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QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS SATISFYING A GENERAL LIPSCHITZ CONDITION

E. O. AYOOLA

Mathematics Section, The Abdus Salam
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT. We establish further results concerning the existence and non-uniqueness of solutions of quantum stochastic differential inclusions in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus. Our results are established by considering a general Lipschitz condition on the coefficients of the inclusion. We present examples of continuous multivalued maps satisfying the general Lipschitz condition in the sense of this paper.

Key Words: Fock space, Exponential vectors, Lipschitzian quantum stochastic differential inclusions

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1. INTRODUCTION

Some very important preoccupations of classical analysis are the numerical and analytical characterizations of solutions of classical differential inclusions defined in finite dimensional Euclidean spaces. Indeed the existence and non uniqueness of solutions of such inclusions have been thoroughly investigated (see, for example, [1, 11, 13, 16]). Indeed, many features of reachable sets, the solution sets and their selection theorems have been studied to a great extent [6, 7, 11, 13, 15, 16].

However, in the non commutative quantum setting, the situation is different. The analysis of quantum stochastic differential inclusions (QSDI) concerns quantum stochastic processes as solutions that live in certain infinite dimensional locally convex spaces. In addition, there are several locally convex operator topologies that may be defined on the space of such processes arising from several theories of noncommutative stochastic integration. There are several variants of topological conditions depending on the underlying properties of the locally convex spaces of the integrands that may be required of the coefficients of the quantum stochastic differential inclusions. The objective of this paper is to further investigate the existence and non-uniqueness of

Permanent Address: Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

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solutions of quantum stochastic differential inclusions of the form:

$$\begin{aligned} X(t) \in X_0 + \int_0^t \big(&E(s,X(s))d\wedge_\pi(s) + F(s,X(s))dA_f(s) \\ &+ G(s,X(s))dA_g^+(s) + H(s,X(s))ds\big), \quad t \in [0,T], \end{aligned} \tag{1.1}$$

under a Lipschitz condition that generalizes similar condition employed in [8]. In the framework of the Hudson and Parthasarathy [12, 14] formulation of quantum stochastic calculus, we consider a more general class of Lipschitzian coefficients E, F, G, H appearing in (1.1). The Lipschiz condition in [8] is a special case of the present formulation. The integral in (1.1) is understood in the sense of Hudson and Parthasarathy [12] and the maps f, g, π belong to appropriate function spaces as described in [8]. The integrator processes $\wedge_\pi$, A_f^+, A_g are the gauge, creation and annihilation processes associated with the basic field operators of quantum field theory. In [8], under the Lipschitz condition of that paper, the existence of solutions and the equivalent form of QSDI (1.1) have been established. We establish a wider class of Lipschitzian QSDI (1.1) to cover some important multivalued maps that are Lipschitzian in the general sense of this paper. This class of maps was not covered by the notion of Lipschitz maps due to [8]. In particular, we present a class of Lipschitzian multivalued maps associated with the space of continuous endomorphisms of the locally convex space of our quantum stochastic processes as an important example of multivalued maps satisfying the Lipschitz condition in our sense. This work therefore extends the class of QSDI investigated in [2, 3, 4, 5, 8]. We remark that a very strong motivation for studying QSDI (1.1) among others, concerns the need for sufficient information and knowledge about the dynamics and fluctuations of the systems described by discontinuous quantum stochastic differential equations which may be reformulated as regularized QSDI. QSDI of the form (1.1) plays a central role in quantum stochastic control theory and quantum dynamical systems (see [3, 4, 8]).

The rest of the paper is organized as follows: We present in Section 2, the description of some very important relevant spaces, some fundamental assumptions and some results. Our main results concerning the existence, and non-uniqueness of solutions of QSDI (1.1) are established in Section 3.

2. PRELIMINARY RESULTS AND ASSUMPTIONS

Our framework in this paper relies largely on the formulation in [8, 9, 10]. Detailed definitions of various spaces that appear below can be found in [8]. In what follows, γ is a fixed Hilbert space, $\mathbb{D}$ is an inner product space with $\mathcal{R}$ as its completion, and $\Gamma(L^2_\gamma(\mathbb{R}_+))$ is the Boson Fock Space determined by the function space $L^2_\gamma(\mathbb{R}_+)$. The set $\mathbb{E}$ is the subset of the Fock space generated by the exponential vectors. If $\mathcal{N}$ is a topological space, then we denote by $clos(\mathcal{N})$ (resp. $comp(\mathcal{N})$), the family of all nonempty closed subsets of $\mathcal{N}$ (resp. compact members of $clos(\mathcal{N})$).

In our formulations, quantum stochastic processes are $\tilde{\mathcal{A}}$-valued maps on $[0,T]$. The space $\tilde{\mathcal{A}}$ is the completion of the linear space

$$\mathcal{A} = L_W^+(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$

endowed with the locally convex operator topology generated by the family of seminorms $\{x \to \|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|,\ \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$. Here, $\mathcal{A}$ consists of linear operators from $\mathbb{D} \underline{\otimes} \mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ with the property that the domain of the operator adjoint contains $\mathbb{D} \underline{\otimes} \mathbb{E}$. We adopt the notation and the definitions of Hausdorff topology on $clos(\tilde{\mathcal{A}})$ as explained in [8, 9, 10].

For any pair of $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ such that $\eta = c \otimes e(\alpha)$,$\xi = d \otimes e(\beta)$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$, we shall in what follows, employ the equivalent form of (1.1) established in [8] and given by the nonclassical ordinary differential inclusion:

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi), \\ &X(0) = X_0, \quad t \in [0, T]. \end{aligned} \tag{2.1}$$

The multivalued map P appearing in (2.1) is of the form

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi\rangle$$

where the map $P_{\alpha\beta} : [0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ is given by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : [0, T] \to \mathbb{C}$ are defined by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma,\ \nu_\beta(t) = \langle f(t), \beta(t)\rangle_\gamma,$$

$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\gamma,\ t \in [0, T]$$

for all $(t, x) \in [0, T]\times\tilde{\mathcal{A}}$ and the coefficients E, F, G, H belong to the space $L^2_{loc}([0, T]\times \tilde{\mathcal{A}})_{mvs}$ of multivalued stochastic processes with closed values.

As explained in [8], the map P cannot in general be written in the form:

$$P(t, x)(\eta, \xi) = \tilde{P}(t, < \eta, x\xi >)$$

for some complex valued multifunction defined on $[0, T] \times \mathbb{C}$, for $t \in [0, T]$, $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2.1. (a) Let Fin$[A]$ denote the family of all finite subsets of a nonempty set A. For $x \in \mathcal{A}$, and $\Theta \in \text{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]$, define $\|x\|_\Theta$ by

$$\|x\|_\Theta = \max_{(\eta,\xi)\in\Theta} \|x\|_{\eta\xi}. \tag{2.2}$$

Then, the set $\{\|\cdot\|_\Theta : \Theta \in \text{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]\}$ is a family of seminorms on $\mathcal{A}$. We denote by τ the topology generated by this family of seminorms and we let $\tilde{\mathcal{A}}'$ represents the completion of the topological space $(\mathcal{A}, \tau)$.

(b) Let $I = [0, T] \subseteq \mathbb{R}_+$. A multivalued map $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ will be called Lipschitzian if for any pair $(\eta, \xi) \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$, the map satisfies an estimate of the type

$$\rho_{\eta\xi}\left(\Phi(t,x), \Phi(t,y)\right) \leq K^{\Phi}_{\eta\xi}(t)\|x-y\|_{\Theta_\Phi(\eta,\xi)} \tag{2.3}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$ and where $K^{\Phi}_{\eta\xi} : I \to (0, \infty)$ lies in $L^1_{loc}(I)$ and Θ_Φ is a map from $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ into $\text{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]$. Similar definition holds for a map of the form $\Phi : I \times \tilde{\mathcal{A}} \to clos(\mathbb{C})$ where the Hausdorff metric $\rho(\cdot,\cdot)$ on $clos(\mathbb{C})$ replaces the pseudo metric $\rho_{\eta\xi}(\cdot,\cdot)$ on $clos(\tilde{\mathcal{A}})$ (see [8]).

Remark. In [8], the map $(\eta, \xi) \to \Theta_\Phi(\eta, \xi)$ that appears in (2.3) is just the identity map. Let $L(\tilde{\mathcal{A}})$ denote the linear space of all continuous endomorphisms of $\tilde{\mathcal{A}}$. Then the above definition enables us to exhibit a class of Lipschitzian multivalued maps which are continuous from the space $\mathbb{R}_+ \times \tilde{\mathcal{A}}$ to the Hausdorff topological space $\left(clos(\tilde{\mathcal{A}}), \tau_H\right)$. The multivalued maps in this class are not Lipschitzian in the sense of [8].

Theorem 2.2. Let $A : \mathbb{R}_+ \to L(\tilde{\mathcal{A}})$ be a single valued map on $\mathbb{R}_+$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and a fixed closed ball $S \in comp(\tilde{\mathcal{A}})$ with centre at the origin, define for any $x \in \tilde{\mathcal{A}}$,

$$F(t,x) = \|A(t)x\|_{\eta\xi} S.$$

Then the map $(t,x) \to F(t,x)$ is Lipschitzian.

Proof: For $x,\ y \in \tilde{\mathcal{A}},\ t \in \mathbb{R}_+$, we employ some basic results similar to Lemma (II.1.5) and Corollary (II.1.2) in [13] as follows:

$$\begin{aligned}
\rho_{\eta\xi}\left(F(t,x), F(t,y)\right) &= \rho_{\eta\xi}\left(\|A(t)x\|_{\eta\xi}S, \|A(t)y\|_{\eta\xi}S\right) \\
&\leq |\|A(t)x\|_{\eta\xi} - \|A(t)y\|_{\eta\xi}|\rho_{\eta\xi}\left(S, \{0\}\right) \\
&\leq \|A(t)x - A(t)y\|_{\eta\xi}\rho_{\eta\xi}\left(S,\ \{0\}\right) \\
&= \|A(t)(x-y)\|_{\eta\xi}\rho_{\eta\xi}\left(S, \{0\}\right) \\
&\leq \|S\|_{\eta\xi} C^A_{\eta\xi}(t)\|x-y\|_{\Theta_A(\eta,\xi)} \\
&= K^F_{\eta\xi}(t)\|x-y\|_{\Theta_A(\eta,\xi)},
\end{aligned}$$

where $\|S\|_{\eta\xi} = \rho_{\eta\xi}\left(S, \{0\}\right)$, $K^F_{\eta\xi}(t) = \|S\|_{\eta\xi}C^A_{\eta\xi}(t)$, Θ_A is a map from $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ into $\text{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ and $C^A_{\eta\xi}(t)$ is a positive function depending on the map $A(t)$ and elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The continuity of the multivalued map $(t,x) \to F(t,x)$ follows from the last inequality.

Remark. (a) Since Θ is a finite set, we see that $\|x\|_\Theta = \|x\|_{\eta'\xi'}$, for some $(\eta', \xi') \in \Theta$. Thus, in what follows, we employ in the proof of our main results the fact that a map $\Phi\ :\ I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ is Lipschitzian if given any $(\eta, \xi) \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$, there corresponds

$(\eta',\xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ such that

$$\rho_{\eta\xi}(\Phi(t,x),\Phi(t,y)) \leq K^{\Phi}_{\eta\xi}(t)\|x-y\|_{\eta'\xi'} \tag{2.4}$$

for all $x, y \in \tilde{\mathcal{A}}$ and $t \in I$.

(b) By the definition of the map $(t,x) \to P(t,x)(\eta,\xi)$ that appears in (2.1), and by the remark above, it is straightforward to show that if the coefficients of (1.1) are Lipschitzian in the sense of (2.4), then the complex valued multifunction $(t,x) \to P(t,x)(\eta,\xi)$ is also Lipschitzian. That is, there exists $(\eta',\xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ such that for all $x, y \in \tilde{\mathcal{A}}$,

$$\rho(P(t,x)(\eta,\xi),P(t,y)(\eta,\xi)) \leq K^{P}_{\eta\xi}(t)\|x-y\|_{\eta'\xi'} \tag{2.5}$$

where the map $K^P_{\eta\xi} : [0,T] \to \mathbb{R}_+$ lies in $L^1_{loc}([0,T])$ and $\rho(\cdot,\cdot)$ is the Hausdorf distance function on $clos(\mathbb{C})$.

(c) Using the definition in (a), we see that if $P : \mathbb{R}_+ \to comp(\tilde{\mathcal{A}})$ such that $P(t)$ is a closed ball with centre at the origin and $(\eta',\xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ is a fixed point, then the multivalued map F defined by

$$F(t,x) = |\langle\eta', x\xi'\rangle| P(t)$$

is Lipschitzian. This follows, since for any $t \in \mathbb{R}_+$, $x, y, \in \tilde{\mathcal{A}}$,

$$\begin{aligned}\rho_{\eta\xi}(F(t,x),F(t,y)) &= \rho_{\eta\xi}\left(\langle\eta', x\xi'\rangle P(t), \langle\eta', y\xi'\rangle P(t)\right)\\ &\leq |\ \|x\|_{\eta'\xi'} - \|y\|_{\eta'\xi'}\ |\rho_{\eta\xi}\left(P(t),\{0\}\right) \leq \|P(t)\|_{\eta\xi}\|x-y\|_{\eta'\xi'},\end{aligned}$$

where

$$\|P(t)\|_{\eta\xi} = \rho_{\eta\xi}(P(t),\{0\}).$$

3. EXISTENCE AND NON UNIQUENESS OF SOLUTIONS

Subject to the conditions below, we shall establish the existence and non-uniqueness of solutions of QSDI (1.1) in this section. By a solution of (1.1) we mean a quantum stochastic process $\Phi : [0,T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (1.1).

In what follows, we consider, without loss of generality, quantum stochastic processes and the related inclusions defined on the interval $[0,1]$. We employ the notion of adaptedness of quantum stochastic processes as explained in [8]. In connection with the subsequent results, we list the following statements and assumptions.

$(\mathcal{S}_{(1)})$ $Z : [0,1] \to \tilde{\mathcal{A}}$ is a stochastic process in $Ad(\tilde{\mathcal{A}})_{wac}$ with the property that for each pair $\eta,\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and almost all $t \in [0,1]$, there exists a positive function $W_{\eta\xi}(t)$ lying in $L^1_{loc}([0,1])$ such that

$$d\left(\frac{d}{dt}\langle\eta, Z(t)\xi\rangle, P(t,Z(t))(\eta,\xi)\right) \leq W_{\eta\xi}(t)$$

$(\mathcal{S}_{(2)})$ $\gamma > 0$ is an arbitrary but fixed number and $Q_{Z,\gamma}$ is the set

$$Q_{Z,\gamma} = \{(t,x) \in [0,1] \times \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \leq \gamma,\ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

$(\mathcal{S}_{(3)})$ Each of the coefficients E, F, G, H appearing in (1.1) is Lipschitzian from $Q_{Z,\gamma}$ to the Hausdorf topological space $(clos(\tilde{\mathcal{A}}), \tau_H)$, i.e, for each $M \in \{E, F, G, H\}$ there exists a positive map $K^M_{\eta\xi} : [0,1] \to \mathbb{R}_+$ lying in $L^1_{loc}([0,1])$ corresponding to each pair η, ξ such that

$$\rho_{\eta\xi}\left(M(t,x), M(t,y)\right) \leq K^M_{\eta\xi}(t)\|x - y\|_{\Theta_M(\eta,\xi)}$$

for some map

$$\Theta_M : (\mathbb{D}\underline{\otimes}\mathbb{E})^2 \to \ \mathrm{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2].$$

$(\mathcal{S}_{(4)})$ For each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$\delta_{\eta\xi} \equiv \|x_0 - Z(0)\|_{\eta\xi} \quad \text{and} \quad \delta_{\eta\xi} \leq \gamma.$$

$(\mathcal{S}_{(5)})$

$$R_{\eta\xi} := \max(\delta_{\eta\xi},\ W_{\eta\xi})$$

for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ where

$$W_{\eta\xi} = ess \sup_{[0,1]} W_{\eta\xi}(t).$$

$(\mathcal{S}_{(6)})$ For any countably infinite sequence of points $\{(\eta_n, \xi_n) \subseteq (\mathbb{D}\underline{\otimes}\mathbb{E})^2, n = 1, 2, \dots\}$,

$$\sup_{n\in\mathbb{N}} \left\{ ess \sup_{t\in[0,1]} K^P_{\eta_n\xi_n}(t) \right\} < \infty.$$

$(\mathcal{S}_{(7)})$ $\{L_{\eta_j\xi_j}\}_{j=1}^{j=\infty}$ is a sequence of positive real numbers indexed by a countably infinite sequence of elements $\{(\eta_j, \xi_j)\}_{j=1}^{\infty} \subseteq (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ that depends on an arbitrary pair $(\eta, \xi) \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and defined as follows:

$$L_{\eta_1\xi_1} = R_{\eta_1\xi_1}$$

and

$$L_{\eta_j\xi_j} := ess \sup_{[0,1]} K^P_{\eta_j\xi_j}(t), \quad j \geq 2.$$

$(\mathcal{S}_{(8)})$ From $(\mathcal{S}_{(7)})$ above, we set

$$L_{\eta\xi,n} = \max_{j=1,2,\dots,n}\{L_{\eta_j\xi_j}\} \ \text{ and } \ L_{\eta\xi} = \sup_{n\in\mathbb{N}}\{L_{\eta\xi,n}\}.$$

$(\mathcal{S}_{(9)})$ For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $t \in [0,1]$, we define

$$\mathcal{E}_{\eta\xi}(t) = 2L_{\eta\xi} + 2L_{\eta\xi}\int_0^t \left(K^P_{\eta\xi}(s)e^{L_{\eta\xi}s}\right) ds,$$

where the constant $L_{\eta\xi}$ is given by $\mathcal{S}_{(8)}$ above.

$(\mathcal{S}_{(10)})$ J is the subset of the interval $[0,1]$ defined by

$$J = \{t \in [0,1] : \mathcal{E}_{\eta\xi}(t) \leq \gamma,\ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

Next we present a proposition which is useful for the proof of the existence result that follows.

Proposition 3.1. Let $\{\Phi_i\}_{i=1}^{\infty}$ be a sequence of weakly absolutely continuous maps from $[0,1]$ to $\tilde{\mathcal{A}}$ which satisfy the following conditions:
(i) $(t,\Phi_i(t)) \in Q_{Z,\gamma},\ i \geq 1,$ for almost all $t \in J$.
(ii) There exists a sequence $\{V_i\}_{i=1}^{\infty}$ such that for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and a constant $L_{\eta\xi} > 0$,
(a) $\Phi_i(t) = X_0 + \int_0^t V_{i-1}(s)ds,\ \ i \geq 1$
(b) $|\frac{d}{dt}\langle\eta,\Phi_i(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_{i-1}(t)\xi\rangle| \leq 2L_{\eta\xi}^{i-1}K_{\eta\xi}^{P}(t)\frac{t^{i-2}}{(i-2)!}$, for almost all $t \in J$. Then,
(c) $\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq 2L_{\eta\xi}\int_0^t K_{\eta\xi}^{P}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!}ds,\ t \in J,\ i \geq 2.$

Proof. Let (i) and (ii) hold. Then

$$\begin{aligned}
\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} &= |\int_0^t \langle\eta,(V_{i-1}(s) - V_{i-2}(s))\xi\rangle ds|, \text{ by (ii)(a)}\\
&= |\int_0^t \{\frac{d}{ds}\langle\eta,\Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta,\Phi_{i-1}(s)\xi\rangle\}ds|, \text{ by (ii)(a)}\\
&\leq \int_0^t |\frac{d}{ds}\langle\eta,\Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta,\Phi_{i-1}(s)\xi\rangle|ds\\
&\leq 2L_{\eta\xi}^{i-1}\int_0^t K_{\eta\xi}^{P}(s)\frac{s^{i-2}}{(i-2)!}ds\\
&= 2L_{\eta\xi}\int_0^t K_{\eta\xi}^{P}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!}ds, \quad t \in J, \quad i \geq 2, \quad \text{ by (ii)(b).}
\end{aligned}$$

This concludes the proof.

Next, we present our result on the existence of solution of QSDI (1.1) subject to the conditions $(\mathcal{S}_{(1)}) - (\mathcal{S}_{(10)})$ above. The result shall be established by employing a similar line of argument as in the proof of Theorem (8.2) in [8].

Theorem 3.2. Suppose that the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(10)}$ hold and the coefficients E,F,G,H are continuous from $[0,1]\times\tilde{\mathcal{A}}$ to $\big(clos(\tilde{\mathcal{A}}),\tau_H\big)$.
Then there exists a solution Φ of (1.1) such that

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq \mathcal{E}_{\eta\xi}(t),\ \ t \in J, \tag{3.1}$$

and

$$|\frac{d}{dt}\langle\eta,\Phi(t)\xi\rangle - \frac{d}{dt}\langle\eta,Z(t)\xi\rangle| \leq L_{\eta\xi}\left(1 + 2K_{\eta\xi}^{P}(t)e^{L_{\eta\xi}t}\right). \tag{3.2}$$

Proof. In what follows, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ are arbitrary elements. Our proof will be established by constructing a Cauchy sequence $\{\Phi_n\}_{n\geq 0}$ in $\tilde{\mathcal{A}}$ of successive approximations of Φ in such a way that the sequence $\{\frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle\}$ is also Cauchy in the field of complex numbers.

Define $\Phi_0(t) = Z$, then Φ_0 is adapted. By Theorem (1.14.2) in [1] (See also [8]), there exists a measurable selection $V_0(\cdot)(\eta,\xi) \in P(\cdot,\Phi_0(\cdot))(\eta,\xi)$ such that

$$\begin{aligned} &|V_0(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| \\ &\qquad = d\left(\frac{d}{dt}\langle \eta, \Phi_0(t)\rangle, P(t,\Phi_0(t))(\eta,\xi)\right) \leq W_{\eta\xi}(t). \end{aligned} \tag{3.3}$$

As the map $(\eta,\xi) \to V_0(t)(\eta,\xi)$ is a sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$, for almost all $t \in J$, then there exists $V_0(t) \in \tilde{\mathcal{A}}$ such that $V_0(t)(\eta,\xi) = \langle \eta, V_0(t)\xi\rangle$. Since $V_0(\cdot)(\eta,\xi)$ is locally absolutely integrable, then $V_0 \in L^1_{loc}(\tilde{\mathcal{A}})$.

Next we define

$$\Phi_1(t) = X_0 + \int_0^t V_0(s)ds, \quad t \in J.$$

As $V_0(t) \in \tilde{\mathcal{A}}$ for almost all $t \in J$, it follows that $\Phi_1(t) \in \tilde{\mathcal{A}}_t$, i.e Φ_1 is adapted.

Furthermore, for $t \in J$,

$$\begin{aligned} \|\Phi_1(t) - \Phi_0(t)\|_{\eta\xi} &\leq \|X_0 - \Phi_0(t_0)\|_{\eta\xi} + \int_0^t |V_0(s)(\eta,\xi) - \frac{d}{ds}\langle \eta, \Phi_0(s)\xi\rangle| ds \\ &\leq \delta_{\eta\xi} + \int_0^t W_{\eta\xi}(s)ds \end{aligned} \tag{3.4}$$

Notice that by (3.3),

$$|\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| \leq W_{\eta\xi}(t). \tag{3.5}$$

Again there exists a measurable selection $V_1(\cdot)(\eta,\xi) \in P(\cdot,\Phi_1(\cdot))(\eta,\xi)$ such that

$$\begin{aligned} |V_1(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle| &= d\left(\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle,\ P(t,\Phi_1(t))(\eta,\xi)\right) \\ &\leq \rho\left(P(t,\Phi_0(t))(\eta,\xi),\ P(t,\Phi_1(t))(\eta,\xi)\right) \\ &\leq K^P_{\eta\xi}(t)\|\Phi_0(t) - \Phi_1(t)\|_{\eta_1\xi_1} \\ &\leq K^P_{\eta\xi}(t)\left(\delta_{\eta_1\xi_1} + \int_0^t W_{\eta_1\xi_1}(s)ds\right), \end{aligned} \tag{3.6}$$

for some $\eta_1,\xi_1 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ that depend on η,ξ.

By a similar argument as for the existence of $V_0(\cdot)$, there exists $V_1 \in L^1_{loc}(\tilde{\mathcal{A}})$ such that for almost all $t \in J$,

$$V_1(t)(\eta,\xi) = \langle \eta, V_1(t)\xi\rangle.$$

Next we define,

$$\Phi_2(t) = X_0 + \int_0^t V_1(s)ds, \quad t \in J.$$

Again, $\Phi_2(t) \in \tilde{\mathcal{A}}_t$ since $V_1(t) \in \tilde{\mathcal{A}}$ for almost all $t \in J$, i.e. Φ_2 is adapted.

Furthermore, for $t \in J$,

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} = \|\int_0^t \left(V_1(s) - V_0(s)\right) ds\|_{\eta\xi}$$

$$= |\int_0^t \langle \eta, (V_1(s) - V_0(s))\,\xi\rangle ds|$$
$$\leq \int_0^t |\langle \eta, V_1(s)\xi\rangle - \langle \eta, V_0(s)\xi\rangle| ds$$
$$\leq \int_0^t \rho\left(P(s,\Phi_1(s))(\eta,\xi),\ P(s,\Phi_0(s))(\eta,\xi)\right) ds$$
$$\leq \int_0^t K^P_{\eta\xi}(s)\|\Phi_1(s) - \Phi_0(s)\|_{\eta_1\xi_1} ds$$

By applying (3.4), we have the estimate

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} \leq \int_0^t \left(K^P_{\eta\xi}(s)\left[\delta_{\eta_1\xi_1} + \int_0^s W_{\eta_1\xi_1}(r)dr\right]\right) ds. \tag{3.7}$$

We may write (3.6) as

$$|\frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle| \leq K^P_{\eta\xi}(t)\left(\delta_{\eta_1\xi_1} + \int_0^t W_{\eta_1\xi_1}(r)dr\right). \tag{3.8}$$

Continuing the procedure, there exists a measurable selection $V_2(\cdot)(\eta,\xi) \in P(\cdot, \Phi_2(\cdot))(\eta,\xi)$ and a pair of elements $\eta_2,\ \xi_2 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ depending on η, ξ such that

$$|V_2(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle| = d\left(\frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle,\ P(t,\Phi_2(t))(\eta,\xi)\right)$$
$$\leq \rho\left(P(t,\Phi_1(t))(\eta,\xi),\ P(t,\Phi_2(t))(\eta,\xi)\right)$$
$$\leq K^P_{\eta\xi}(t)\|\Phi_1(t) - \Phi_2(t)\|_{\eta_2\xi_2}$$
$$\leq K^P_{\eta\xi}(t)\int_0^t \left(K^P_{\eta_2\xi_2}(s)\left[\delta_{\eta_1\xi_1} + \int_0^s W_{\eta_1\xi_1}(r)dr\right]\right) ds, \tag{3.9}$$

on account of (3.7).

Again, (3.9) may be written as

$$|\frac{d}{dt}\langle \eta, \Phi_3(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle|$$
$$\leq K^P_{\eta\xi}(t)\delta_{\eta_1\xi_1}\int_0^t K^P_{\eta_2\xi_2}(s)ds + K^P_{\eta\xi}(t)\int_0^t K^P_{\eta_2\xi_2}(s)\int_0^s W_{\eta_1\xi_1}(r)dr\, ds \tag{3.10}$$

As before, it is straightforward to show that there exist $V_3, V_2 \in L^1_{loc}(\tilde{\mathcal{A}})$ defining adapted processes $\Phi_3,\ \Phi_4$ for $t \in J$ by

$$\Phi_3(t) = X_0 + \int_0^t V_2(s)ds,\ t \in J$$
$$\Phi_4(t) = X_0 + \int_0^t V_3(s)ds,\ t \in J \tag{3.11}$$

and satisfy the following inequalities

$$\|\Phi_3(t)-\Phi_2(t)\|_{\eta\xi}$$
$$\leq \int_0^t \left(K^P_{\eta\xi}(s) \left[\int_0^s K^P_{\eta_2\xi_2}(s^{'}) \left[\delta_{\eta_1\xi_1} + \int_0^{s^{'}} W_{\eta_1\xi_1}(r)dr \right] ds^{'} \right] \right) ds$$
$$= \int_0^t K^P_{\eta\xi}(s) \int_0^s \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s^{'}) ds^{'} ds$$
$$+ \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_2\xi_2}(s') \int_0^{s'} W_{\eta_1\xi_1}(r)dr\, ds' ds. \tag{3.12}$$

and

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi}$$
$$\leq \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_3\xi_3}(s') \int_0^{s'} \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s'') ds'' ds' ds$$
$$+ \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_3\xi_3}(s') \int_0^{s'} K^P_{\eta_2\xi_2}(s'') \int_0^{s''} W_{\eta_1\xi_1}(r)dr ds^{''} ds^{'} ds. \tag{3.13}$$

Furthermore

$$|\frac{d}{dt}\langle \eta, \Phi_4(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_3(t)\xi\rangle| \leq K^P_{\eta\xi}(t) \int_0^t K^P_{\eta_3\xi_3}(s) \int_0^s \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s') ds' ds$$
$$+ K^P_{\eta\xi}(t) \int_0^t K^P_{\eta_3\xi_3}(s) \int_0^s K^P_{\eta_2\xi_2}(s') \int_0^{s'} W_{\eta_1\xi_1}(r) dr ds' ds \tag{3.14}$$

so that from (3.13) and (3.14)

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi} \leq \int_0^t K^P_{\eta\xi}(s) \int_0^s L_{\eta_3\xi_3} \int_0^{s'} \delta_{\eta_1\xi_1} L_{\eta_2\xi_2} ds'' ds' ds$$
$$+ \int_0^t K^P_{\eta\xi}(s) \int_0^s L_{\eta_3\xi_3} \int_0^{s'} L_{\eta_2\xi_2} \int_{t_0}^{s''} W_{\eta_1\xi_1} dr\, ds'' ds' ds$$
$$\leq 2L^3_{\eta\xi} \int_0^t K^P_{\eta\xi}(s) \frac{s^2}{2} ds,$$

and

$$|\frac{d}{dt}\langle \eta, \Phi_4(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_3(t)\xi\rangle|$$
$$\leq K^P_{\eta\xi}(t) \left[\delta_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \int_0^t \int_0^s ds' ds \right.$$
$$\left. + W_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \int_0^t \int_0^s \int_0^{s'} dr\, ds' ds \right]$$
$$= K^P_{\eta\xi}(t) \left[\delta_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \frac{t^2}{2} + W_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \frac{t^3}{6} \right]$$
$$\leq K^P_{\eta\xi}(t) \left[L^3_{\eta\xi,3} \frac{t^2}{2} + L^3_{\eta\xi,3} \frac{t^3}{6} \right] \leq 2K^P_{\eta\xi}(t) L^3_{\eta\xi} \frac{t^2}{2}, \qquad t \in [0,1]. \tag{3.15}$$

Indeed, there exists a sequence $\{\Phi_i\}_{i\geq 0}$ of weakly absolutely continuous processes from $[0,1]$ to $\tilde{\mathcal{A}}$ satisfying the hypothesis (i) and (ii) of Proposition (3.1) and hence its conclusion.

To prove this claim, we assume that the sequence $\{\Phi_i\}$ has already been defined and satisfies the hypothesis (i) and (ii) of the proposition for $i = 0, 1, 2 \ldots, n$. We shall show that there exists a map $\Phi_{n+1} : J \to \tilde{\mathcal{A}}$ for which (i) and (ii) of the proposition also hold.

Again by Theorem (1.14.2) in [1], there exists

$$V_n(\cdot)(\eta,\xi) \in P(\cdot, \Phi_n(\cdot))(\eta,\xi)$$

such that

$$|\frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle - V_n(t)(\eta,\xi)| = d\left(\frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle,\ P(t,\Phi_n(t))(\eta,\xi)\right), \text{ a.e. on } J.$$

As $(\eta,\xi) \to V_n(t)(\eta,\xi)$ is a sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$, for almost all $t \in J$, there exist $V_n \in L^1_{loc}(\tilde{\mathcal{A}})$ such that

$$V_n(t)(\eta,\xi) = \langle \eta, V_n(t)\xi\rangle, \text{ a.e on } J$$

Define

$$\Phi_{n+1}(t) = X_0 + \int_0^t V_n(s)ds,\ t \in J.$$

Then, for some pair of elements $\eta_n,\ \xi_n \in \mathbb{D}\underline{\otimes}\mathbb{E}$ depending on $\eta,\ \xi$, we have the following estimates:

$$\begin{aligned}
|\frac{d}{dt}\langle \eta, \Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle| &= |\langle \eta, V_n(t)\xi\rangle - \langle \eta, V_{n-1}(t)\xi\rangle| \\
&\leq \rho\left(P(t,\Phi_n(t))(\eta,\xi),\ P(t,\Phi_{n-1}(t))(\eta,\xi)\right) \\
&\leq K^P_{\eta\xi}(t)\|\Phi_n(t) - \Phi_{n-1}(t)\|_{\eta_n\xi_n} \\
&\leq K^P_{\eta\xi}(t)\left[2L_{\eta\xi}\int_0^t K^P_{\eta_n\xi_n}(s)\frac{(L_{\eta\xi}s)^{n-2}}{(n-2)!}ds\right] \\
&\leq 2L^n_{\eta\xi}K^P_{\eta\xi}(t)\frac{t^{n-1}}{(n-1)!},
\end{aligned}$$

which proves (ii)(b) of Proposition (3.1).

Furthermore, for $t \in J$,

$$\begin{aligned}
\|\Phi_{n+1}(t) - \Phi_0(t)\|_{\eta\xi} &\leq \|\Phi_1(t) - \Phi_0(t)\|_{\eta\xi} + \|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} \\
&\quad + \cdots + \|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \\
&\leq 2L_{\eta\xi} + 2L_{\eta\xi}\sum_{k=0}^{n-1}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^k}{k!}ds \\
&\leq 2L_{\eta\xi}\left(1 + \int_0^t K^P_{\eta\xi}(s)e^{L_{\eta\xi}s}ds\right) \leq \gamma. \qquad (3.16)
\end{aligned}$$

This shows that $(t, \Phi_{n+1}(t)) \in Q_{Z,\gamma}$ and therefore proves (ii)(c) of Proposition 3.1. It follows that the sequence $\{\Phi_n(t)\}$ is a τ_ω-Cauchy sequence and therefore converges to some $\Phi(t) \in \tilde{\mathcal{A}}$. We conclude that $\Phi(t)$ is a solution of (1.1) for almost all $t \in J$ in the same way as in the proof of Theorem 8.2 in [8].

Finally, by using (ii)(b) of Proposition 3.1, we have the following:

$$|\frac{d}{dt}\langle\eta, \Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle| \leq |\frac{d}{dt}\langle\eta, \Phi_1(t)\xi\rangle - \frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle| + \sum_{k=0}^{n-1} 2L_{\eta\xi}K^P_{\eta\xi}(t)\frac{[L_{\eta\xi}t]^k}{k!} \leq L_{\eta\xi} + 2L_{\eta\xi}K^P_{\eta\xi}(t)e^{L_{\eta\xi}t}.$$

Taking the limit as $n \to \infty$, we obtain inequality (3.2). Similarly, inequality (3.1) follows from (3.16) above.

Corollary 3.3. Suppose that the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(10)}$ hold in the region

$$Q_{X_0,\gamma} = \{(t, x) \in [0, 1] \times \tilde{\mathcal{A}} : \|x - X_0\|_{\eta\xi} \leq \gamma\},$$

then the solution $X(t)$ of (1.1) exists on the segment.

Proof. The conditions of Theorem (3.2) will be satisfied if we set $Z(t) \equiv X_0$, a trivially adapted quasi solution, and the function

$$W_{\eta\xi}(t) = d\,(0,\ P(t, X_0)(\eta, \xi))$$

is continuous, by the continuity of the map $(t, x) \to P(t, x)(\eta, \xi)$.

Our next result shows that new solutions of QSDI (1.1) exist in some neighbourhoods of a solution. This establishes the nonuniqueness of solutions as in the case of Lipschitz differential inclusions in finite dimensional Euclidean spaces (see [1]).

Theorem 3.4. Let $\Phi_0(t)$ be a solution of problem (1.1). Suppose that in the region Q_{Φ_0,ϵ_0}, the conditions of Theorem (3.2) are satisfied with Lipschitz constant $K_{\eta\xi}$ that depends only on arbitrary elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, for some constant $\epsilon_0 > 0$.

Then for any

$$\epsilon > 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1) \tag{3.17}$$

valid for all $\eta,\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, a solution $\Phi(t)$ of QSDI (1.1) exists such that

$$\|\Phi(t) - \Phi_0(t)\|_{\eta\xi} < \epsilon, \quad \text{on } [0, 1].$$

Suppose in addition that the map $t \to \frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle$ is continuous on the interval $[0, 1]$, then there exists a constant $M_{\eta\xi} > 0$ depending on η, ξ such that

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle| < M_{\eta\xi}, \ \text{ almost all } \ t \in [0, 1]. \tag{3.18}$$

Proof. We employ an adaptation of the argument in the proof of Theorem 2 in [11] as follows:

We consider the region $Q_{\Phi_0,\epsilon}$ for ϵ_0 big enough such that $0 < \epsilon < \epsilon_0$ in view of the constraint (3.17). By the continuity of the map $(t,x) \to d\,(0, P(t,x)(\eta,\xi))$ on the region Q_{Φ_0,ϵ_0}, we have

$$\sup_{Q_{\Phi_0,\epsilon_0}} d\,(0,\; P(t,x)(\eta,\xi)) = S_{\eta\xi} < \infty.$$

Define the number

$$A_{\eta\xi} = \frac{2\epsilon}{\epsilon - 2L_{\eta\xi} - 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1)}.$$

Then in view of (3.17), $A_{\eta\xi} > 0$. Thus, by a similar reason as in [11], we can find numbers $b \geq K_{\eta\xi}\epsilon$, $b \geq S_{\eta\xi}$ such that

$$\int_B |\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| dt < \frac{\epsilon}{A_{\eta\xi}} \tag{3.19}$$

where

$$B = \{t \in [0,1] : |\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| > b\}.$$

By the argument in the proof of Theorem 3.2, since $\Phi_0(t)$ is a solution of (1.1), there exists an element $V_0 \in L^1_{loc}(\tilde{\mathcal{A}})$ such that

$$\Phi_0(t) = X_0 + \int_0^t V_0(s)ds$$

and

$$\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle = \langle \eta, V_0(t)\xi\rangle, \text{ almost all } \; t \in [0,1].$$

Next we define

$$\begin{aligned} V(t) &= V_0(t), \;\; t \in ([0,1]\backslash B) \\ &= 0, \qquad t \in B. \end{aligned}$$

and

$$Y(t) = X_0 + \int_0^t V(s)ds.$$

We note here that the process Y lies in $Ad(\tilde{\mathcal{A}})_{wac}$.

For $t \in ([0,1]\backslash B)$,

$$\langle \eta, Y(t)\xi\rangle = \langle \eta, X_0\xi\rangle + \int_0^t \langle \eta, V(s)\xi\rangle ds = \langle \eta, \Phi_0(t)\xi\rangle.$$

For $t \in B$,

$$\langle \eta, Y(t)\xi\rangle = \langle \eta, X_0\xi\rangle.$$

Therefore we have for both cases using (3.19)

$$|\langle \eta, Y(t)\xi\rangle - \langle \eta, \Phi_0(t)\xi\rangle| = \|Y(t) - \Phi_0(t)\|_{\eta\xi} \leq \frac{\epsilon}{A_{\eta\xi}},$$

and

$$d\left(\frac{d}{dt}\langle \eta, Y(t)\xi\rangle,\; P(t, Y(t))(\eta,\xi)\right) = W_{\eta\xi}(t),$$

almost everywhere on $[0,1]$.

Furthermore,

$$W_{\eta\xi}(t) \leq S_{\eta\xi} < b, \quad \text{for} \ \ t \in B, \tag{3.20}$$

For $t \in ([0,1]\backslash B)$, we have

$$\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle \in P(t, \Phi_0(t))(\eta,\xi).$$

Thus

$$W_{\eta\xi}(t) = d\left(\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle, \ P(t, Y(t))(\eta,\xi)\right) = 0 \leq K_{\eta\xi}\epsilon. \tag{3.21}$$

Hence by Theorem 3.2, there exists a solution Φ of (1.1) satisfying

$$\|\Phi(t) - Y(t)\|_{\eta\xi} \leq \mathcal{E}_{\eta\xi}(t), \quad t \in J,$$

and where $\mathcal{E}_{\eta\xi}(t)$ is given by $\mathcal{S}_{(9)}$.

By the definition of the set $B \subseteq [0,1]$, and the estimate (3.19) above, we have

$$\int_B b ds < \int_B |\frac{d}{ds}\langle \eta, \Phi_0(s)\xi\rangle| ds \leq \frac{\epsilon}{A_{\eta\xi}}. \tag{3.22}$$

By (3.19), (3.22) and $\mathcal{S}_{(9)}$, we have,

$$\begin{aligned}\mathcal{E}_{\eta\xi}(t) &< \int_B S_{\eta\xi} ds + 2L_{\eta\xi} + 2L_{\eta\xi}\int_0^t \left(K_{\eta\xi}e^{L_{\eta\xi}s}\right) ds \\ &< \frac{\epsilon}{A_{\eta\xi}} + 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\Phi(t) - \Phi_0(t)\|_{\eta\xi} &\leq \|\Phi(t) - Y(t)\|_{\eta\xi} + \|Y(t) - \Phi_0(t)\|_{\eta\xi} \\ &\leq \frac{2\epsilon}{A_{\eta\xi}} + 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1) = \epsilon.\end{aligned}$$

Again by Equation (3.2), $\Phi(t)$ satisfies

$$\begin{aligned}|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Y(t)\xi\rangle| &\leq L_{\eta\xi}(1 + 2K_{\eta\xi}e^{L_{\eta\xi}t}) \\ &\leq L_{\eta\xi}(1 + 2K_{\eta\xi}U_{\eta\xi}) := N_{\eta\xi},\end{aligned} \tag{3.23}$$

where

$$U_{\eta\xi} = \sup_{t\in[0,1]}(e^{L_{\eta\xi}t}).$$

Thus by definition, $\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = 0$ for $t \in B$ and for $t \in ([0,1]\backslash B)$,

$$\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle.$$

Putting

$$\sup_{[0,1]}|\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| = T_{\eta\xi},$$

then from (3.23)

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle| \leq N_{\eta\xi}, \quad t \in B,$$

and

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle| \leq T_{\eta\xi} + N_{\eta\xi}, \quad t \in ([0,1]\backslash B).$$

Inequality (3.18) follows by defining

$$M_{\eta\xi} = T_{\eta\xi} + N_{\eta\xi}.$$

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