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ON MONOTONIC SOLUTIONS OF A SINGULAR QUADRATIC INTEGRAL EQUATION WITH SUPREMUM

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ABSTRACT. We prove an existence theorem for a singular quadratic integral equation with supremum. The quadratic integral equation studied below contains as a special case numerous integral equations encountered in the theory of radiative transfer and in the kinetic theory of gases. We show that the singular quadratic integral equations with supremum has a monotonic solution in C[0, 1]. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

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1. INTRODUCTION

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially , the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications (cf. [2, 3, 5, 8, 9, 10, 13]). On the other hand, integral equations of Volterra type with supremum have been studied in [1, 12], among others. These equations can be considered with connection to the Cauchy problem [7]:

$$x'(t) = f(t) \cdot \max_{[0,t]} |x(\tau)|, \quad x(0) = 0.$$

More recently, Caballero et al [7] investigated the so-called quadratic integral equations of Volterra type with supremum and proved the existence of monotonic solutions in C[0, 1].

In this paper we will study the existence of nondecreasing solutions of singular quadratic integral equation of Volterra type with supremum, namely

$$(1.1) \quad x(t) = h(t) + \frac{(\mathrm{T}\, x)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds,\ 0 < \alpha \leq 1,\ 0 \leq t \leq 1.$$

Using the technique associated with measures of noncompactness we show that equation (1.1) has a solution belonging to $C[0,1]$ and nondecreasing on the interval $[0,1]$.

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In fact, our results in this paper are motivated by the extensions and generalization of the work of Caballero et al [7] based on the a measure of noncompactness and fixed point theorem due to Darbo.

2. AUXILIARY FACTS AND RESULTS

This section is devoted to collect some definitions and results which will be needed further on. Assume that $(\mathrm{E}, \|\cdot\|)$ is a real Banach space with zero element 0. Let $B(x,r)$ denotes the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(0,r)$.

If X is a subset of E, then $\bar{\mathrm{X}}$ and $Conv\mathrm{X}$ denote the closure and convex closure of X, respectively. The symbols $\lambda\,\mathrm{X}$ and $\mathrm{X}+\mathrm{Y}$ denote the usual algebraic operators on sets. Moreover, we denote by $\mathcal{M}_{\mathrm{E}}$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_{\mathrm{E}}$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [6]:

Definition 2.1. A mapping $\mu : \mathcal{M}_{\mathrm{E}} \to [0,+\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\mathrm{Ker}\mu = \{X \in \mathcal{M}_{\mathrm{E}} : \mu(X) = 0\}$ is nonempty and $\mathrm{Ker}\mu \subset \mathcal{N}_{\mathrm{E}}$.
2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3) $\mu(\bar{X}) = \mu(Conv X) = \mu(X)$.
4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\,\mu(X) + (1-\lambda)\,\mu(Y)$ for $0 \leq \lambda \leq 1$.
5) If $X_n \in \mathcal{M}_{\mathrm{E}}$, $X_n = \bar{X}_n$, $X_{n+1} \subset X_n$ for $n = 1,\ 2,\ 3,\ \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $\cap_{n=1}^{\infty} X_n \neq \phi$.

The family Ker μ described above is called the kernel of the measure of noncompactness μ.

In what follows we will work in the Banach space C[0, 1] consisting of all real functions defined and continuous on $[0,1]$. For convenience, we write I and $C(\mathrm{I})$ instead of $[0,1]$ and C[0, 1], respectively. The space $C(\mathrm{I})$ is equipped with the standard norm

$$\|\, x \,\| = \max\{|x(t)| : t \geq 0\}$$

Now, we recollect the construction of the measure of noncompactness in C(I) which will be used in the next section (see [3], [4]).

Let us fix a nonempty and bounded subset X of $C(\mathrm{I})$. For $x \in X$ and $\varepsilon \geq 0$ denoted by $\omega(x,\varepsilon)$, the modulus of continuity of the function x, i.e.,

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in \mathrm{I},\ |t-s| \leq \varepsilon\}$$

Further, let us put

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon).$$

Define

$$d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t,\ s \in \mathrm{I},\ t \le s\}$$

and

$$d(X) = \sup\{d(x) : x \in X\}.$$

Observe that, all functions belonging to X are nondecreasing on I if and only if $d(X) = 0$.

Now, let us define the function μ on the family $\mathcal{M}_{\mathrm{C(I)}}$ by the formula

$$\mu(X) = \omega_0(X) + d(X).$$

The function μ is a measure of noncompactness in the space C(I) [4].

We will make use of the following fixed point theorem due to Darbo [11]. To quote this theorem, we need the following definition

Definition 2.2. Let M be a nonempty subset of a Banach space E, and $\mathcal{P} : M \to \mathrm{E}$ be a continuous operator that transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with constant $\beta \ge 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(\mathcal{P}\, X) \le \beta\, \mu(X).$$

If $\mathcal{P}$ satisfies the Darbo condition with $\beta < 1$ then it is called a contraction operator with respect to μ.

Theorem 2.3 ([11]). *Let Q be a nonempty, bounded, closed and convex subset of the space* E *and let*

$$\mathcal{P} : Q \to Q$$

be a contraction with respect to the measure of noncompactness μ.

Then $\mathcal{P}$ has a fixed point in the set Q.

Remark 2.4 ([6]). Under the assumptions of the above theorem it can be shown that the set Fix $\mathcal{P}$ of fixed points of $\mathcal{P}$ belonging to Q is a member of Ker μ.

Finally, we will need the following two lemmas [7].

Lemma 2.5. *Suppose $x \in C(\mathrm{I})$ and $r : \mathrm{I} \to \mathrm{I}$ is a continuous function. Define*

$$(Fx)(t) = \max_{[0,r(t)]} |x(\tau)| \text{ for } t \in \mathrm{I},$$

then $Fx \in C(\mathrm{I})$.

Lemma 2.6. *Let $(x_n) \subset C(\mathrm{I})$ and $x \in C(\mathrm{I})$. Suppose that $x_n \to x$ in $C(\mathrm{I})$. Then $Fx_n \to Fx$ uniformly on* I.

3. MAIN THEOREM

In this section, we will study Eq. (1.1) assuming that the following assumptions are satisfied:

(a_1) $h : \mathrm{I} \longrightarrow \mathbb{R}$ is a continuous, nondecreasing and nonnegative function on I.

(a_2) The operator $\mathrm{T} : \mathrm{C(I)} \to \mathrm{C(I)}$ is continuous and satisfies the Darbo condition for the measure of noncompactness μ with a constant q. Moreover, T is a positive operator, i.e., $\mathrm{T}x \geq 0$ if $x \geq 0$.

(a_3) There exist nonnegative constants a and b such that

$$|(\mathrm{T}\, x)(t)| \leq a + b\, \|x\|$$

for each $x \in \mathrm{C(I)}$ and $t \in \mathrm{I}$.

(a_4) $k : \mathrm{I}\times\mathrm{I} \longrightarrow \mathbb{R}_+$ is continuous on $\mathrm{I}\times\mathrm{I}$ and $k(t,s)$ is nondecreasing for each variable t and s, separately.

(a_5) $r : \mathrm{I} \longrightarrow \mathrm{I}$ is a continuous and nondecreasing function on I.

(a_6) There exists $r_0 > 0$ such that

$$\|h\|\, \Gamma(\alpha+1) + (a + b\, r_0)\, k^*\, r_0 \leq r_0\, \Gamma(\alpha+1) \tag{3.1}$$

and $q\, k^*\, r_0 < \Gamma(\alpha+1)$, where $k^* = \sup\{k(t,s) : (t,s) \in \mathrm{I}\times\mathrm{I}\}$.

Now, we are in a position to state and prove our main result.

Theorem 3.1. *Let the assumptions* $(a_1) - (a_6)$ *be satisfied. Then Eq.* (1.1) *has at least one solution* $x \in \mathrm{C(I)}$ *which is nondecreasing on the interval* I.

Proof. Let $\mathcal{K}$ and $\mathcal{F}$ be the two operators defined on the space C(I) by

$$(\mathcal{K}\, x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \tag{3.2}$$

and

$$(\mathcal{F}\, x)(t) = h(t) + (\mathrm{T}x)(t)\cdot(\mathcal{K}\, x)(t), \tag{3.3}$$

respectively. Solving Eq. (1.1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space C(I).

First, we prove that $\mathcal{F}$ transforms the space C(I) into itself. To do this it suffices to show that if $x \in \mathrm{C(I)}$ then $\mathcal{K}\, x \in \mathrm{C(I)}$. Fix $\varepsilon > 0$, let $x \in \mathrm{C(I)}$ and $t_1,\ t_2 \in \mathrm{I}$ such that $t_2 \geq t_1$ and $|t_2 - t_1| \leq \varepsilon$. Then

$$\begin{aligned}
|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \right.
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
& + \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
& - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
& + \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
& - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{|k(t_2,s)-k(t_1,s)|}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|k(t_1,s)|}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |k(t_1,s)|\, |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| \max_{[0,r(s)]} |x(\tau)|\, ds.
\end{aligned}
$$

Therefore, if

$$\omega_k(\varepsilon,.) = \sup\{|k(t,s)-k(\tau,s)| : \; t,\; \tau \in \mathrm{I} \text{ and } |t-\tau| \leq \varepsilon\}$$

we obtain

$$
\begin{aligned}
|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \;\; \leq \;\; & \frac{\|x\|}{\Gamma(\alpha)}\, \omega_k(\varepsilon,.) \int_0^{t_2} (t_2-s)^{\alpha-1}\, ds \\
& + \frac{k^*\, \|x\|}{\Gamma(\alpha)} \left\{ \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] ds \right. \\
& \qquad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right\} \\
\leq \;\; & \frac{\|x\|}{\Gamma(\alpha+1)}\, \omega_k(\varepsilon,.)\; t_2^{\alpha} + \frac{k^*\, \|x\|}{\Gamma(\alpha+1)} [t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha}] \\
\leq \;\; & \frac{\|x\|}{\Gamma(\alpha+1)}\, \omega_k(\varepsilon,.) + \frac{2\, k^*\, \|x\|}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha} \\
\leq \;\; & \frac{\|x\|}{\Gamma(\alpha+1)}\, \omega_k(\varepsilon,.) + \frac{2\, k^*\, \|x\|}{\Gamma(\alpha+1)}\, \varepsilon^{\alpha} \qquad (3.4)
\end{aligned}
$$

In the view of the uniform continuity of the function k on $\mathrm{I}\times\mathrm{I}$ we have that $\omega_k(\varepsilon,.) \to 0$ as $\varepsilon \to 0$. Thus $\mathcal{K}x \in \mathrm{C(I)}$, and consequently, $\mathcal{F}x \in \mathrm{C(I)}$. Moreover, for each $t \in \mathrm{I}$ we have

$$
\begin{aligned}
|(\mathcal{F}x)(t)| \;\; \leq \;\; & \left| h(t) + \frac{(\mathrm{T}x)(t)}{\Gamma(\alpha)} \int_0^{t} \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \right| \\
\leq \;\; & \|h\| + \frac{a + b\, \|x\|}{\Gamma(\alpha)} \int_0^{t} \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
\leq \;\; & \|h\| + \frac{a + b\, \|x\|}{\Gamma(\alpha+1)}\, k^*\, \|x\|
\end{aligned}
$$

Hence

$$\|\mathcal{F}x\| \leq \|h\| + \frac{a + b\,\|x\|}{\Gamma(\alpha+1)}\, k^* \,\|x\|$$

Thus, if $\|x\| \leq r_0$ we obtain from assumption (a_6) that

$$\|\mathcal{F}x\| \leq \|h\| + \frac{a + b\,r_0}{\Gamma(\alpha+1)}\, k^* \,r_0 \leq r_0$$

Consequently, the operator F transforms the ball B_{r_0} into itself.

In what follows we will consider the operator $\mathcal{F}$ on the subset $B^+_{r_0}$ of the ball B_{r_0} defined by

$$B^+_{r_0} = \{x \in B_{r_0} : x(t) \geq 0, \text{ for } t \in \mathrm{I}\}.$$

Obviously, the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. In view of this facts and assumptions (a_1), (a_3) and (a_5), we deduce that $\mathcal{F}$ transforms the set $B^+_{r_0}$ into itself.

Next, we prove that the operator $\mathcal{F}$ is continuous on $B^+_{r_0}$. To do this, let us fix a sequence (x_n) in $B^+_{r_0}$ such that $x_n \to x$ and we will prove that $\mathcal{F}x_n \to \mathcal{F}x$. In fact, for each $t \in \mathrm{I}$ we have

$$\begin{aligned}
|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)| &= \Bigg| \frac{(\mathrm{T}x_n)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x_n(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\leq \Bigg| \frac{(\mathrm{T}x_n)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x_n(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x_n(\tau)|\, ds \Bigg| \\
&\quad + \Bigg| \frac{(\mathrm{T}x)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x_n(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t)}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} |(\mathrm{T}x_n)(t) - (\mathrm{T}x)(t)| \int_0^t \frac{|k(t,s)|}{(t-s)^{1-\alpha}} \max_{[0,r(s)]} |x_n(\tau)|\, ds \\
&\quad + \frac{|(\mathrm{T}x)(t)|}{\Gamma(\alpha)} \int_0^t \frac{|k(t,s)|}{(t-s)^{1-\alpha}} \left| \max_{[0,r(s)]} |x_n(\tau)| - \max_{[0,r(s)]} |x(\tau)| \right| ds
\end{aligned}$$

In virtue of Lemma 2.6, we get

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \frac{k^*\, r_0}{\Gamma(\alpha+1)} \|\mathrm{T}x_n - \mathrm{T}x\| + \frac{k^*\,(a + b\,r_0)}{\Gamma(\alpha+1)} \|x_n - x\|. \tag{3.5}$$

As T is a continuous operator, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ we have

$$\|\mathrm{T}x_n - \mathrm{T}x\| \leq \frac{\varepsilon\, \Gamma(\alpha+1)}{2\, k^*\, r_0}.$$

Moreover, we can find $n_2 \in \mathbb{N}$ such that for $n \geq n_2$ we have

$$\|x_n - x\| \leq \frac{\varepsilon\,\Gamma(\alpha+1)}{2\,k^*\,(a+b\,r_0)}.$$

Now, if we take $n \geq \max\{n_1, n_2\}$, from (3.5) we obtain

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \varepsilon.$$

This proves that $\mathcal{F}$ is continuous in $B^+_{r_0}$.

Now, let us take a nonempty set $X \subset B^+_{r_0}$. Fix arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $t_1,\ t_2 \in \mathrm{I}$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_2 \geq t_1$. Then, in view of our assumptions, we obtain

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &\leq |h(t_2) - h(t_1)| + |(\mathrm{T}x)(t_2)\,(\mathcal{K}x)(t_2) - (\mathrm{T}x)(t_2)\,(\mathcal{K}x)(t_1)| \\
&\quad + |(\mathrm{T}x)(t_2)\,(\mathcal{K}x)(t_1) - (\mathrm{T}x)(t_1)\,(\mathcal{K}x)(t_1)| \\
&\leq \omega(h,\varepsilon) + |(\mathrm{T}x)(t_2)|\ |(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\quad + |(\mathrm{T}x)(t_2) - (\mathrm{T}x)(t_1)|\ |(\mathcal{K}x)(t_1)|\,.
\end{aligned}$$

By using our assumptions and inequality (3.4), we have

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &\leq \omega(h,\varepsilon) + \frac{(a+b\,\|x\|)}{\Gamma(\alpha+1)}\,[\|x\|\,\omega_k(\varepsilon,.) + 2\,k^*\,\|x\|\,\varepsilon^\alpha] \\
&\quad + \frac{\omega(\mathrm{T}x,\varepsilon)}{\Gamma(\alpha+1)}\,\|x\|\,k^* \\
&\leq \omega(h,\varepsilon) + \frac{r_0\,(a+b\,r_0)}{\Gamma(\alpha+1)}[\omega_k(\varepsilon,.) + 2\,k^*\,\varepsilon^\alpha] \\
&\quad + \frac{k^*\,r_0}{\Gamma(\alpha+1)}\omega(\mathrm{T}x,\varepsilon).
\end{aligned}$$

Hence,

$$\omega(\mathcal{F}x,\varepsilon) \leq \omega(h,\varepsilon) + \frac{r_0\,(a+b\,r_0)}{\Gamma(\alpha+1)}[\omega_k(\varepsilon,.) + 2\,k^*\,\varepsilon^\alpha] + \frac{k^*\,r_0}{\Gamma(\alpha+1)}\omega(\mathrm{T}x,\varepsilon).$$

Consequently,

$$\omega(\mathcal{F}\,\mathrm{X},\varepsilon) \leq \omega(h,\varepsilon) + \frac{r_0\,(a+b\,r_0)}{\Gamma(\alpha+1)}[\omega_k(\varepsilon,.) + 2\,k^*\,\varepsilon^\alpha] + \frac{k^*\,r_0}{\Gamma(\alpha+1)}\omega(\mathrm{T}\,\mathrm{X},\varepsilon).$$

In the view of the uniform continuity of the function k on $\mathrm{I} \times \mathrm{I}$ and the continuity of the function h on I then from the last inequality, we have

$$\omega_0(\mathcal{F}\,\mathrm{X}) \leq \frac{k^*\,r_0}{\Gamma(\alpha+1)}\,\omega_0(\mathrm{T}\,\mathrm{X}). \tag{3.6}$$

In what follows, fix arbitrary $x \in \mathrm{X}$ and $t_1,\ t_2 \in \mathrm{I}$ with $t_2 > t_1$. Then, taking into account our assumptions, we have

$$
\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)] \\
&\quad = \Bigg| h(t_2) + \frac{(\mathrm{T}x)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - h(t_1) - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\qquad - \Bigg[h(t_2) + \frac{(\mathrm{T}x)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - h(t_1) - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg] \\
&\quad \le \{ |h(t_2) - h(t_1)| - [h(t_2) - h(t_1)] \} \\
&\qquad + \Bigg| \frac{(\mathrm{T}x)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\qquad + \Bigg| \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\qquad - \Bigg\{ \Bigg[\frac{(\mathrm{T}x)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg] \\
&\qquad + \Bigg[\frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg] \Bigg\} \\
&\quad \le \{ |(\mathrm{T}x)(t_2) - (\mathrm{T}x)(t_1)| - [(\mathrm{T}x)(t_2) - (\mathrm{T}x)(t_1)] \} \\
&\quad \times \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad + \frac{(\mathrm{T}x)(t_1)}{\Gamma(\alpha)} \Bigg\{ \Bigg| \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg| \\
&\qquad - \Bigg[\int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \Bigg] \Bigg\} \qquad (3.7)
\end{aligned}
$$

Now, we will prove that

$$\int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \geq 0.$$

In fact, we have

$$\begin{aligned}
&\int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&= \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad + \int_0^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad + \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&= \int_0^{t_2} \frac{(k(t_2,s)-k(t_1,s))}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds + \int_{t_1}^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad + \int_0^{t_1} k(t_1,s)\, [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \max_{[0,r(s)]} |x(\tau)|\, ds.
\end{aligned}$$

Since $k(t,s)$ is nondecreasing with respect to t, we have that $k(t_2,s) \geq k(t_1,s)$, then

$$\int_0^{t_2} \frac{(k(t_2,s)-k(t_1,s))}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \geq 0. \tag{3.8}$$

On the other hand, since the term $(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}$ is negative for $0 \leq s < t_1$, thus

$$\begin{aligned}
&\int_0^{t_1} k(t_1,s)\, [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \max_{[0,r(s)]} |x(\tau)|\, ds + \int_{t_1}^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \\
&\qquad \geq \int_0^{t_1} k(t_1,t_1)\, [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \max_{[0,r(t_1)]} |x(\tau)|\, ds \\
&\qquad\quad + \int_{t_1}^{t_2} \frac{k(t_1,t_1)}{(t_2-s)^{1-\alpha}} \max_{[0,r(t_1)]} |x(\tau)|\, ds \\
&\qquad = k(t_1,t_1) \max_{[0,r(t_1)]} |x(\tau)| \left[\int_0^{t_2} \frac{ds}{(t_2-s)^{1-\alpha}} - \int_0^{t_1} \frac{ds}{(t_1-s)^{1-\alpha}}\right] \\
&\qquad = k(t_1,t_1)\, \frac{t_2^\alpha - t_1^\alpha}{\alpha} \max_{[0,r(t_1)]} |x(\tau)| \\
&\qquad \geq 0. \qquad (3.9)
\end{aligned}$$

Finally, (3.8) and (3.9) imply that

$$\int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds - \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds \geq 0.$$

This together with (3.7) yields

$$\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)] \\
&\qquad = \{|(\mathrm{T}x)(t_2) - (\mathrm{T}x)(t_1)| - [(\mathrm{T}x)(t_2) - (\mathrm{T}x)(t_1)]\}
\end{aligned}$$

$$\times \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,r(s)]} |x(\tau)|\, ds$$

$$\leq \frac{k^*\, r_0}{\Gamma(\alpha+1)}\, d(\mathrm{T}\, x). \tag{3.10}$$

Therefore,

$$d(\mathcal{F}\, x) \leq \frac{k^*\, r_0}{\Gamma(\alpha+1)}\, d(\mathrm{T}\, x)$$

and consequently,

$$d(\mathcal{F}\, \mathrm{X}) \leq \frac{k^*\, r_0}{\Gamma(\alpha+1)}\, d(\mathrm{T}\, \mathrm{X}). \tag{3.11}$$

Finally, from (3.6) and (3.11) and the definition of the measure of noncompactness μ, we obtain

$$\begin{aligned} \mu(\mathcal{F}\, X) &\leq \frac{k^*\, r_0}{\Gamma(\alpha+1)}\, \mu(\mathrm{T}\, X) \\ &\leq \frac{k^*\, r_0\, q}{\Gamma(\alpha+1)}\, \mu(X). \end{aligned}$$

Now, the above obtained inequality together with the fact that $k^*\, r_0\, q < \Gamma(\alpha+1)$ enable us to apply Theorem 2.3, then Eq. (1.1) has at least one solution $x \in \mathrm{C(I)}$. This completes the proof. □

4. **EXAMPLE**

Consider the following singular quadratic integral equation of Abel type with supremum, $\alpha = \frac{1}{2}$,

$$x(t) = t^3 + \frac{1}{4\,\Gamma(\frac{1}{2})} \int_0^t \frac{e^t}{(1+s^2)\sqrt{t-s}} \max_{[0,\sqrt{s}]} |x(\tau)|\, ds. \tag{4.1}$$

In this example, we have that $h(t) = t^3$ and this function satisfies assumption (a_1) and $\|h\| = 1$. Moreover, $k(t,s) = \frac{e^t}{1+s^2}$ verifies assumption (a_4) and $k^* = e$. The function r is defined by $r(s) = \sqrt{s}$ and satisfies assumption (a_5). Also, $(\mathrm{T}x)(t) = \frac{1}{4}$ and satisfies assumptions (a_2) and (a_3) with $q = 0$, $a = \frac{1}{4}$ and $b = 0$. In this case the inequality (3.1) has the form

$$\Gamma(\frac{3}{2}) + \frac{1}{4}\, e\, r_0 \leq r_0\, \Gamma(\frac{3}{2})$$

or

$$\Gamma(\frac{1}{2}) + \frac{e}{2}\, r_0 \leq r_0\, \Gamma(\frac{1}{2})$$

and this admits

$$r_0 = \frac{2\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{1}{2}) - e}$$

as a positive solution since $\Gamma(\frac{1}{2}) \simeq 1.77245$. Moreover, as $q = 0$,

$$q\, r_0\, k^* < \Gamma(\frac{3}{2}).$$

Theorem 3.1 guarantees that equation (4.1) has a nondecreasing solution.

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