous, F^* is bounded (say by M) on $L^2(\Omega)$. Furthermore, for $u \in H_0^1(\Omega)$

$$\langle Lu + \lambda u - F^* u, u\rangle = a(u, u) + \lambda \|u\|^2 \ge (\mu + \lambda)\|u\|^2_{L^2(\Omega)} - M\|u\|_{L^2(\Omega)} > 0.$$

By Theorem 3.1, (4.3) has a solution $u \in H_0^1(\Omega)$. We claim that $\alpha \le u \le \beta$. If not, then either $T_\beta u > \beta$ or $T^\alpha u < \alpha$. If $T_\beta u > \beta$ let $z = u - \beta$. Observe that, since $\gamma_0\beta \ge 0$ and $\gamma_0 u = 0$, $\gamma_0 z \le 0$. Therefore, $\gamma_0 T_0 z = 0$ and $T_0 z \in H_0^1(\Omega)$. Also $\|T_0 z\|_{L^2(\Omega)} > 0$ since $T_0 z \ne 0$. Now, using Lemma 4.3 and the assumption that $\mu \ge 0$,

$$\begin{aligned} 0 &\le \langle Lz, T_0 z\rangle = \langle F^* u - \lambda u - L\beta, T_0 z\rangle \\ &= \langle F\beta - \lambda z - L\beta, T_0 z\rangle \\ &\le -\lambda\langle z, T_0 z\rangle = -\lambda\|T_0 z\|^2_{L^2(\Omega)} < 0, \end{aligned}$$

a contradiction. We get a similar contradiction if $T^\alpha u_m < \alpha$ holds. □

5. THE QUASILINEARIZATION METHOD

In this section we discuss the QSL method for (2.1). Let $\mathcal{K} \subset H^1(\Omega)$ be the cone of nonnegative functions. For $\xi, \eta \in H^{-1}(\Omega)$ we will say that $\xi \preceq \eta$ if

$$\langle \xi, z \rangle \leq \langle \eta, z \rangle \quad \forall z \in \mathcal{K}.$$

A similar meaning is given to the relation $\xi \succeq \eta$. If $\xi, \eta \in L^2(\Omega)$ then the notions $\xi \preceq \eta$ and $\xi \leq \eta$ coincide. We start with the following definition of p-convex operators.

Definition 5.1. Suppose $C \subset L^2(\Omega)$ is a convex set and $J : C \to H^{-1}(\Omega)$. We will say that J is p-convex if

$$J((1-\theta)u + \theta v) \preceq (1-\theta) Ju + \theta Jv. \tag{5.1}$$

for all $\theta \in [0,1]$, $u, v \in C$.

We can easily show that (see also [4], Theorem 4.3.16) an operator $J : C \to H^{-1}(\Omega)$ with Gateaux derivative J' is p-convex if and only if

$$Ju \succeq Jv + J'u(v-u)$$

for all $u, v \in C$.

From this point on we will assume that (2.1) has a lower solution $\alpha_0$and an upper solution β_0 such that $\alpha_0 \leq \beta_0$. Let $C = [\alpha_0, \beta_0]$ and let $J : D_F \to L^2(\Omega)$ be weakly continuous, p-convex on C and has two weakly continuous Fréchet derivatives J', J''. (For example, J could be of the form $Ju = \gamma(\langle u, u_0 \rangle)^2 w_0$, where $w_0 \geq 0, \gamma > 0$.) Observe that, considered as an operator on $H^1(\Omega)$ into $H^{-1}(\Omega)$, J is still weakly continuous and with two weakly continuous Fréchet derivatives. Define the weakly continuous operators $\Phi : D_F \to H^{-1}(\Omega)$ and $G : D_F \times D_F \to H^{-1}(\Omega)$ by

$$\Phi u = Ju - Fu,$$

and

$$G(u,v) = Fv + J'v(u-v) - [\Phi u - \Phi v].$$

Suppose that, for a given $v \in D_F$, u is a solution of

$$Lu = G(u,v).$$

Then it is straightforward to check that u is a lower solution of (2.1). On the other hand, if β is an upper solution of (2.1) then it is also an upper solution of the above equation. Consider the differential equation

$$Lu = G(u, \alpha_0). \tag{5.2}$$

Since α_0, β_0 are lower and upper solutions of (5.2), respectively, Theorem 4.4 tells us that (5.2) has a solution $\alpha_1 \in C$. Repeating the process with α_0 replaced by α_1 and so on we obtain a sequence $\alpha_k \in C_k := [\alpha_{k-1}, \beta_0] \cap H_0^1(\Omega)$, $k = 1, 2, \ldots$.

Theorem 5.2. *Assume* (A) *holds. Assume further that* (4.1) *holds with* $\mu > 0$ *and that* $\alpha_0, \beta_0 \in H^1(\Omega)$ *are lower and upper solutions , respectively of* (2.1) *such that* $\alpha_0 \leq \beta_0$. *Then the sequence* $\{\alpha_k\}$ *generated as discussed above converges in* $L^2(\Omega)$ *to a solution of* (2.1).

Moreover, if F *is monotone decreasing, with range in* $L^2(\Omega)$, *then*

- **(a):** $\{\alpha_k\}$ *converges in* $H_0^1(\Omega)$ *and*
- **(b):** *the convergence in* $L^2(\Omega)$ *is quadratic.*

Proof. The sequence $\{\alpha_k\}$ converges in $L^2(\Omega)$ to a function $\alpha \in C$. For any $v \in H_0^1(\Omega)$,

$$\mu \langle \nabla\alpha_k, \nabla v\rangle \leq a(\alpha_k, v) = \langle L\alpha_k, v\rangle = \langle G(\alpha_k, \alpha_{k-1}), v\rangle .$$

Since G is bounded on C, $\{\alpha_k\}$ is bounded in $H_0^1(\Omega)$. Therefore, (a subsequence) $\alpha_k \rightharpoonup \beta \in H_0^1(\Omega)$. Furthermore, the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ implies that (a subsequence) $\alpha_k \to \beta$ in $L^2(\Omega)$. By uniqueness of the limit, $\beta = \alpha$. By Corollary 2.2, (a subsequence) $L\alpha_k \rightharpoonup L\alpha$. Hence, for any $v \in H_0^1(\Omega)$,

$$\langle L\alpha, v\rangle = \lim \langle L\alpha_k, v\rangle = \lim \langle G(\alpha_k; \alpha_{k-1}), v\rangle = \langle F\alpha, v\rangle .$$

Thus, $L\alpha = F\alpha$.

To show (a), observe first that if F has its range in $L^2(\Omega)$, then the same is true for G and both functions are still norm bounded, say by M, on C. Now

$$\begin{aligned} \mu\langle \nabla\alpha_k - \nabla\alpha, \nabla\alpha_k - \nabla\alpha\rangle &\leq a(\nabla\alpha_k - \nabla\alpha, \nabla\alpha_k - \nabla\alpha) \\ &= \langle L(\alpha_k - \alpha), \alpha_k - \alpha\rangle \\ &= \langle G(\alpha_k; \alpha_{k-1}) - F\alpha, \alpha_k - \alpha\rangle \\ &\leq 2M \|\alpha_k - \alpha\|_{L^2(\Omega)} . \end{aligned}$$

This shows that $\alpha_k \to \alpha$ in $H_0^1(\Omega)$.

To show (b), observe first that J has the representation

$$Ju = Jv + J'v(u - v) + \int_0^1 (1-\tau) J''(v + \tau(u - v))(u - v)^2 \, d\tau$$

for all $u, v \in D_F$. Using this representation for J, the operator G can be written as

$$G(u, v) = Fu - \int_0^1 (1-\tau) J''(v + \tau(u - v))(u - v)^2 \, d\tau.$$

Let $e_k = \alpha - \alpha_k$. Then

$$\begin{aligned} Le_k &= F\alpha - G(\alpha_k, \alpha_{k-1}) \\ &= F\alpha - F\alpha_k + \int_0^1 (1-\tau) J''(\alpha_{k-1} + \tau(\alpha_k - \alpha_{k-1}))(\alpha_k - \alpha_{k-1})^2 \, d\tau. \end{aligned}$$

Therefore, using the monotonicity assumption on F, we get

$$\begin{aligned}\langle Le_k, e_k\rangle &\leq \int_0^1 (1-\tau)^2 \left\|J''\left(\alpha_{k-1}+\tau\left(\alpha_k-\alpha_{k-1}\right)\right)\right\| \left\|\alpha_k-\alpha_{k-1}\right\|_{L^2(\Omega)}^2 \left\|e_k\right\|_{L^2(\Omega)} d\tau \\ &\leq \frac{1}{6}\max_{u\in C}\left\|J''(u)\right\| \left\|\alpha_k-\alpha_{k-1}\right\|_{L^2(\Omega)}^2 \left\|e_k\right\|_{L^2(\Omega)} \\ &\leq \frac{1}{6}\max_{u\in C}\left\|J''(u)\right\| \left\|e_{k-1}\right\|_{L^2(\Omega)}^2 \left\|e_k\right\|_{L^2(\Omega)}.\end{aligned}$$

On the other hand, using the Poincaré inequality, we have

$$\begin{aligned}\left\|e_k\right\|_{L^2(\Omega)}^2 &\leq A\left\|e_k\right\|_{H_0^1(\Omega)}^2 \leq \frac{A}{\mu} a\left(e_k, e_k\right) \\ &= \frac{A}{\mu}\langle Le_k, e_k\rangle \leq A_1 \left\|e_{k-1}\right\|_{L^2(\Omega)}^2 \left\|e_k\right\|_{L^2(\Omega)}.\end{aligned}$$

This establishes the quadratic convergence of the iterates. □

We strengthen the assumptions on F to:

(B): In addition to (A), assume further that F has two weakly continuous Fréchet derivatives on C and $F' \preceq 0$.

Let B be the set

$$B = \{u \in \mathcal{K} : \|u\| = 1\}.$$

Lemma 5.3. *There exists an operator $J : D_F \to L^2(\Omega)$ such that $J''uvw \geq 0$ and $F''uvw - J''uvw \preceq 0$ for all $u \in C$ and $v, w \in \mathcal{K}$.*

Proof. Let S be a linear, finite rank and positive [14] operator on $H^1(\Omega)$ (for example, S could be an orthogonal projection in the direction of a positive function $u_0 \in H^1(\Omega)$). Assume that the cone $S\mathcal{K}$ lies strictly on one side of a supporting hyperplane $\mathcal{M}$ in $S\mathcal{H}$. Since SB is compact and $\mathcal{M}$ is closed, $d(SB, \mathcal{M}) = \delta > 0$. Let z_1 be a unit vector in $SH^1(\Omega)$ that is normal to the hyperplane $\mathcal{M}$. Then

$$\mathcal{M} = \left\{z \in SH^1(\Omega) : \langle z_1, z\rangle = 0\right\}.$$

Since $S\mathcal{K}$ lies on one side of $\mathcal{M}$ we may assume that $\langle z_1, z\rangle \geq 0$ for all $z \in S\mathcal{K}$. For any $u \in B$,

$$0 < \delta \leq d(Su, \mathcal{M}) = \langle Su, z_1\rangle.$$

Define the operator $J : D_F \to L^2(\Omega)$ by

$$Ju = \gamma\left(\langle Su, z_1\rangle\right)^2 S^* z_1.$$

Then $J'uv = 2\gamma\langle Su, z_1\rangle\langle Sv, z_1\rangle S^* z_1$ and $J''uvw = 2\gamma\langle Sw, z_1\rangle\langle Sv, z_1\rangle S^* z_1$. Now for $v, w, z \in B$ and $u \in C$,

$$\begin{aligned}\langle F''uvw - J''uvw, z\rangle &\leq M - 2\gamma\langle Sw, z_1\rangle\langle Sv, z_1\rangle\langle z_1, Sz\rangle \\ &\leq M - 2\gamma\delta^3 \leq 0\end{aligned}$$

for sufficiently large γ. For general nonzero vectors $v, w, z \in \mathcal{K}$, the result follows by writing

$$\langle F''uvw - J''uvw, z\rangle = \|v\|^2 \|w\|^2 \|z\| \left\langle F''u\frac{v}{\|v\|}\frac{w}{\|w\|} - J''u\frac{v}{\|v\|}\frac{w}{\|w\|}, \frac{z}{\|z\|}\right\rangle.$$

□

Assume now that (B) holds and choose an operator $J : D_F \to L^2(\Omega)$ with the properties stated in Lemma 5.3. Let $\Phi = F - J$. Then the inequalities $J''u \succeq 0$ and $\Phi''(u) \preceq 0$ easily give $Ju \succeq Jv + J'v(u-v)$ and $\Phi u \preceq \Phi v + \Phi' v(u-v)$, which in turn give

$$Fu \succeq Fv + [J'v + \Phi' u](u-v) \tag{5.3}$$

for all $u, v \in C$. For given $\alpha, \beta \in D_F$ consider the linear problems

$$Lu = G(u; \alpha, \beta), \tag{I}$$

$$Lu = D(u; \alpha, \beta) \tag{II}$$

where $G(u;\alpha,\beta) = F\alpha + [J'\alpha + \Phi'\beta](u-\alpha)$ and $D(u;\alpha,\beta) = F\beta + [J'\alpha + \Phi'\beta](u-\beta)$. Let us verify that a solution u of (I) is a lower solution of (II). Using (5.3) repeatedly, and that Φ' is decreasing and J' is increasing on C with respect to the relation $\preceq$, we have

$$\begin{aligned} Lu &= F\alpha + [J'\alpha + \Phi'\beta](u-\alpha) \\ &\preceq Fu \preceq F\beta - [J'u + \Phi'\beta](\beta - u) \\ &\preceq F\beta - [J'\alpha + \Phi'\beta](\beta-u) = D(u;\alpha,\beta). \end{aligned}$$

Similarly, we can show that a solution u of (II) is an upper solution of (I). Also, α_0, β_0 are lower and upper solutions, respectively, of both (I) and (II). Theorem 4.4 then tells us that the equation

$$Lu = G(u; \alpha_0, \beta_0)$$

has a solution $\alpha_1 \in [\alpha_0, \beta_0] \cap H_0^1(\Omega)$ and that the equation

$$Lu = D(u; \alpha_1, \beta_0)$$

has a solution $\beta_1 \in [\alpha_1, \beta_0] \cap H_0^1(\Omega)$. Repeating this process with α_0, β_0 replaced by α_1, β_1 and so on, we obtain two sequences $\{\alpha_k\}, \{\beta_k\}$ in $H_0^1(\Omega)$ of solutions of (the appropriate adjustments of) (I), (II), respectively, with

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le \beta_k \le \cdots \le \beta_1 \le \beta_0 \ \forall k.$$

Theorem 5.4. *Assume* (B) *holds. Assume also that* (4.1) *holds with* $\mu > 0$ *and* $\alpha_0, \beta_0 \in H^1(\Omega)$ *are lower and upper solutions, respectively, of* (2.1) *such that* $\alpha_0 \leq \beta_0$. *Then the sequences* $\{\alpha_k\}, \{\beta_k\}$ *generated as discussed above converge in* $L^2(\Omega)$ *to a solution of* (2.1).

Moreover, if F *has its range in* $L^2(\Omega)$, *then*

1. $\{\alpha_k\}, \{\beta_k\}$ *converge in* $H_0^1(\Omega)$;
2. *the convergence in* $L^2(\Omega)$ *is quadratic.*

Proof. The Lebesgue dominated convergence theorem gives $\alpha_k \to \alpha$ and $\beta_k \to \beta$ in $L^2(\Omega)$. As in the proof of Theorem 5.2, we can show that $\alpha, \beta \in H_0^1(\Omega)$, $L\alpha = F\alpha$ and $L\beta = F\beta$. It follows from the strong positivity of L and assumption (B) that $\alpha = \beta$. To show the quadratic rate of convergence, define the error functions $e_k = \alpha - \alpha_k$ and $r_k = \beta_k - \alpha$. Then

$$\begin{aligned} F\alpha - G(\alpha_k; \alpha_{k-1}, \beta_{k-1}) &= F\alpha - \{F\alpha_{k-1} + [J'\alpha_{k-1} + \Phi'\beta_{k-1}](\alpha_k - \alpha_{k-1})\} \\ &= \{J\alpha - J\alpha_{k-1} - J'\alpha_{k-1}(\alpha - \alpha_{k-1})\} + \{\Phi\alpha - \Phi\alpha_{k-1} - \Phi'\beta_{k-1}(\alpha - \alpha_{k-1})\} \\ &\quad + [J'\alpha_{k-1} + \Phi'\beta_{k-1}](\alpha - \alpha_k) + [J'\alpha_{k-1} + \Phi'\alpha_{k-1}] e_k. \end{aligned}$$

Now

$$J\alpha - J\alpha_{k-1} - J'\alpha_{k-1}(\alpha - \alpha_{k-1}) = \int_0^1 (1-\tau) J''(\alpha_{k-1} + \tau e_{k-1}) e_{k-1}^2 d\tau$$

and

$$\begin{aligned} \Phi\alpha - \Phi\alpha_{k-1} - \Phi'\beta_{k-1}(\alpha - \alpha_{k-1}) &= \int_0^1 (1-\tau) \Phi''(\alpha_{k-1} + \tau e_{k-1}) e_{k-1}^2 d\tau \\ &\quad - \{\Phi'\beta_{k-1} - \Phi'\alpha_{k-1}\} e_{k-1} \leq -\{\Phi'\beta_{k-1} - \Phi'\alpha_{k-1}\} e_{k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} F\alpha - G(\alpha_k; \alpha_{k-1}, \beta_{k-1}) &\leq \int_0^1 (1-\tau) J''(\alpha_{k-1} + \tau e_{k-1}) e_{k-1}^2 d\tau \\ &\quad - \{\Phi'\beta_{k-1} - \Phi'\alpha_{k-1}\} e_{k-1} + [J'\alpha_{k-1} + \Phi'\alpha_{k-1}] e_k \\ &= \int_0^1 (1-\tau) J''(\alpha_{k-1} + \tau e_{k-1}) e_{k-1}^2 d\tau \\ &\quad - \int_0^1 (1-\tau) \Phi''(\alpha_{k-1} + \tau(\beta_{k-1} - \alpha_{k-1}))(\beta_{k-1} - \alpha_{k-1}) e_{k-1} d\tau \\ &\quad + F'\alpha_{k-1} e_k \\ &\leq \int_0^1 (1-\tau) \big\{ J''(\alpha_{k-1} + \tau e_{k-1}) e_{k-1}^2 \\ &\quad - \Phi''(\alpha_{k-1} + \tau(\beta_{k-1} - \alpha_{k-1}))(e_{k-1} + r_{k-1}) e_{k-1} \big\} d\tau. \end{aligned}$$

Since $e_k \geq 0$, using the boundedness of J'', Φ'' on C, we have

$$\begin{aligned}\langle Le_k, e_k\rangle &\leq \int_0^1 (1-\tau)\Big\langle J''(\alpha_{k-1}+\tau e_{k-1})\, e_{k-1}^2 \\ &\qquad - \Phi''(\alpha_{k-1}+\tau(\beta_{k-1}-\alpha_{k-1}))\,(e_{k-1}+r_{k-1})\, e_{k-1}, e_k\Big\rangle d\tau \\ &\leq M\left(\|e_{k-1}\|^2 + \|e_{k-1}\|\,\|r_{k-1}\|\right)\|e_k\| \\ &\leq M\left(\|e_{k-1}\|^2 + \|r_{k-1}\|^2\right)\|e_k\|.\end{aligned}$$

Since L is bounded below by μ,

$$\|e_k\| \leq \mu^{-1} M\left(\|e_{k-1}\|^2 + \|r_{k-1}\|^2\right).$$

Similarly,

$$\|r_k\| \leq \mu^{-1} M\left(\|e_{k-1}\|^2 + \|r_{k-1}\|^2\right).$$

This establishes the quadratic convergence of the iterates. □

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