Dynamic Systems and Applications 17 (2008) 515-538

# POSITIVE ALMOST AUTOMORPHIC SOLUTIONS FOR SOME NONLINEAR INFINITE DELAY INTEGRAL EQUATIONS

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**ABSTRACT.** We state sufficient conditions for the existence of positive bounded, almost automorphic or almost periodic solutions of the following nonlinear infinite delay integral equation:

$$x(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) \, ds.$$

Then we apply these results to a finite delay integral equation when the delay is time-dependent and for a delay differential equation.

**AMS (MOS) Subject Classification.** 42A75; 44A35, 45G10

## 1. INTRODUCTION

For a continuous map $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we consider the following nonlinear integral equation:

$$x(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) \, ds, \tag{1.1}$$

where $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a map such that $a(t, .)$ is nonnegative integrable function on $\mathbb{R}^+$, for each $t \in \mathbb{R}$. In this paper we give sufficient conditions for the existence of positive bounded solutions of Equation (1.1). We also treat almost periodic solutions and the almost automorphic solutions. Then we apply these results to the following finite delay integral equation:

$$x(t) = \int_{t-\sigma(t)}^{t} f(s, x(s)) \, ds, \tag{1.2}$$

when the delay is time-dependent. Also, we apply our results to the delay differential equation:

$$x'(t) + \alpha(t) x(t) = f(t, x(t-\tau)) \tag{1.3}$$

Received May 12, 2007 

where $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tau \geq 0$.

Almost automorphic functions are an extension of almost periodic functions. The notion of almost automorphy has been introduced in the literature by Bochner [8] and more recently, it was developed by N'Guerekata [22].

Similar equations were considered, notably in connection with epidemic problems, by Cooke and Kaplan [12], Nussbaum [23], Busenberg and Cooke [10], Kaplan, Sorg and Yorke [19], Leggett and Williams [20, 21], Smith [24], Guo and Lakshmikantham [18], Burton and Hatvani [9] and Ait Dads, Arino and Ezzinbi [1], all those authors are considered the periodic case. The extensions of the periodic case were treated by Fink and Gatica [17], Torrejón [26], Chen and Torrejón [11], Ait Dads et al [2, 3, 4, 5, 6], Ezzinbi and Hachimi [15], more recently by Xu and Yuan [27, 28]. All those works are concerned with almost periodic type solutions.

Ait Dads and Ezzinbi [5] state sufficient conditions for the existence of positive pseudo almost periodic solutions for the following infinite delay integral equation:

$$x(t) = \int_{-\infty}^{t} b(t-s) f(s, x(s))\, ds, \tag{1.4}$$

that is a particular case of Equation (1.1). In this work it is assumed that the function $f(t, .)$ is nondecreasing on $\mathbb{R}^+$. Then to avoid the hypothesis of monotony of the function $f(t, .)$, Xu and Yuan [28] construct a new fixed point theorem in a cone. In [28], the authors state the existence of positive almost periodic type solutions of equations (1.2) and (1.4). They do not assume that $f(t, .)$ is nondecreasing, but only that $f(t, x) = f_1(t, x) + f_2(t, x)$ where $f_1(t, .)$ (respectively $f_2(t, .)$) is nondecreasing (respectively nonincreasing). For Equation (1.2) when the delay is constant, Xu and Yuan [27] established similar results.

The purpose of our work is to state a generalization of hypotheses on the function $f(t, .)$ done in Ait Dads and Ezzinbi [5], Ezzinbi and Hachimi [15] and Xu and Yuan [28], and to extend theses results to Equation (1.1).

In the nonlinear case, Favard-type conditions ensure that an almost periodic differential equation has an almost periodic solution as soon as it has a bounded solution, under stability or Favard's separation conditions [29]. However the Favard's approach cannot be applied for the existence of an almost periodic solution of the nonlinear integral equation (1.1). The Favard's approach is generalized to the compact almost automorphic differential equation, but not to the almost automorphic case, because the convergence uniform on any compact subset of $\mathbb{R}$ which appears in the definition of the almost periodic or compact almost automorphic function, plays a crucial role in this theory. In this paper, we do not use Favard-type conditions, on the only result (Proposition 6.5) where it is applicable, because it is a corollary of the study of the existence of almost periodic or almost automorphic solution of integral equation (1.1). Moreover hypotheses of Proposition 6.5 to get the existence of bounded solution of the semilinear differential (1.3), permit us to state the almost periodicity of the bounded solution without using Favard's approach.

The paper is organized as follows: in Section 2 we recall some notations and definitions on almost periodic and almost automorphic, then we recall the main notions related on the Hilbert's projective metric. In this section we give the list of hypotheses which are used in this paper. In Section 3, we state results on the positive bounded solutions for equations (1.1) and (1.2). We treat the almost automorphic case in Section 4 and the almost periodic case in Section 5. We will compare some results of Xu and Yuan in [28] on almost periodic solutions in Section 5. Section 6 is concerned with application of these results to Equation (1.3).

## 2. NOTATION AND DEFINITIONS

2.1. **Some results on almost periodic type functions.** Let $E$ and $F$ be two metric sets, $C(E,F)$ (respectively $C_b(E,F)$) denotes the space of continuous (respectively continuous and bounded) functions defined on $E$ with values in $F$. In the particular case where $F=\mathbb{R}$, we denote $C(E,\mathbb{R})$ (respectively $C_b(E,\mathbb{R})$) by $C(E)$ (respectively $C_b(E)$). We denote by $L^\infty(\mathbb{R})$ the space of essentially bounded measurable functions in $\mathbb{R}$ and $L^1(\mathbb{R}^+)$ the Lebesgue space of order one in $\mathbb{R}^+$. Let $(X, \| \,.\, \|)$ a Banach space. Throughout the paper $X$ will be $\mathbb{R}$ or $L^1(\mathbb{R}^+)$ with the norm

$$\| g\|_{L^1(\mathbb{R}^+)} = \int_0^{+\infty} | \, g(t) \, | \; dt.$$

Let $g \in C(\mathbb{R}, X)$ (respectively $C(\mathbb{R}\times\mathbb{R}^+)$). Define the linear shift operator $\tau_s$ for some $s\in\mathbb{R}$ by $\tau_s g(t) = g(t+s)$ for each $t\in\mathbb{R}$, (respectively $\tau_s g(t,x) = g(t+s,x)$), for each $(t,x)\in\mathbb{R}\times\mathbb{R}^+$).

**Definition 2.1.** A function $g\in C(\mathbb{R},X)$ (respectively $C(\mathbb{R}\times\mathbb{R}^+)$) is called almost periodic (respectively almost periodic in $t$ uniformly with respect to $x\in\mathbb{R}^+$), if for each $\epsilon>0$ (respectively $\epsilon>0$ and compact $K\subset\mathbb{R}^+$), there exists $l_\epsilon>0$ such that every interval of length $l_\epsilon$ contains a number $\mu$ with the property that

$$\sup_{t\in\mathbb{R}} \| \, \tau_\mu g(t) - g(t)\|_X < \epsilon$$

$$\text{(respectively} \qquad \sup_{(t,x)\in\mathbb{R}\times K} | \, \tau_\mu g(t,x) - g(t,x) \, | < \epsilon).$$

Denote $AP(X)$ (respectively $AP(\mathbb{R}\times\mathbb{R}^+)$) the set of all such functions.

Every $g\in AP(X)$ possesses a *mean value*

$$M\{g(t)\}_t := \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} g(t)dt.$$

For each $\omega \in \mathbb{R}$, $a(g,\omega) := M\{g(t)e^{-i\omega t}\}_t$ is the *Fourier-Bohr coefficient* of $g$ associated at $\omega$ and $\Lambda(g) := \{\omega \in \mathbb{R}; a(g,\omega) \neq 0\}$ is the set of *exponents* of $g$. The *module* of $g$ denoted by $mod(g)$, is the additive group generated by $\Lambda(g)$. Similarly, if $g \in AP(\mathbb{R} \times \mathbb{R}^+)$, the module of $g$, denoted also by $mod(g)$, is the additive group generated by

$$\Lambda(g) := \cup_{x\in\mathbb{R}^+}\{\omega \in \mathbb{R}; M\{g(t,x)e^{-i\omega t}\}_t \neq 0\}.$$

**Theorem 2.2** (Bochner [7]). *Let $g \in C(\mathbb{R},X)$. A function $g \in AP(X)$ if and only if for any pair of sequences of real numbers $(t'_n)_n$ and $(s'_n)_n$, there exists a common subsequence of $(t'_n)_n$ and $(s'_n)_n$, denoted $(t_n)_n$ and $(s_n)_n$ such that*

$$\forall t \in \mathbb{R}, \quad \lim_{m\to+\infty}\lim_{n\to+\infty} g(t+t_n+s_m) = \lim_{n\to+\infty} g(t+t_n+s_n).$$

*The limits above mean that for each $t \in \mathbb{R}$, $h(t) = \lim_{n\to+\infty} g(t+t_n)$ is well-defined and $\lim_{n\to+\infty} g(t+t_n+s_n) = \lim_{m\to+\infty} h(t+s_m)$.*

For some preliminary results on almost periodic functions, we refer to [13, 16, 29].

**Definition 2.3.** Let $g \in C(\mathbb{R},X)$ (respectively $C(\mathbb{R},\mathbb{R}\times\mathbb{R}^+)$) is called almost automorphic (respectively almost automorphic in $t$ uniformly with respect to $x \in \mathbb{R}^+$) if for any sequence of real numbers $(t'_n)_n$, there exists a subsequence of $(t'_n)_n$, denoted $(t_n)_n$ such that for each $t \in \mathbb{R}$

$$\lim_{m\to+\infty}\lim_{n\to+\infty} g(t+t_n-t_m) = g(t)$$

$$(\text{respectively } \forall x \in \mathbb{R}^+ \quad \lim_{m\to+\infty}\lim_{n\to+\infty} g(t+t_n-t_m,x) = g(t,x)).$$

Denote $AA(X)$ (respectively $AA(\mathbb{R}\times\mathbb{R}^+)$) the set of all such functions.

**Remark 2.4.** Because the convergence is point-wise, the function

$$g^*(t) = \lim_{n\to+\infty} g(t+t_n)$$

is in $L^\infty(\mathbb{R},X)$, but it is not necessarily continuous. It is also clear from the definition above that almost periodic functions are almost automorphic.

For some details on almost automorphic functions, we refer to [22]. With these definitions, we have the following inclusions:

$$AP(X) \subset AA(X) \quad \text{and} \quad AP(\mathbb{R}\times\mathbb{R}^+) \subset AA(\mathbb{R}\times\mathbb{R}^+).$$

2.2. **Hilbert's projective metric.** Let $X$ be a real Banach space. A closed convex set $K$ in $X$ is called a convex cone if the following conditions are satisfied:

$$(i) \qquad \text{if} \quad x \in K, \text{ then } \lambda x \in K \qquad \text{for } \lambda \geq 0$$

$$(ii) \qquad \text{if} \quad x \in K \text{ and } -x \in K, \text{ then } x = 0.$$

A cone $K$ induces a partial ordering $\leq$ in $X$ by

$$x \leq y \qquad \text{if and only if} \qquad y - x \in K.$$

A cone $K$ is called *normal* if there exists a constant $k$ such that

$$0 \leq x \leq y \qquad \text{implies that} \quad \| x \| \leq k \| y \|$$

where $\| \, . \, \|$ is the norm on $X$. If $K$ is now a general cone in a Banach space $X$ and $x$ and $y$ are elements of $K^* = K - \{0\}$, we say that $x$ and $y$ are *comparable* if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha x \leq y \leq \beta x.$$

This define an equivalence relation on $K^*$ and divides $K^*$ into disjoint subsets which we call *components of* $K$. If $x$ and $y$ are comparable, we define the numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) := \sup \{\alpha > 0; \alpha x \leq y\} \tag{2.1}$$

$$M(y/x) := \inf \{\beta > 0; y \leq \beta x\}. \tag{2.2}$$

We define a metric which was introduced by Thompson [25]. If $x$ and $y \in K^*$ are comparable, define $d(x, y)$ by

$$d(x, y) := \max (\log M(y/x), \log M(x/y))$$

$$= \max(\log(M(y/x), -\log m(y/x)). \tag{2.3}$$

If $C$ is a component of $K$, one can easily prove (see [25]) that $d$ gives a metric on $C$. Moreover Thompson proves the following result.

**Theorem 2.5** (Thompson [25]). *Let $K$ be a normal cone in a Banach space $X$ and let $C$ be a component of $K$. Then $C$ is a complete metric space with respect to the metric $d$.*

**Proposition 2.6** (Thompson [25]). *Let $K$ be a normal cone in a Banach space $X$ with nonempty interior $\overset{\circ}{K}$. Then $\overset{\circ}{K}$ is a component of $K$.*

It follows that if $K$ is a normal cone with nonempty interior $\overset{\circ}{K}$, then $\overset{\circ}{K}$ is a complete metric space with respect to the metric $d$.

**Theorem 2.7** (Deimling [14])**.** *Let $E$ be a complete space with respect to the metric $d$. If $f$ be a mapping from $E$ into $E$ satisfying*

$$d(f(x), f(y)) \leq \Phi(d(x,y)) \qquad \text{for all } x \text{ and } y \in E,$$

*where $\Phi$ is a positive nondecreasing function continuous on $[0, +\infty[$ and verifying $\Phi(r) < r$ for every $r > 0$, $\Phi(0) = 0$, then $f$ has exactly one fixed point in $E$.*

Now we give a list of hypotheses which are used. From $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ and $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we formulate the following hypotheses.

**(H1)** *$f \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ and there exists $x_1 > 0$ such that $f(., x_1) \in C_b(\mathbb{R})$.*

**(H2)** *There exists a continuous map $\phi : (0,1) \longrightarrow \mathbb{R}$ satisfying $\phi(\lambda) > \lambda$ and for each $x$ and $y > 0$, $t \in \mathbb{R}$ and $\lambda \in (0,1)$, one has*

$$\lambda x \leq y \leq \lambda^{-1} x \Longrightarrow f(t,y) \geq \phi(\lambda) f(t,x).$$

**(H3)** *For each $t \in \mathbb{R}$, $a(t,.) \in L^1(\mathbb{R}^+)$ and there exists $x_2 > 0$ such that*

$$\inf_{t \in \mathbb{R}} \int_0^{+\infty} a(t,s) f(t-s, x_2)\, ds > 0.$$

**(H4)** *The function $t \mapsto a(t,.)$ is in $C_b(\mathbb{R}, L^1(\mathbb{R}^+))$.*

**(H5)** *$f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is an almost automorphic function in $t$ uniformly with respect to $x \in \mathbb{R}^+$.*

**(H6)** *The function $t \mapsto a(t,.)$ is in $AA(L^1(\mathbb{R}^+))$.*

**(H7)** *$f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is an almost periodic function in $t$ uniformly with respect to $x \in \mathbb{R}^+$.*

**(H8)** *The function $t \mapsto a(t,.)$ is in $AP(L^1(\mathbb{R}^+))$.*

## 3. EXISTENCE OF POSITIVE BOUNDED SOLUTIONS

In this section, we state some results of existence and uniqueness of the continuous and bounded solution with a positive infinimum.

**Theorem 3.1.** *Suppose that* (H1)–(H4) *hold. Then Equation* (1.1) *has a unique continuous and bounded solution on $\mathbb{R}$ with a positive infinimum.*

An easy consequence of this last result for the finite delay integral Equation (1.2) is the following result.

**Corollary 3.2.** *Suppose that* (H1) *and* (H2) *hold. In addition, we assume that*

i) *$\sigma$ is a positive continuous and bounded function on $\mathbb{R}$,*

ii) *there exists* $x_2 > 0$ *such that*

$$\inf_{t\in\mathbb{R}} \int_{t-\sigma(t)}^{t} f(s, x_2)\, ds > 0. \tag{3.1}$$

*Then Equation* (1.2) *has a unique continuous and bounded solution on* $\mathbb{R}$ *with a positive infinimum.*

*Proof.* We use Theorem 3.1 with the function $a(t,s) := 1_{[0,\sigma(t)]}(s)$ (where $1_{[0,\sigma(t)]}$ is the function defined by $1_{[0,\sigma(t)]}(s) = 1$ if $0 \le s \le \sigma(t)$ and 0 elsewhere). (H3) follows from ii). By using $\| 1_{[0,\sigma(t)]} \|_{L^1(\mathbb{R}^+)} = \sigma(t)$ and

$$\| 1_{[0,\sigma(t+\tau)]} - 1_{[0,\sigma(t)]} \|_{L^1(\mathbb{R}^+)} = | \sigma(t+\tau) - \sigma(t) |, \tag{3.2}$$

we deduce that (H4) is satisfied. □

For the proof of Theorem 3.1, we use the following lemmas.

**Lemma 3.3.** *Suppose that* (H1) *and* (H2) *hold. Then one has*

i) $\forall x, y > 0$, $\forall t \in \mathbb{R}$, $f(t,y) \ge \min\left(\frac{x}{y}, \frac{y}{x}\right) f(t,x)$.

ii) *For each* $[a,b] \subset ]0,+\infty[$, $f$ *is bounded on* $\mathbb{R}\times[a,b]$.

iii) *For each* $[a,b] \subset ]0,+\infty[$, $\exists L \ge 0$, $\forall x, y \in [a,b]$, $\forall t \in \mathbb{R}$,

$$| f(t,x) - f(t,y) | \le L \,| x - y | \,.$$

*Proof.* Let $x$ and $y > 0$. We can assume $x \neq y$, by taking $\lambda = \min\left(\frac{x}{y}, \frac{y}{x}\right)$, we obtain $\lambda x \le y \le \lambda^{-1} x$, then

$$f(t,y) \ge \min\left(\frac{x}{y}, \frac{y}{x}\right) f(t,x).$$

**ii)** By i) one has for $x \in [a,b]$,

$$f(t,x_1) \ge \min\left(\frac{x}{x_1}, \frac{x_1}{x}\right) f(t,x) \ge \min\left(\frac{a}{x_1}, \frac{x_1}{b}\right) f(t,x),$$

the result is a consequence of (H1).

**iii)** For each $t \in \mathbb{R}$ and $x, y \in [a,b]$, with i) of this lemma and

$$\min\left(\frac{x}{y}, \frac{y}{x}\right) - 1 = -\frac{| x - y |}{\max(x,y)},$$

we deduce that

$$f(t,y) - f(t,x) \ge -\frac{| x-y |}{\max(x,y)} f(t,x).$$

If we denote $L := \frac{1}{a} \sup_{t\in\mathbb{R}} \sup_{a\le z\le b} f(t,z) < +\infty$, then we obtain

$$f(t,y) - f(t,x) \ge -L \,| x - y | \,.$$

By interchanging the roles of $x$ and $y$, the result follows. □

**Lemma 3.4.** *Let $(t_n)_n$ be a sequence of real numbers. Let $\alpha$ and $\beta : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be such that $\alpha(t,.)$ and $\beta(t,.)$ are in $L^1(\mathbb{R}^+)$ for each $t \in \mathbb{R}$. Let $u$ and $v \in L^\infty(\mathbb{R})$. We denote by $h$ and $k$ the functions defined by*

$$h(t) := \int_{-\infty}^{t} \alpha(t, t-s)u(s)\ ds \quad \text{and} \quad k(t) := \int_{-\infty}^{t} \beta(t, t-s)v(s)\ ds.$$

*If $\lim_{n\to+\infty} u(t+t_n) = v(t)$ and $\lim_{n\to+\infty} \| \alpha(t+t_n,.) - \beta(t,.)\|_{L^1(\mathbb{R}^+)} = 0$ for each $t \in \mathbb{R}$, then $\lim_{n\to+\infty} h(t+t_n) = k(t)$ for each $t \in \mathbb{R}$.*

*Proof.* The functions $h$ and $k$ are also equal to:

$$h(t) = \int_0^{+\infty} \alpha(t,s)u(t-s)\ ds \quad \text{and} \quad k(t) = \int_0^{+\infty} \beta(t,s)v(t-s)\ ds.$$

From the following inequality

$$\begin{aligned} | h(t+t_n) - k(t) | \leq & \int_0^{+\infty} | \alpha(t+t_n,s) - \beta(t,s) || u(t+t_n-s) |\ \ ds \\ & + \int_0^{+\infty} | \beta(t,s) || u(t+t_n-s) - v(t-s) |\ \ ds, \end{aligned}$$

we obtain

$$\begin{aligned} | h(t+t_n) - k(t) | \leq & \| u \|_\infty \| \alpha(t+t_n,.) - \beta(t,.) \|_{L^1(\mathbb{R}^+)} \\ & + \int_0^{+\infty} F_n(t,s)\ ds, \end{aligned} \tag{3.3}$$

where $F_n(t,s) :=| \beta(t,s) || u(t+t_n-s) - v(t-s) |$. Then $\lim_{n\to+\infty} F_n(t,s) = 0$ and $0 \leq F_n(t,s) \leq (\| u \|_\infty + \| v \|_\infty) | \beta(t,s) |$ where $\beta(t,.) \in L^1(\mathbb{R}^+)$, so by the Lebesgue dominated convergence theorem, we deduce that $\lim_{n\to+\infty} \int_0^{+\infty} F_n(t,s)\ ds = 0$. By using (3.3), we deduce the conclusion. ∎

**Lemma 3.5.** *Let $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be such that the function $t \mapsto a(t,.)$ is in $C_b(\mathbb{R}, L^1(\mathbb{R}^+))$. If $f \in C_b(\mathbb{R})$, then the function*

$$h(t) = \int_{-\infty}^{t} a(t, t-s)f(s)\ ds$$

*is also continuous and bounded on $\mathbb{R}$.*

*Proof.* The function $h$ satisfies

$$| h(t) | = | \int_0^{+\infty} a(t,s)f(t-s)\ ds | \leq \| f \|_\infty \sup_{t\in\mathbb{R}} \| a(t,.) \|_{L^1(\mathbb{R}^+)} < +\infty,$$

since the function $t \mapsto a(t,.)$ is bounded, therefore $h$ is bounded.

Let $(t_n)_n$ be a sequence of real numbers such that $\lim_{n\to+\infty} t_n = 0$. By continuity of the functions $t \mapsto a(t,.)$ and $f$, we have

$$\lim_{n\to+\infty} \| a(t+t_n,.) - a(t,.)\|_{L^1(\mathbb{R}^+)} = 0 \quad \text{and} \quad \lim_{n\to+\infty} f(t+t_n) = f(t).$$

By using Lemma 3.4 with $\alpha = \beta = a$ and $u = v = f$, we obtain $\lim_{n\to+\infty} h(t+t_n) = h(t)$, which proves that $h$ is continuous, and this completes the proof. $\square$

**Lemma 3.6.** *Suppose that* (H1), (H2) *and* (H4) *hold. If* $x \in C_b(\mathbb{R})$ *and* $x$ *has a positive infinimum, then the function*

$$F(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s))\; ds$$

*is also continuous and bounded on* $\mathbb{R}$*.*

*Proof.* Let $x \in C_b(\mathbb{R})$ such that $\inf_{t\in\mathbb{R}} x(t) > 0$. Then there exist $a$ and $b \in \mathbb{R}$ such that $0 < a \le x(t) \le b$, for all $t \in \mathbb{R}$. By Lemma 3.3, we deduce that $t \mapsto f(t, x(t))$ is continuous and bounded. The hypotheses of Lemma 3.5 are satisfied, then one has $F \in C_b(\mathbb{R})$. $\square$

*Proof.* Here, we prove Theorem 3.1. We apply the results of the preceding section in order to prove the existence and uniqueness of the continuous and bounded solution of Equation (1.1) with a positive infinimum. Let $X = C_b(\mathbb{R})$ be the Banach space of continuous and bounded functions endowed with the norm of convergence uniform on $\mathbb{R}$: $\| f\|_\infty = \sup_{t\in\mathbb{R}} | f(t) |$. Let $K$ be the cone of nonnegative functions in $C_b(\mathbb{R})$. Then $K$ is a normal convex cone. Furthermore, one has

$$0 \le x \le y \qquad \Longrightarrow \quad \| x\|_\infty \le \| y\|_\infty.$$

The interior of $K$ is given by $\overset{\circ}{K} = \{x \in C_b(\mathbb{R}) \;;\; \inf_{t\in\mathbb{R}} x(t) > 0\}$. We denote by $T$ the operator associated with the right-hand side of Equation (1.1), namely

$$(Tx)(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s))\; ds. \tag{3.4}$$

Note that the solutions of Equation (1.1) defined on the whole line are fixed points of $T$.

Now, we prove that $T$ maps $\overset{\circ}{K}$ into itself. Let $x \in \overset{\circ}{K}$. Then there exists $\epsilon > 0$ such that $\epsilon \le x(t) \le \epsilon^{-1}$, for each $t \in \mathbb{R}$. By Lemma 3.3, one has

$$\begin{aligned}(Tx)(t) &\ge \int_{-\infty}^{t} a(t, t-s) \min\left(\frac{x(s)}{x_2}, \frac{x_2}{x(s)}\right) f(s, x_2)\; ds \\ &\ge \epsilon \min\left(\frac{1}{x_2}, x_2\right) \int_{-\infty}^{t} a(t, t-s) f(s, x_2)\; ds.\end{aligned}$$

So

$$(Tx)(t) \ge \epsilon \min\left(\frac{1}{x_2}, x_2\right) \inf_{t\in\mathbb{R}} \int_{0}^{+\infty} a(t, s) f(t-s, x_2)\; ds > 0.$$

Furthermore, by Lemma 3.6, $Tx \in C_b(\mathbb{R})$. Then $Tx \in \overset{\circ}{K}$ for all $x \in \overset{\circ}{K}$.

To have a fixed point of $T$ in $\overset{\circ}{K}$, we use Theorem 2.7. We know that $(\overset{\circ}{K}, d)$ is a complete metric space with $d$ defined by (2.3), (c.f. Proposition 2.6). By (H3), there exists $t_0 \in \mathbb{R}$ such that $f(t_0, x_2) > 0$ and by (H2), one has $f(t_0, x_2) \geq \phi(\lambda) f(t_0, x_2)$ and $\phi(\lambda) > \lambda$ for all $\lambda \in (0,1)$, then $\lim_{\lambda \to 1} \phi(\lambda) = 1$. Now we consider that the function $\phi$ is defined and continuous on $]0,1]$. We can assume that $\phi$ is nondecreasing, for that change $\phi$ by $\phi_1(\lambda) = \inf\{\phi(\mu) \ ; \ \lambda \leq \mu \leq 1\})$. Indeed $\phi_1$ is nondecreasing and $\phi_1(\lambda) \leq \phi(\lambda)$. For $\lambda \in ]0,1]$, there exists $\mu \in [\lambda, 1]$ such that $\phi_1(\lambda) = \phi(\mu)$ and by (H2), $\phi(\mu) > \mu$, therefore $\phi_1(\lambda) > \lambda$. For the continuity of $\phi_1$, we consider a sequence $(\lambda_n)_n$ such that $\lim\limits_{n \to +\infty} \lambda_n = \lambda$. If $(\lambda_n)_n$ is increasing, then one has $\lambda_n < \lambda$, therefore $\lim\limits_{n \to +\infty} \phi_1(\lambda_n) \leq \phi_1(\lambda)$. Moreover $\phi_1$ is lower semi-continuous, then one has $\phi_1(\lambda) \leq \liminf\limits_{n \to +\infty} \phi_1(\lambda_n)$, we deduce that $\lim\limits_{n \to +\infty} \phi_1(\lambda_n) = \phi_1(\lambda)$. If $(\lambda_n)_n$ is decreasing, then $\lambda_n > \lambda$ and

$$\lim_{n \to +\infty} \phi_1(\lambda_n) = \inf_{n\mathbb{N}} \inf_{\lambda_n \leq \mu \leq 1} \phi(\mu) = \inf_{\lambda < \mu \leq 1} \phi(\mu) = \phi_1(\lambda).$$

Consequently $\phi_1$ is continuous. Then $\phi_1$ satisfies (H2). Let $x$ and $y \in \overset{\circ}{K}$, $\lambda \in (0,1)$ such that $\lambda x \leq y \leq \lambda^{-1} x$. By (H2), one has

$$\forall t \in \mathbb{R}, \ f(t, y(t)) \geq \phi(\lambda) f(t, x(t)).$$

We also have $\lambda y \leq x \leq \lambda^{-1} y$, then

$$\forall t \in \mathbb{R}, \ \phi(\lambda) f(t, x(t)) \leq f(t, y(t)) \leq (\phi(\lambda))^{-1} f(t, x(t)),$$

thus

$$\phi(\lambda) Tx \leq Ty \leq (\phi(\lambda))^{-1} Tx,$$

therefore

$$d(Tx, Ty) \leq \ln\left(\frac{1}{\phi(\lambda)}\right).$$

For $\lambda = \left(\max\left(M(\frac{y}{x}), M(\frac{x}{y})\right)\right)^{-1}$, we have $d(x,y) = \ln(\lambda^{-1})$. If we choose the function $\Phi(r) := -\ln(\phi(e^{-r}))$ for $r \geq 0$, we deduce that

$$d(Tx, Ty) \leq \Phi\left(d(x,y)\right).$$

Furthermore $\Phi$ is a positive, continuous and nondecreasing function on $[0, +\infty[$ satisfying $\Phi(r) < r$ for all $r > 0$ and $\Phi(0) = 0$, then $T$ has exactly one fixed point in $\overset{\circ}{K}$ which is a continuous and bounded solution of Equation (1.1) with a positive infinimum. This ends the proof of Theorem 3.1. □

**Remark 3.7.** We have proved Theorem 3.1 by a similar method which is done in [5] and [15]. Notably, the same function: $\Phi(r) := -\ln(\phi(e^{-r}))$ for $r \geq 0$ appeared in [15]. We have adapted the proof to avoid the hypothesis about the monotonicity of the function $f(t,.)$.

## 4. THE ALMOST AUTOMORPHIC CASE

In this section, we state some results of existence and uniqueness of the almost automorphic solution with a positive infinimum.

**Theorem 4.1.** *Suppose that* (H2), (H3), (H5) *and* (H6) *hold. Then Equation* (1.1) *has a unique almost automorphic solution with a positive infinimum.*

An easy consequence of this last result for the finite delay integral Equation (1.2) is the following result.

**Corollary 4.2.** *Suppose that* (H2) *and* (H5) *hold. In addition, we assume that*

i) $\sigma$ *is a positive almost automorphic function,*

ii) *there exists* $x_2 > 0$ *such that* (3.1) *holds.*

*Then Equation* (1.2) *has a unique almost automorphic solution with a positive infinimum.*

*Proof.* We use Theorem 4.1 with the function $a(t,s) := 1_{[0,\sigma(t)]}(s)$. (H3) follows from ii). For $\sigma_1$ and $\sigma_2 \in L^\infty(\mathbb{R})$, we have

$$\| 1_{[0,\sigma_1(t]} - 1_{[0,\sigma_2(t)]} \|_{L^1(\mathbb{R}^+)} = | \sigma_1(t) - \sigma_2(t) | \, . \tag{4.1}$$

Let $(t_n)_n$ a sequence of real numbers. By help of (4.1), we obtain that the two following assertions are equivalent:

$$\lim_{n\to+\infty} \sigma_1(t+t_n) = \sigma_2(t)$$

$$\lim_{n\to+\infty} \| 1_{[0,\sigma_1(t+t_n]} - 1_{[0,\sigma_2(t)]} \|_{L^1(\mathbb{R}^+)} = 0.$$

By using the equivalence above and the definition of an almost automorphic function, we deduce that i) implies (H6). ☐

In the almost automorphic case, it is possible to improve (H3) for Equation (1.2) (condition ii) of Corollary 4.2) by showing the existence of a threshold phenomenon.

**Proposition 4.3.** *Suppose that* (H2) *and* (H5) *hold. In addition, we assume that* $f$ *is not the zero function. Then there exists* $\sigma_* > 0$ *such that for each almost automorphic function* $\sigma$ *satisfying*

$$\inf_{t\in\mathbb{R}} \sigma(t) \geq \sigma_* \, , \tag{4.2}$$

*Equation* (1.2) *has a unique almost automorphic solution with a positive infinimum.*

We state Theorem 4.1 before Proposition 4.3. For the proof of Theorem 4.1 we use the following lemmas

**Lemma 4.4.** *Let $a : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ such that the function $t \mapsto a(t, .)$ is in $AA(L^1(\mathbb{R}^+))$. If $f \in AA(\mathbb{R})$, then the function*

$$h(t) = \int_{-\infty}^{t} a(t, t-s)f(s)\ ds$$

*is also almost automorphic.*

*Proof.* By Lemma 3.5, $h$ is continuous and bounded. To check that $h$ is in $AA(\mathbb{R})$, we have to prove that if $(t_n)_n$ is any sequence of real numbers, then one can pick up a subsequence of $(t_n)_n$ such that

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} h(t + t_n) = k(t), \tag{4.3}$$

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} k(t - t_n) = h(t). \tag{4.4}$$

In fact by assumption, we can choose a subsequence of $(t_n)_n$ such that

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} \| a(t + t_n, .) - b(t, .) \|_{L^1(\mathbb{R}^+)} = 0,$$

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} \| b(t - t_n, .) - a(t, .) \|_{L^1(\mathbb{R}^+)} = 0,$$

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} f(t + t_n) = g(t),$$

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} g(t - t_n) = f(t).$$

Let $k(t) = \displaystyle\int_{-\infty}^{t} b(t, t-s)g(s)\ ds$. By using Lemma 3.4 with $\alpha = a$, $\beta = b$, $u = f$ and $v = g$, we obtain (4.3) and we state (4.4) by using Lemma 3.4 with the sequence $(-t_n)_n$, $\alpha = b$, $\beta = a$, $u = g$ and $v = f$. This ends the proof. □

**Lemma 4.5.** *Suppose that* (H2), (H5) *and* (H6) *hold. If $x$ is in $AA(\mathbb{R})$ and $x$ has a positive infinimum, then the function*

$$F(t) = \int_{-\infty}^{t} a(t, t-s)f(s, x(s))\ ds$$

*is also almost automorphic.*

*Proof.* There exist $a$ and $b \in \mathbb{R}$ such that $0 < a \leq x(t) \leq b$, for all $t \in \mathbb{R}$. By Lemma 3.3, we obtain $| f(t, x_1) - f(t, x_2) | \leq L | x_1 - x_2 |$ for all $t \in \mathbb{R}$, $x_1$ and $x_2 \in [a, b]$. Since $x \in AA(\mathbb{R})$ and $f$ satisfies (H5), by composition theorem of almost automorphic functions, we deduce that $t \mapsto f(t, x(t))$ is almost automorphic ([22], Theorem 2.2.6, p. 22). The hypotheses of Lemma 4.4 are satisfied, then $F \in AA(\mathbb{R})$. □

*Proof.* The Proof of Theorem 4.1 is similar to the one given in Theorem 3.1, by considering $K$ the cone of nonnegative functions in the Banach space $AA(\mathbb{R})$ endowed with the norm defined by $\| f\|_\infty = \sup_{t\in\mathbb{R}} | f(t) |$. Lemma 4.5 permits us to state that the operator $T$ defined by (3.4) on $\overset{\circ}{K} = \{x \in AA(\mathbb{R})\ ;\ \inf_{t\in\mathbb{R}} x(t) > 0\}$ maps $\overset{\circ}{K}$ into itself. □

For the proof of Proposition 4.3, we use the following lemmas.

**Lemma 4.6.** *Let $t_* \in \mathbb{R}$. If $\phi \in AA(X)$ then*

$$\overline{\phi([t_*, +\infty[)} = \overline{\phi(\mathbb{R})}. \tag{4.5}$$

*In particular, if $\phi \in AA(\mathbb{R})$, then $\inf_{t\geq t_*} \phi(t) = \inf_{t\in\mathbb{R}} \phi(t)$.*

*Proof.* Consider the sequence of natural numbers $(n)_n$. Then there exists a subsequence $(n_k)_k$ of $(n)_n$ such that

$$\forall t \in \mathbb{R}, \quad \lim_{k\to+\infty} \phi(t + n_k) = \psi(t),$$
$$\forall t \in \mathbb{R}, \quad \lim_{k\to+\infty} \psi(t - n_k) = \phi(t).$$

Obviously, one has $\overline{\phi(\mathbb{R})} = \overline{\psi(\mathbb{R})}$. Let $t \in \mathbb{R}$. Since $\lim_{k\to+\infty} t + n_k = +\infty$, there exists $k_0 \in \mathbb{N}$ such that $t + n_k \geq t_*$ for each $k \geq k_0$. With $\lim_{k\to+\infty} \phi(t + n_k) = \psi(t)$, it yields

$$\overline{\psi(\mathbb{R})} \subset \overline{\phi([t_*, +\infty[)}$$

and with $\overline{\phi(\mathbb{R})} = \overline{\psi(\mathbb{R})}$, we deduce that (4.5) holds. Obviously, if $\phi \in AA(\mathbb{R})$, one has $\inf_{t\geq t_*} \phi(t) = \inf_{t\in\mathbb{R}} \phi(t)$. □

**Lemma 4.7.** *Let $\phi \in AA(\mathbb{R})$ such that $\phi \geq 0$. If*

$$\forall r > 0, \quad \inf_{t\in\mathbb{R}} \int_{t-r}^{t} \phi(s)\ ds = 0, \tag{4.6}$$

*then $\phi$ is the zero function.*

*Proof.* Let $\Phi_r(t) := \int_{t-r}^{t} \phi(s)\ ds$. Applying Lemma 4.4 with $a(t, s) := 1_{[0,r]}(s)$, we deduce that $\Phi_r$ is almost automorphic. Assume that (4.6) holds: $\inf_{t\in\mathbb{R}} \Phi_r(t) = 0$. By Lemma 4.6, we obtain that $\inf_{t\geq 0} \Phi_r(t) = 0$ for all $r > 0$, then there exists a nondecreasing sequence $(r_n)_n$ with positive terms such that $\inf_{t\geq 0} \int_{t-r_n}^{t} \phi(s)\ ds = 0$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} r_n = +\infty. \tag{4.7}$$

We deduce the existence of a sequence $(t_n)_n$ such that $t_n \geq 0$ and

$$\forall n \in \mathbb{N}, \quad 0 \leq \int_{t_n-r_n}^{t_n} \phi(s)\ ds \leq \frac{1}{n+1},$$

therefore

$$\lim_{n\to+\infty} \int_{t_n-r_n}^{t_n} \phi(s)\ ds = 0. \tag{4.8}$$

Since $\phi$ is almost automorphic, there exists a subsequence of $(t_n)_n$ such that

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} \phi(t+t_n) = \psi(t), \tag{4.9}$$

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} \psi(t-t_n) = \phi(t). \tag{4.10}$$

Let $k \in \mathbb{N}^* = \mathbb{N} - \{0\}$. By using (4.8), we deduce that

$$\lim_{n\to+\infty} \int_{-r_k}^{0} \phi(s+t_n)\, ds = \lim_{n\to+\infty} \int_{t_n-r_k}^{t_n} \phi(s)\, ds = 0 \tag{4.11}$$

since $\phi \geq 0$ and the sequence $(r_n)_n$ is nondecreasing. By using the Lebesgue dominated convergence theorem applied to the function $t \mapsto \phi(t+t_n)$ on $[-r_k, 0]$, with (4.9) and (4.11), we obtain $\int_{-r_k}^{0} \psi(s)\, ds = 0$, therefore $\psi(t) = 0$ a.e. on $[-r_k, 0]$ because $\psi \geq 0$. With (4.7), we obtain

$$\psi(t) = 0 \quad \text{a.e. on } \mathbb{R}^-. \tag{4.12}$$

With $t_n \geq 0$, (4.10) and (4.12), we deduce that $\phi(t) = 0$ for all $t \leq 0$, then by almost automorphicity, $\phi$ is the zero function ([22], Theorem 2.1.8, p. 17). □

*Proof.* Here, we prove Proposition 4.3. We use Corollary 4.2. It suffices to prove that Hypothesis ii) is satisfied. Since $f$ is not the zero function, there exists $x_2 > 0$ such that $f(.,x_2)$ is not the zero function. By using Lemma 4.7 with $\phi(t) = f(t,x_2)$, we deduce that $\exists \sigma_* > 0$, such that $\delta := \inf_{t\in\mathbb{R}} \int_{t-\sigma_*}^{t} f(s,x_2)\, ds > 0$. Consequently for each $\sigma$ such that $\inf_{t\in\mathbb{R}} \sigma(t) \geq \sigma_*$, (3.1) holds, therefore Hypothesis ii) is satisfied. □

## 5. THE ALMOST PERIODIC CASE

In this section, we state some results of existence and uniqueness of the almost periodic solution with a positive infinimum.

**Theorem 5.1.** *Suppose that* (H2), (H3), (H7) *and* (H8) *hold. Then Equation* (1.1) *has a unique almost periodic solution $x$ with a positive infinimum. Furthermore, we have*

$$mod(x) \subset mod(f) + mod(\tilde{a}), \tag{5.1}$$

*where $\tilde{a}$ denotes the function defined by $\tilde{a}(t) := a(t,.)$.*

**Remark.** In the periodic case, namely the functions $t \mapsto f(t,.)$ and $t \mapsto a(t,.)$ are $T$-periodic, by the module containment formula , we deduce that the almost periodic solution is $T$-periodic.

An easy consequence of this last result for the finite delay integral Equation (1.2) is the following result.

**Corollary 5.2.** *Suppose that* (H2) *and* (H7) *hold. In addition, we assume that*

i) $\sigma$ *is a positive almost periodic function,*

ii) *there exists* $x_2 > 0$ *such that* (3.1) *holds.*

*Then Equation* (1.2) *has a unique almost periodic solution with a positive infinimum. Furthermore, we have*

$$mod(x) \subset mod(f) + mod(\sigma). \tag{5.2}$$

*Proof.* We use Theorem 5.1 with the function $a(t,s) := 1_{[0,\sigma(t)]}(s)$. (H3) follows from ii). By using (3.2) we deduce that (H8) holds. For the module containment formula (5.2), it remains to show that $mod(1_{[0,\sigma(.)]}) \subset mod(\sigma)$, for that we use ([16], Theorem 4.5, p. 61), which can easily be seen to be true for almost periodic functions with values in a Banach space. Assume that $\lim_{n\to+\infty} \sigma(t+t_n) = \sigma_*(t)$ for each $t \in \mathbb{R}$. By using (4.1) we deduce that $\lim_{n\to+\infty} \| 1_{[0,\sigma(t+t_n)]} - 1_{[0,\sigma_*(t)]} \|_{L^1(\mathbb{R}^+)} = 0$ for each $t \in \mathbb{R}$, so the inclusion $mod(1_{[0,\sigma(.)]}) \subset mod(\sigma)$ is established. This completes the proof. ☐

In the almost periodic case, it is possible to improve (H3) for Equation (1.2) (condition ii) of Corollary 5.2) by showing the existence of a threshold phenomenon.

**Proposition 5.3.** *Suppose that* (H2) *and* (H7) *hold. In addition, we assume that* $f$ *is not the zero function. Then there exists* $\sigma_* > 0$ *such that for each almost periodic function* $\sigma$ *satisfying* (4.2), *Equation* (1.2) *has a unique almost periodic solution with a positive infinimum. Furthermore, the module containment formula* (5.2) *holds.*

*Proof.* Since an almost periodic function is almost automorphic, Hypothesis ii) is satisfied, see proof of Proposition 4.3. ☐

Before to start the proof of Theorem 5.1, we compare our result (Corollary 5.2) with a result on almost periodic solutions of Xu and Yuan [28].

**Remark.** Here we explain how Corollary 5.2 improves on Theorem 2 of Xu and Yuan [28]. Let us first recall their result. For that we complete the list of hypotheses on $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ where $f(t,x) := f_1(t,x) + f_2(t,x)$ with $f_1 : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $f_2 : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$.

**(A1)** *The function* $x \mapsto f_1(t,x)$ *is nondecreasing and the function* $x \mapsto f_2(t,x)$ *is nonincreasing, for each* $t \in \mathbb{R}$.

**(A2)** *There exists a continuous map* $\phi : (0,1) \longrightarrow \mathbb{R}$ *satisfying* $\phi(\lambda) > \lambda$ *and for each* $x > 0$, $t \in \mathbb{R}$ *and* $\lambda \in (0,1)$, *one has*

$$f_1(t,\lambda x) \geq \phi(\lambda) f_1(t,x) \quad \text{and} \quad f_2(t,\frac{x}{\lambda}) \geq \phi(\lambda) f_2(t,x).$$

**(A3)** *There exists* $x_2 > 0$ *such that* $\inf_{t\in\mathbb{R}} \int_{t-\sigma(t)}^{t} f_1(s,x_2)\, ds > 0.$

The result of Xu and Yuan ([28], Theorem 2) is as follow: *suppose that* $f_1$ *and* $f_2$ *are almost periodic in* $t$ *uniformly with respect to* $x \in \mathbb{R}^+$ *and* $f_1,\ f_2 \geq 0$. *If the conditions* (A1)–(A3) *and i) (of Corollary* 5.2*) hold, then Equation* (1.2) *has a unique almost periodic solution with a positive infinimum and the module containment formula* (5.2) *holds.*

Hypotheses (A1) and (A2) imply (H2), Hypothesis (A3) and $f_2 \geq 0$ imply ii) (of Corollary 5.2), then our Corollary 5.2 gives the same result of ([28], Theorem 2).

If we consider $\sigma$ to be an almost periodic function with a positive infinimum and the function $f$ defined on $\mathbb{R} \times \mathbb{R}^+$ by

$$f(t,x) = p(t)\left(1_{[0,1]}(x)\sqrt{x} + 1_{]1,+\infty[}(x)\frac{1}{\sqrt{x}}\right), \tag{5.3}$$

where $p$ is the almost periodic function defined by

$$p(t) = \cos^2 t + \cos^2 \pi t, \tag{5.4}$$

then the hypotheses of Corollary 5.2 hold with $\phi(\lambda) = \sqrt{\lambda}$, therefore our result permits us to conclude the existence of almost periodic solutions while theirs is not appropriate, because (A1) is not satisfied. In fact (A1) and $f_1,\ f_2 \geq 0$ imply that $f(t,.)$ is nondecreasing if $f(t,0) = 0$, (because $\sup\limits_{x\geq 0} f_2(t,x) = f_2(t,0) \leq f(t,0)$) and $f(t,.)$ is nonincreasing if $\lim\limits_{x\to+\infty} f(t,x) = 0$, because

$$\sup_{x\geq 0} f_1(t,x) = \lim_{x\to+\infty} f_1(t,x) \leq \lim_{x\to+\infty} f(t,x).$$

In conclusion, Corollary 5.2 provides an improvement on ([28], Theorem 2).

For the proof of Theorem 5.1, we use the following lemmas.

**Lemma 5.4.** *Let* $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be such that the function* $t \mapsto a(t,.)$ *is in* $AP(L^1(\mathbb{R}^+))$. *If* $f \in AP(\mathbb{R})$, *then the function*

$$h(t) = \int_{-\infty}^{t} a(t, t-s) f(s)\ ds$$

*is also almost periodic.*

*Proof.* By Lemma 3.5, $h$ is continuous and bounded. To check that $h$ is in $AP(\mathbb{R})$, we have to prove that if $(t_n)_n$ and $(s_n)_n$ is a pair of sequences of real numbers, then one can pick up a common subsequence of $(t_n)_n$ and $(s_n)_n$ such that for each $t \in \mathbb{R}$,

$$\lim_{n\to+\infty}\lim_{m\to+\infty} h(t+t_n+s_m) = \lim_{n\to+\infty} h(t+t_n+s_n). \tag{5.5}$$

In fact by assumption, we can choose a common subsequence of $(t_n)_n$ and $(s_n)_n$ such that for each $t \in \mathbb{R}$

$$\lim_{n\to+\infty}\lim_{m\to+\infty} a(t+t_n+s_m,.) = \lim_{n\to+\infty} a(t+t_n+s_n,.) \quad \text{in } L^1(\mathbb{R}^+),$$

$$\lim_{n\to+\infty}\lim_{m\to+\infty} f(t+t_n+s_m) = \lim_{n\to+\infty} f(t+t_n+s_n).$$

The proof of Lemma 5.4 is similar to the one given in Lemma 4.4, by using three times Lemma 3.4. □

**Lemma 5.5.** *Suppose that* (H7) *and* (H8) *hold. If* $x \in AP(\mathbb{R})$ *and* $x$ *has a positive infinimum, then the function*

$$F(t) = \int_{-\infty}^{t} a(t,t-s) f(s,x(s))\, ds$$

*is also almost periodic.*

*Proof.* Since $x \in AP(\mathbb{R})$ and $f$ satisfies (H7), $t \mapsto f(t,x(t))$ is almost periodic ([29], Theorem 2.7, p. 16). The hypotheses of Lemma 5.4 are satisfied, then $F \in AP(\mathbb{R})$. □

*Proof.* Here, we prove Theorem 5.1. The proof of the existence and uniqueness of the almost periodic solution with a positive infinimum, is similar to the one given in Theorem 4.1, by considering the Banach space $AP(\mathbb{R})$ instead of $AA(\mathbb{R})$ endowed with the same norm.

Now we prove the formula of the modules. Let $(t_n)_n$ be a numerical sequence such that for all compact $K$ of $\mathbb{R}^+$

$$f(t+t_n,x) \to g(t,x) \quad \text{as} \quad n \to +\infty \tag{5.6}$$

uniformly on $\mathbb{R} \times K$ and

$$\| a(t+t_n,.) - b(t,.)\|_{L^1(\mathbb{R}^+)} \to 0 \quad \text{as} \quad n \to +\infty \tag{5.7}$$

uniformly on $\mathbb{R}$ where $b \in AP(L^1(\mathbb{R}^+))$. To state the module containment formula, it suffices to prove that the sequence $(x(.+t_n))_n$ converges uniformly on $\mathbb{R}$ ([29], Theorem 2.8, p. 18). By using Lemma 3.4 with $\alpha = a$, $\beta = b$, $u = f(.,x_2)$ and $v = g(.,x_2)$, we deduce that

$$\lim_{n\to+\infty} \int_0^{+\infty} a(t+t_n,s) f(t+t_n-s,x_2)\, ds = \int_0^{+\infty} b(t,s) g(t-s,x_2)\, ds,$$

therefore $b$ and $g$ satisfy all hypotheses of Theorem 5.1, then Equation

$$y(t) = \int_{-\infty}^{t} b(t,t-s) g(s,y(s))\, ds, \tag{5.8}$$

has a unique almost periodic solution $y$ with a positive infinimum. Let a subsequence of $(x(.+t_n))_n$ which we denote by the similar manner. Since this last subsequence is with values in $AP(\mathbb{R})$, it has a cluster point $x_*$ in $AP(\mathbb{R})$, so we have:

$$x(t+t_n) \to x_*(t) \quad \text{as} \quad n \to +\infty \tag{5.9}$$

uniformly on $\mathbb{R}$. From (5.6) and (5.9), we deduce that

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} f(t+t_n, x(t+t_n)) = g(t,x_*(t)). \tag{5.10}$$

On the other hand, by using (5.7), (5.10) and Lemma 3.4 with $\alpha = a$, $\beta = b$, $u(t) = f(t, x(t))$ and $v(t) = g(t, x_*(t))$, we obtain

$$\forall t \in \mathbb{R}, \quad \lim_{n \to +\infty} x(s + t_n) = \int_{-\infty}^{t} b(t, t-s) g(s, x_*(s)) \, ds,$$

thus with (5.9), we deduce that $x_*$ is an almost periodic solution of Equation (5.8) with a positive infinimum. By uniqueness of this last solution, we have $x_* = y$. We deduce that $(x(t+t_n))_n$ converges uniformly on $\mathbb{R}$. In conclusion, we have the desired result. □

## 6. APPLICATION TO A DIFFERENTIAL EQUATION

In this section, we apply our results for the existence of the almost automorphic and almost periodic solutions with a positive infinimum to the following first order semilinear differential Equation (1.3). Let $\alpha \in C_b(\mathbb{R})$ and $\tau \geq 0$. Recall that the homogeneous linear equation

$$x'(t) + \alpha(t)x(t) = 0 \tag{6.1}$$

has an exponential dichotomy if there exist $k$ and $c > 0$ such that

$$\exp\left(-\int_s^t \alpha(\xi) \, d\xi\right) \leq k e^{-c(t-s)}, \quad \forall t \geq s. \tag{6.2}$$

If Equation (6.1) has an exponential dichotomy, then for any $p \in C_b(\mathbb{R})$, the linear equation

$$x'(t) + \alpha(t)x(t) = p(t)$$

has a unique bounded solution which is given by

$$x(t) = \int_{-\infty}^{t} \exp\left(-\int_s^t \alpha(\xi) \, d\xi\right) p(s) \, ds.$$

Similarly, if Equation (6.1) has an exponential dichotomy and if $f$ is bounded on every $\mathbb{R} \times K$ where $K$ is a compact subset of $\mathbb{R}^+$, then $x$ is a bounded solution of Equation (1.3) if and only if $x$ is a bounded solution of

$$x(t) = \int_{-\infty}^{t} \exp\left(-\int_s^t \alpha(\xi) \, d\xi\right) f(s, x(s-\tau)) \, ds. \tag{6.3}$$

By making the change of variables of $s$ to $s + \tau$, one can rewrite Equation (6.3) as

$$x(t) = \int_{-\infty}^{t} \exp\left(-\int_{s+\tau}^t \alpha(\xi) \, d\xi\right) 1_{[\tau,+\infty]}(t-s) f(s+\tau, x(s)) \, ds. \tag{6.4}$$

To start, we give a result on the exponential dichotomy of Equation (6.1) in the almost automorphic case.

**Lemma 6.1.** *Let $\alpha \in C_b(\mathbb{R})$.*

i) *If there exists $r_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_{t-r_0}^{t} \alpha(\xi)\ d\xi > 0, \tag{6.5}$$

*then Equation* (6.1) *has an exponential dichotomy.*

ii) *In particular if $\alpha \in AA(\mathbb{R})$, $\alpha \geq 0$ and $\alpha$ is not the zero function, then Equation* (6.1) *has an exponential dichotomy.*

**Remark.** In fact Equation (6.1) has an exponential dichotomy if and only if there exists $r_0 > 0$ such that (6.5) holds.

*Proof.* i) Denote by

$$\delta_0 := \inf_{t\in\mathbb{R}} \int_{t-r_0}^{t} \alpha(\xi)\ d\xi > 0.$$

Let $s \geq 0$. Then there exists $n_0 \in \mathbb{N}$ such that $n_0 r_0 \leq s < (n_0+1)r_0$. By using the following inequalities

$$\int_{t-n_0r_0}^{t} \alpha(\xi)\ d\xi \geq n_0\delta_0 \geq \left(\frac{s}{r_0} - 1\right)\delta_0$$

and

$$\left| \int_{t-s}^{t-n_0r_0} \alpha(\xi)\ d\xi \right| \leq r_0 \parallel \alpha\|_\infty,$$

we deduce that

$$\forall s \geq 0, \quad \int_{t-s}^{t} \alpha(\xi)\ d\xi \geq \frac{\delta_0}{r_0}s - (\delta_0 + r_0 \parallel \alpha\|_\infty),$$

therefore (6.2) holds with $k = \exp(\delta_0 + r_0 \parallel \alpha\|_\infty)$ and $c = \dfrac{\delta_0}{r_0}$.

ii) By using Lemma 4.7, we can assert that there exists $r_0 > 0$ such that (6.5) holds. The first sentence of this lemma permits us to conclude. This ends the proof. $\square$

**Proposition 6.2.** *We assume that $f$ satisfies* (H2), (H5) *and $f$ is not the zero function. In addition we suppose that $\alpha \in AA(\mathbb{R})$ and there exists $r_0 > 0$ such that* (6.5) *holds. Then Equation* (1.3) *has a unique almost automorphic solution with a positive infinimum.*

**Remark.** In Proposition 6.2, we can replace hypotheses on $\alpha$ by $\alpha \in AA(\mathbb{R})$ such that $\alpha \geq 0$ and $\alpha$ is not the zero function (see Lemma 6.1).

For the proof of Proposition 6.2, we use the following lemmas.

**Lemma 6.3.** *Let $c > 0$ and let $\phi \in AA(\mathbb{R})$ such that $\phi \geq 0$. If*

$$\inf_{t \in \mathbb{R}} \int_0^{+\infty} e^{-cs} \phi(t-s)\, ds = 0, \tag{6.6}$$

*then $\phi$ is the zero function.*

*Proof.* Let $r > 0$, by following inequalities

$$\int_{t-r}^{t} \phi(s)\, ds \leq e^{cr} \int_{-\infty}^{t} e^{-c(t-s)} \phi(s)\, ds = e^{cr} \int_0^{+\infty} e^{-cs} \phi(t-s)\, ds$$

and by (6.6) we deduce that

$$\forall r > 0, \qquad \inf_{t \in \mathbb{R}} \int_{t-r}^{t} \phi(s)\, ds = 0.$$

Thank to Lemma 4.7, we obtain the result. □

**Lemma 6.4.** *Let $(t_n)_n$ be a sequence of real numbers. Let $u$ and $v \in L^\infty(\mathbb{R})$. We denote by $A$ and $B$ the functions defined by*

$$A(t,s) := \exp\left( - \int_{t+\tau-s}^{t} u(\xi)\, d\xi \right) 1_{[\tau,+\infty[}(s),$$

$$B(t,s) := \exp\left( - \int_{t+\tau-s}^{t} v(\xi)\, d\xi \right) 1_{[\tau,+\infty[}(s).$$

*We assume that there exist $k$ and $c > 0$ such that*

$$\exp\left( - \int_{t-s}^{t} u(\xi)\, d\xi \right) \leq k e^{-cs}, \quad \forall s \geq 0. \tag{6.7}$$

*If for each $t \in \mathbb{R}$*

$$\lim_{n \to +\infty} u(t + t_n) = v(t), \tag{6.8}$$

*then*

$$\exp\left( - \int_{t-s}^{t} v(\xi)\, d\xi \right) \leq k e^{-cs}, \quad \forall s \geq 0 \tag{6.9}$$

*and*

$$\lim_{n \to +\infty} \| A(t + t_n, .) - B(t, .) \|_{L^1(\mathbb{R}^+)} = 0. \tag{6.10}$$

*Proof.* By Lebesgue dominated convergence theorem and by (6.8), we obtain

$$\forall c, d \in \mathbb{R}, \quad \lim_{n \to +\infty} \int_{c+t_n}^{d+t_n} u(\xi)\, d\xi = \int_c^d v(\xi)\, d\xi. \tag{6.11}$$

For $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $s \geq 0$, we denote $F_n(t,s) := A(t + t_n, s)$. The function $F_n$ satisfies $0 \leq F_n(t,s) \leq k e^{-c(s-\tau)}$ and

$$\lim_{n \to +\infty} F_n(t,s) = B(t,s). \tag{6.12}$$

By using Lebesgue theorem, we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty} \mid F_n(t,s)-B(t,s)\mid \ ds=0,$$

so (6.10) is fulfilled. Obviously (6.9) follows from (6.7) and (6.11). This ends the proof. □

*Proof.* Here, we prove Proposition 6.2. By Lemma 6.1, Equation (6.1) admits an exponential dichotomy, then an almost automorphic function $x$ is a solution of Equation (1.3) if and only if $x$ is a solution of Equation (6.4). To state Proposition 6.2, we use Theorem 4.1 with the functions

$$a(t,s):=\exp\left(-\int_{t+\tau-s}^{t}\alpha(\xi)\ d\xi\right)1_{[\tau,+\infty[}(s), \tag{6.13}$$

and

$$(t,x)\mapsto f(t+\tau,x). \tag{6.14}$$

It suffices to prove that hypotheses (H3) and (H6) are satisfied. First we state (H3). Since $f$ is not the zero function, there exists $x_2>0$ such that $f(t,x_2)$ is not the zero function. Moreover $\|\alpha\|_\infty>0$, because $\alpha$ is not the zero function. By using Lemma 6.3, we obtain

$$\delta:=\inf_{t\in\mathbb{R}}\int_0^{+\infty} e^{-s\|\alpha\|_\infty}f(t-s,x_2)\ ds>0,$$

then (H3) is fulfilled because

$$\inf_{t\in\mathbb{R}}\int_0^{+\infty} a(t,s)f(t+\tau-s,x_2)\ ds\geq\delta>0.$$

For Hypothesis (H6), we use Lemma 6.4. Remark that, for each $t\in\mathbb{R}$, the function $s\mapsto a(t,s)\in L^1(\mathbb{R}^+)$ because Equation (6.1) admits an exponential dichotomy. To check that the function $t\mapsto a(t,.)$ is in $AA(L^1(\mathbb{R}^+))$, we have to prove that if $(t_n)_n$ is any sequence of real numbers, then one can pick up a subsequence of $(t_n)_n$ such that

$$\forall t\in\mathbb{R},\quad \lim_{n\to+\infty}\parallel a(t+t_n,.)-b(t,.)\parallel_{L^1(\mathbb{R}^+)}=0, \tag{6.15}$$

$$\forall t\in\mathbb{R},\quad \lim_{n\to+\infty}\parallel b(t-t_n,.)-a(t,.)\parallel_{L^1(\mathbb{R}^+)}=0, \tag{6.16}$$

In fact by assumption, we can choose a subsequence of $(t_n)_n$ such that

$$\forall t\in\mathbb{R},\quad \lim_{n\to+\infty}\alpha(t+t_n)=\beta(t),$$
$$\forall t\in\mathbb{R},\quad \lim_{n\to+\infty}\beta(t-t_n)=\alpha(t).$$

Let $b(t,s)=\exp\left(-\int_{t+\tau-s}^{t}\beta(\xi)\ d\xi\right)1_{[\tau,+\infty[}(s)$. By using Lemma 6.4 with $u=\alpha$ and $v=\beta$, we obtain (6.15) and we state (6.16) by using Lemma 6.4 with the sequence $(-t_n)_n$, $u=\beta$ and $v=\alpha$. This ends the proof. □

**Proposition 6.5.** *We assume that $f$ satisfies* (H2), (H7) *and $f$ is not the zero function. In addition we suppose that $\alpha \in AP(\mathbb{R})$ such that $M\{\alpha(t)\}_t > 0$. Then Equation* (1.3) *has a unique almost periodic solution with a positive infinimum. Furthermore, we have*

$$mod(x) \subset mod(f) + mod(\alpha). \tag{6.17}$$

*Proof.* Since $\alpha \in AP(\mathbb{R})$ such that $M\{\alpha(t)\}_t > 0$, then

$$\lim_{r\to+\infty} \frac{1}{r}\int_{t-r}^{t} \alpha(s)\ ds = M\{\alpha(s)\}_s > 0$$

uniformly with respect to $t \in \mathbb{R}$. Consequently, there exist $L > 0$ and $r_0 > 0$ such that

$$\forall r \geq r_0,\ \forall t \in \mathbb{R}, \quad \frac{1}{r}\int_{t-r}^{t} \alpha(s)\ ds \geq L,$$

therefore (6.5) holds. By Lemma 6.1, we can assert that Equation (6.1) has an exponential dichotomy. To state Proposition 6.5, we use Theorem 5.1 with the functions (6.13) and (6.14). Since an almost periodic function is almost automorphic, Hypothesis (H3) is satisfied. It remains to prove that hypotheses (H8) is fulfilled. Remark that, for each $t \in \mathbb{R}$, the function $s \mapsto a(t,s) \in L^1(\mathbb{R}^+)$ because Equation (6.1) admits an exponential dichotomy. To check that $t \mapsto a(t,.)$ is in $AP(L^1(\mathbb{R}^+))$, we have to prove that if $(t_n)_n$ and $(s_n)_n$ is a pair of sequences of real numbers, then one can pick up a common subsequence of $(t_n)_n$ and $(s_n)_n$ such that for each $t \in \mathbb{R}$,

$$\lim_{n\to+\infty}\lim_{m\to+\infty} a(t+t_n+s_m,.) = \lim_{n\to+\infty} a(t+t_n+s_n,.) \quad \text{in } L^1(\mathbb{R}^+).$$

In fact by assumption, we can choose a common subsequence of $(t_n)_n$ and $(s_n)_n$ such that for each $t \in \mathbb{R}$

$$\lim_{n\to+\infty}\lim_{m\to+\infty} \alpha(t+t_n+s_m) = \lim_{n\to+\infty} \alpha(t+t_n+s_n).$$

The proof of Proposition 6.5 is similar to the one given in Proposition 6.2, by using three times Lemma 6.4. For the module containment formula (6.17), it remains to show that

$$mod(\tilde{a}) \subset mod(\alpha), \tag{6.18}$$

where $\tilde{a} = a(t,.)$. Assume that $\lim_{n\to+\infty} \alpha(t+t_n) = \alpha_*(t)$ for each $t \in \mathbb{R}$. By using Lemma 6.4 with $u = \alpha$ and $v = \alpha_*$ we deduce that

$$\lim_{n\to+\infty} \| a(t+t_n,.) - a_*(t,.) \|_{L^1(\mathbb{R}^+)} = 0$$

for each $t \in \mathbb{R}$ with $a_*(t,s) = \exp\left(-\int_{t+\tau-s}^{t} \alpha_*(\xi)\ d\xi\right) 1_{[\tau,+\infty[}(s)$. Using ([16], Theorem 4.5, p. 61), we deduce (6.18), so the formula (6.17) is established. This ends the proof. $\square$

**Acknowledgements.** The authors wish to thank the referee for his remarks and suggestions to improve the presentation of the paper.

This paper was written while the second author was visiting the LMDP of Faculté des Sciences Semlalia, Marrakech. He wishes to express his gratitude for the invitation.

**Support.** This research is supported by a Grant from Development Research Institute, IRD-Paris.

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