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HARDY-KNOPP-TYPE INEQUALITIES ON TIME SCALES

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ABSTRACT. This paper deals with a time scale version of the Hardy-Knopp-Type and the two-dimensional Hardy-Knopp-type inequalities. Moreover, Hardy inequality for several functions is presented on time scales.

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1. INTRODUCTION

In a note published in 1920, Hardy [6] stated that if $a > 0$, $p > 1$, $f(x) \geq 0$ and $\int_a^\infty f^p(x)dx$ is convergent, then

$$\int_a^\infty \left[\frac{1}{x}\int_a^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(x)dx. \tag{1.1}$$

In [7] Hardy stated and proved that the result above hold in fact in the following more precise form:

$$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,\ p > 1. \tag{1.2}$$

This inequality is usually called Hardy's inequality in the literature and it has later on been extensively studied and used as a model example for the investigation of more general integral inequalities.

In [8] Kaijser pointed out that inequality (1.1) is just special case of the much more general Hardy-Knopp-type inequality for positive functions f,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \leq \int_0^\infty \Phi\left(f(x)\right)\frac{dx}{x} \tag{1.3}$$

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where Φ is a convex function on $(0,\infty)$. By choosing $\Phi(u) = u^p$ we find that (1.3) implies Hardy inequality in particular form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p \frac{dx}{x} \le \int_0^\infty f^p(x)\frac{dx}{x},\ p>1, \tag{1.4}$$

which can be rewritten in the usual form

$$\int_0^\infty \left[\frac{1}{x}\int_0^x g(t)dt\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x)dx,\ p>1, \tag{1.5}$$

where $g(x) = f(x^{\frac{p-1}{p}})x^{-\frac{1}{p}}$.

Čižmešija [5] has given a strengthened Hardy-Knopp-type inequality which can be written as the following inequality: Suppose $0 < b \le \infty$, $u : (0,b) \to \mathbb{R}$ is a nonnegative function such that the function $x \to \frac{u(x)}{x^2}$ is locally integrable in $(0,b)$, and the function v is defined by

$$v(t) = t\int_t^b \frac{u(x)}{x^2}dx, \quad t \in (0,b).$$

If the real-valued function Φ is convex on (a,c), where $-\infty \le a \le c \le \infty$, then the inequality

$$\int_0^b u(x)\Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^b v(x)\Phi\left(f(x)\right)\frac{dx}{x} \tag{1.6}$$

holds for all integrable functions $f : (0,b) \to \mathbb{R}$, such that $f(x) \in (a,c)$ for all $x \in (0,b)$, and so Čižmešija obtained generalization of Hardy-Knopp-type inequality (1.3). Moreover, Kaijser obtained a multidimensional Hardy-Knopp-type inequality in [9]. On the other hand, Bougoffa has given Hardy integral inequality involving many functions in [4].

Recently, Řehák [11] has given the time scale version of the Hardy inequality as follows:

$$\int_a^\infty \left(\frac{\int_a^{\sigma(t)} f(s)\Delta s}{\sigma(t)-a}\right)^p \Delta t < \left(\frac{p}{p-1}\right)^p \int_a^\infty (f(t))^p\,\Delta t, \tag{1.7}$$

where $p > 1$, f is a nonnegative function.

The aim of this paper is to extend a Hardy-Knopp-type inequality to an arbitrary time scale. A one-dimensional and a two-dimensional version are established. In particular, extensions of Fubini's theorem and Jensen's inequality are utilized. Moreover, Hardy inequality for several functions is presented on time scales.

We first briefly introduce the time scales theory. By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$ with order and topological structure present in canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale; then two mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called the jump operators. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say t is *right-scattered*. If $\rho(t) < t$, $t \in \mathbb{T}$, we say t is *left-scattered*. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say t is *right-dense*. If $\rho(t) = t$, $t \in \mathbb{T}$, we say t is *left-dense*.

A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous if*

(i) f is continuous at each right-dense point or maximal point of $\mathbb{T}$;

(ii) at each left-dense point $t \in \mathbb{T}$,

$$\lim_{s \to t^-} g(s) = g(t^-)$$

exists.

The set of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Let

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that for any given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \leq \varepsilon \left| \sigma(t) - s \right|$$

for all $s \in U$. In this case $f^\Delta(t)$ is called the *delta derivative of* $f(t)$ at t. If f is differentiable at each $t \in \mathbb{T}$, then f is called *delta differentiable* on $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$, and in this case, we define the integral of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

for all $a, b \in \mathbb{T}$, and we say that f is integrable on $\mathbb{T}$.

Also let us recall some essentials about partial derivatives on time scales:

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. For $i = 1, 2$ let σ_i, ρ_i and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively, on $\mathbb{T}_i$. Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half-closed bounded interval in $\mathbb{T}_2$. Let us introduce a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b) \times [c, d) = \{(t_1, t_2) : t_1 \in [a, b), t_2 \in [c, d)\}.$$

Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. At $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we say that f has a "Δ_1 partial derivative" $f^{\Delta_1}(t_1, t_2)$ (with respect to t_1) if for each $\varepsilon > 0$ there exists a neighborhood U_{t_1}, (open in the relative topology of $\mathbb{T}_1$), of t_1 such that

$$\left|f(\sigma_1(t_1), t_2) - f(s, t_2) - f^{\Delta_1}(t_1, t_2)(\sigma_1(t_1) - s)\right| \leq \varepsilon \left|\sigma_1(t_1) - s\right|$$

for all $s \in U_{t_1}$. At $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we say that f has a "Δ_2 partial derivative" $f^{\Delta_2}(t_1, t_2)$ (with respect to t_2) if for each $\varepsilon > 0$ there exists a neighborhood U_{t_2}, of t_2 such that

$$\left|f(t_1, \sigma_2(t_2)) - f(t_1, l) - f^{\Delta_2}(t_1, t_2)(\sigma_2(t_2) - l)\right| \leq \varepsilon \left|\sigma_2(t_2) - l\right|$$

for all $l \in U_{t_2}$.

Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. The function f is called *rd-continuous* in t_2 if for every $\alpha_1 \in \mathbb{T}_1$, the function $f(\alpha_1, t_2)$ is *rd-continuous* on $\mathbb{T}_2$. The function f is called *rd-continuous* in t_1 if for every $\alpha_2 \in \mathbb{T}_2$, the function $f(t_1, \alpha_2)$ is *rd-continuous* on $\mathbb{T}_1$.

Let CC_{rd} denote the set of functions $f(t_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the properties

• f is *rd-continuous* in t_1,

• f is *rd-continuous* in t_2,

• if $(x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1 right-dense or maximal and x_2 right-dense or maximal, then f is continuous at (x_1, x_2),

• if x_1 and x_2 are both left-dense, then the limit of $f(t_1, t_2)$ exists as (t_1, t_2) approaches (x_1, x_2) along any path in the region

$$R_{LL}(x_1, x_2) = \{(t_1, t_2) : t_1 \in [a, x_1] \cap \mathbb{T}_1,\ t_2 \in [c, x_2] \cap \mathbb{T}_2\}.$$

Let CC_{rd}^1 be the set of all functions in CC_{rd} for which both the Δ_1 partial derivative and the Δ_2 partial derivative exist and are in CC_{rd}.

In order to obtain our results, we need the following theorem in [3].

Theorem 1.1 *(The standard calculus version of Fubini's Theorem).* Let f be bounded and delta integrable over R and suppose that the single integral

$$I(t) = \int_c^d f(t, s)\Delta_2 s$$

exists for each $t \in [a,b)$. Then the iterated integral

$$\int_a^b I(t)\Delta_1 t = \int_a^b \Delta_1 t \int_c^d f(t,s)\Delta_2 s$$

exists and the equality

$$\iint_R f(t,s)\Delta_1 t\Delta_2 s = \int_a^b \Delta_1 t \int_c^d f(t,s)\Delta_2 s \tag{1.8}$$

holds.

It is evident from the Theorem 1.1 that we can interchange the roles t and s, that is, we may assume the existence of the double integral and existence of the single integral

$$K(s) = \int_a^b f(t,s)\Delta_1 t$$

for each $s \in [c,d)$. Then the theorem will state the existence of the iterated iterated integral

$$\int_c^d K(s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,s)\Delta_1 t$$

and the equality

$$\iint_R f(t,s)\Delta_1 t\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,s)\Delta_1 t. \tag{1.9}$$

If together with the double integral $\iint_R f(t,s)\Delta_1 t\Delta_2 s$ there exist both single integrals I and K, then the formulas (1.8) and (1.9) will hold simultaneously, i.e.,

$$\int_a^b \Delta_1 t \int_c^d f(t,s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,s)\Delta_1 t.$$

We refer the reader to [2] for a comprehensive development of the calculus of the Δ derivative and we refer the reader to [1,3] for an account of the calculus of the partial derivative and double integral.

2. A STRENGTHENED HARDY-KNOPP-TYPE INEQUALITY

Throughout this section, we suppose that $\mathbb{T}$ is a particular time scale, $0 \le a < b \le \infty$ are points in $\mathbb{T}$. Thus our result reads as follows.

Theorem 2.1. Suppose $u \in C_{rd}([a,b),\mathbb{R})$ is a nonnegative function such that the delta integral $\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x$ exists as a finite number, and the function v is defined by

$$v(t) = (t-a)\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x,\ t \in [a,b).$$

If $\Phi : (c,d) \to \mathbb{R}$ is continuous and convex, where $c, d \in \mathbb{R}$, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \le \int_a^b v(x)\Phi\left(f(x)\right)\frac{\Delta x}{x-a} \tag{2.1}$$

holds for all delta integrable functions $f \in C_{rd}([a,b),\mathbb{R})$ such that $f(x) \in (c,d)$.

Proof. Let $f : [a,b) \to \mathbb{R}$ is rd-continuous function with values in (c,d). Applying Jensen's inequality [2] and Fubini's Theorem [1,3] we obtain

$$\begin{aligned}\int_a^b u(x)\Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} &\le \int_a^b u(x)\left(\int_a^{\sigma(x)}\Phi(f(t))\Delta t\right)\frac{\Delta x}{(x-a)(\sigma(x)-a)}\\ &= \int_a^b \Phi(f(t))\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x\Delta t\\ &= \int_a^b v(t)\Phi\left(f(t)\right)\frac{\Delta t}{t-a}\end{aligned}$$

and the proof is complete. □

Now, we give some applications of Theorem 2.1.

Corollary 2.1. If the weighted function u is chosen to be $u(x) \equiv 1$, in Theorem 2.1, then we have

$$v(x) = \begin{cases} (x-a)\int_x^b \frac{\Delta t}{(t-a)(\sigma(t)-a)} = 1 - \frac{x-a}{b-a}, & b < \infty \\ 1, & b = \infty \end{cases}$$

so in the case when $b < \infty$ inequality (2.1) reads

$$\int_a^b \Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \le \int_a^b \left(1-\frac{x-a}{b-a}\right)\Phi\left(f(x)\right)\frac{\Delta x}{x-a}, \tag{2.2}$$

while for $b = \infty$ it becomes

$$\int_a^\infty \Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \le \int_a^\infty \Phi\left(f(x)\right)\frac{\Delta x}{x-a}. \tag{2.3}$$

Corollary 2.2. Let $p > 1$ be a constant, a function f be a nonnegative and such that the delta integral $\int_a^b f^p(x)\frac{\Delta x}{x-a}$ exists as a finite number. If the convex function Φ is chosen to be $\Phi(x) = x^p$ in inequality (2.2), then we have

$$\int_a^b \left(\frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\Delta t \right)^p \frac{\Delta x}{x-a} \leq \int_a^b \left(1 - \frac{x-a}{b-a}\right) f^p(x) \frac{\Delta x}{x-a}, \tag{2.4}$$

unless $f \equiv 0$.

So we find that inequality (2.2) implies a time scale version of Hardy's inequality in the particular form inequality (2.4). Meanwhile we obtain time scale versions of (1.3) and (1.6).

Corollary 2.3. Let a function f be a nonnegative and such that the delta integral $\int_a^b f(x)\frac{\Delta x}{x-a}$ exists as a finite number. If the convex function Φ is chosen to be $\Phi(x) = e^x$ and by replacing $f(x)$ with $\ln f(x)$ in inequality (2.2), then we have

$$\int_a^b \exp\left(\frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} \ln f(t)\Delta t \right) \frac{\Delta x}{x-a} \leq \int_a^b f(x) \frac{\Delta x}{x-a}, \tag{2.5}$$

unless $f \equiv 0$.

3. A TWO-DIMENSIONAL HARDY-KNOPP-TYPE INEQUALITY

Throughout this section, we suppose that

(a) $\mathbb{T}_1$ is a time scale, $0 \leq a < b$ are points in $\mathbb{T}_1$,

(b) $\mathbb{T}_2$ is a time scale, $0 \leq c < d$ are points in $\mathbb{T}_2$,

(c) R is a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ defined by

$$R = [a,b) \times [c,d) = \{(t,s) : t \in [a,b), s \in [c,d)\}.$$

In order to obtain our result in this section, we need the following theorem.

Theorem 3.1 *(Jensen's inequality).* Let $t, s \in R$ and $-\infty \leq m < n \leq \infty$. If $f \in CC^1_{rd}(R,(m,n))$ and $\Phi : (m,n) \to \mathbb{R}$ is convex, then

$$\Phi\left(\frac{\int_a^b \int_c^d f(t,s)\Delta_1 t \Delta_2 s}{\int_a^b \int_c^d \Delta_1 t \Delta_2 s} \right) \leq \frac{\int_a^b \int_c^d \Phi\left(f(t,s)\right)\Delta_1 t \Delta_2 s}{\int_a^b \int_c^d \Delta_1 t \Delta_2 s}. \tag{3.1}$$

Proof. This theorem is a direct extension of the Theorem 6.17 in [2]. □

Theorem 3.2. Let R be a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ and f be a delta integrable function over R and $f \in CC^1_{rd}(R, \mathbb{R})$ such that $-\infty < \alpha < f(t,s) < \beta < \infty$. If Φ is convex and positive function on (α, β), then

$$\int_a^b \int_c^d \Phi\left(\frac{1}{(\sigma(x)-a)(\tau(y)-c)} \int_a^{\sigma(x)} \int_c^{\tau(y)} f(t,s)\Delta_1 t \Delta_2 s\right) \frac{\Delta_1 x \Delta_2 y}{(x-a)(y-c)} \tag{3.2}$$
$$\leq \int_a^b \int_c^d \Phi(f(t,s)) \left(1 - \frac{t-a}{b-a}\right)\left(1 - \frac{s-c}{d-c}\right) \frac{\Delta_1 t \Delta_2 s}{(t-a)(s-c)}.$$

Proof. Let Φ be convex. Then, according to Jensen's inequality and Fubini theorem we have

$$\int_a^b \int_c^d \Phi\left(\frac{1}{(\sigma(x)-a)(\tau(y)-c)} \int_a^{\sigma(x)} \int_c^{\tau(y)} f(t,s)\Delta_1 t \Delta_2 s\right) \frac{\Delta_1 x \Delta_2 y}{(x-a)(y-c)}$$
$$\leq \int_a^b \int_c^d \left(\int_a^{\sigma(x)} \int_c^{\tau(y)} \Phi(f(t,s))\Delta_1 t \Delta_2 s\right) \frac{\Delta_1 x \Delta_2 y}{(x-a)(\sigma(x)-a)(y-c)(\tau(y)-c)}$$
$$= \int_a^b \int_c^d \Phi(f(t,s)) \left(\int_t^b \int_s^d \frac{\Delta_1 x \Delta_2 y}{(x-a)(\sigma(x)-a)(y-c)(\tau(y)-c)}\right) \Delta_1 t \Delta_2 s$$
$$= \int_a^b \int_c^d \Phi(f(t,s)) \left(1 - \frac{t-a}{b-a}\right)\left(1 - \frac{s-c}{d-c}\right) \frac{\Delta_1 t \Delta_2 s}{(t-a)(s-c)}$$

and the proof is complete. □

Corollary 3.1. Let $p > 1$ be a constant, a function f be a nonnegative on R. If the convex function Φ is chosen to be $\Phi(u) = u^p$ in Theorem 3.2, then we have

$$\int_a^b \int_c^d \left(\frac{1}{(\sigma(x)-a)(\tau(y)-c)} \int_a^{\sigma(x)} \int_c^{\tau(y)} f(t,s)\Delta_1 t \Delta_2 s\right)^p \frac{\Delta_1 x \Delta_2 y}{(x-a)(y-c)} \tag{3.3}$$
$$\leq \int_a^b \int_c^d f^p(t,s) \left(1 - \frac{t-a}{b-a}\right)\left(1 - \frac{s-c}{d-c}\right) \frac{\Delta_1 t \Delta_2 s}{(t-a)(s-c)}.$$

unless $f \equiv 0$.

Corollary 3.2. Let a function f be a nonnegative on R. If the convex function Φ is chosen to be $\Phi(u) = e^u$ and replacing f by $\ln f$ in Theorem 3.2, then we have

$$\int_a^b \int_c^d \exp\left(\frac{1}{(\sigma(x)-a)(\tau(y)-c)} \int_a^{\sigma(x)} \int_c^{\tau(y)} \ln f(t,s)\Delta_1 t \Delta_2 s\right) \frac{\Delta_1 x \Delta_2 y}{(x-a)(y-c)} \tag{3.4}$$
$$\leq \int_a^b \int_c^d f(t,s)\left(1-\frac{t-a}{b-a}\right)\left(1-\frac{s-c}{d-c}\right)\frac{\Delta_1 t \Delta_2 s}{(t-a)(s-c)}.$$

unless $f \equiv 0$.

4. TIME SCALE HARDY INTEGRAL INEQUALITY FOR SEVERAL FUNCTIONS

Our purpose in this section is to prove the Hardy integral inequality for several functions on time scale.

Theorem 4.1. Let $a \geq 0$ and $f_1, f_2, \ldots, f_n$ be nonnegative integrable functions. Denote $F_k(t) := \int_a^t f_k(s)\Delta s$, $k = 1, 2, \ldots, n$. Then

$$\int_a^\infty \left(\frac{F_1^\sigma(t)F_2^\sigma(t)\ldots F_n^\sigma(t)}{(\sigma(t)-a)^n}\right)^{\frac{p}{n}} \Delta t < \left(\frac{p}{np-n}\right)^p \int_a^\infty (f_1(t)+f_2(t)+\cdots+f_n(t))^p \Delta t. \tag{4.1}$$

Proof. By using Jensen's inequality [10]

$$(F_1^\sigma(t)F_2^\sigma(t)\cdots F_n^\sigma(t))^{\frac{1}{n}} \leq \frac{\sum_{k=1}^n F_k^\sigma(t)}{n} \tag{4.2}$$

and so,

$$(F_1^\sigma(t)F_2^\sigma(t)\cdots F_n^\sigma(t))^{\frac{p}{n}} \leq \frac{\left(\sum_{k=1}^n F_k^\sigma(t)\right)^p}{n^p}. \tag{4.3}$$

Divide both sides of (4.3) by $(\sigma(t)-a)^p$ and integrate resulting the inequality to get

$$\int_a^\infty \left(\frac{F_1^\sigma(t)F_2^\sigma(t)\cdots F_n^\sigma(t)}{(\sigma(t)-a)^n}\right)^{\frac{p}{n}} \Delta t \leq \frac{1}{n^p}\int_a^\infty \left(\frac{F_1^\sigma(t)+F_2^\sigma(t)+\cdots+F_n^\sigma(t)}{\sigma(t)-a}\right)^p \Delta t. \tag{4.4}$$

Applying inequality (1.7) to the right hand side of (4.4), we obtain

$$\int_a^\infty \left(\frac{F_1^\sigma(t)F_2^\sigma(t)\cdots F_n^\sigma(t)}{(\sigma(t)-a)^n}\right)^{\frac{p}{n}} \Delta t < \left(\frac{p}{np-n}\right)^p \int_a^\infty (f_1(t)+f_2(t)+\cdots+f_n(t))^p \Delta t.$$

□

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