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OPTIMAL CHOICE OF NONLINEAR OUTPUT FEEDBACK CONTROL LAW FOR A CLASS OF UNCERTAIN PARABOLIC SYSTEMS

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Abstract. In this paper we consider optimal output feedback boundary control problems for a class of semilinear uncertain parabolic systems. The uncertain initial boundary value problem is converted into an equivalent Cauchy problem described by a differential inclusion in appropriate Banach spaces. We follow game-theoretic formalism and prove existence of saddle points giving optimal strategies. This is an extension of a recent result of the author from linear to a class of nonlinear feedback operators. The paper is concluded with a brief description of open problems and future directions.

Key words: Parabolic Evolution Equations, Uncertain Systems, Output Feedback, Optimal Boundary Control, Optimal Feedback Operators.

AMS (MOS) Subject Classification: 49J20, 49J24, 49N35, 93C25, 93C20, 93B52,65N21, 47A62

1. INTRODUCTION

This paper presents an extension of a recent result of the author [1] on the existence of saddle points (optimal strategies) of a games problem arising from control problem of an uncertain parabolic system. Optimal control of uncertain systems described by differential inclusions on Banach spaces have been widely considered in the literature [5, 6, 7] (see also the references therein). These papers present results on existence of optimal (open loop) strategies. Recently Mordukhovich [4] considered the problem of optimal design of output feedback controller for a class of uncertain systems described by second order parabolic equations with Dirichlet boundary control. The design variable here is the feedback control law mapping output into control actions on the boundary. In a recent paper of the author [1], a general class of uncertain parabolic systems was considered with the class of output feedback control laws chosen from the class of bounded linear operators furnished with the strong operator topology. Here we consider a class of continuous nonlinear operators furnished with the topology of pointwise convergence [11]. The problem considered is the question of existence of saddle points and the corresponding optimal strategies.

The rest of the paper is organized as follows: In section 2 we present the system model on output feedback boundary control of uncertain parabolic systems. In section 3 we formulate this as an abstract differential inclusion on a Banach space and conclude with the problem formulation as a games problem where an optimal feedback operator from the space of observations to the control space is sought. In section 4 we prove existence of saddle points. In section 5 we present an extension of this result to systems with state dependent multifunction representing state dependent uncertainty. We conclude the paper pointing out some open problems and future directions.

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2. UNCERTAIN PARABOLIC BOUNDARY CONTROL PROBLEM

Let Ω be an open bounded connected domain in R^n with smooth boundary $\partial\Omega$, $I = [0, T], T < \infty$, and L an elliptic (partial) differential operator and $\mathcal{B}$ a suitable boundary operator compatible with the differential operator L and F is a multifunction. The system is governed by the following parabolic inclusion representing a boundary controlled uncertain system:

$$\partial\phi/\partial t + L\phi \in F(t, \xi)\ (t, \xi) \in I \times \Omega \tag{1}$$

$$\mathcal{B}\phi = u,\ (t, \xi) \in I \times \partial\Omega, \tag{2}$$

$$\phi(0, \cdot) = \phi_0. \tag{3}$$

The function u defined on $I \times \partial\Omega$ is the control and ϕ_0 is the initial state.

Let E and Y denote any pair of suitable Banach spaces of functions or generalized functions defined on Ω and $\partial\Omega$ respectively with ϕ taking values from E and control u taking values from Y at any given time $t \in I$. In addition to the system equation, there is the output equation given by

$$z = G(\phi), (\text{measured output}), \tag{4}$$

with z taking values from another suitable Banach space Z representing the output space and $G : E \longrightarrow Z$ is a continuous map giving the output. Here $F : I \times \Omega \longrightarrow 2^R \setminus \emptyset$ is a suitable multifunction representing the uncertainty. Loosely stated, the problem is to find an output-feedback control law $f : Z \longrightarrow Y$ giving

$$u = f(G(\phi)) \tag{5}$$

that minimizes the maximum risk or loss which is given by a suitable functional of the measured output as the observation.

3. SEMIGROUP SETTING OF THE BASIC PROBLEM

It is shown in [2, p. 59–66] and [3, p. 214–220] that the controlled initial-boundary problem as stated above can be formulated as a controlled initial value (Cauchy) problem in the Banach space E (state space) as follows:

$$\dot{x} + Ax \in F + A\mathcal{R}u, x(0) = x_0, \tag{6}$$

$$u \equiv f(G(x)), f \in C(Z, Y), t \in I, \tag{7}$$

where

$$A \equiv L|_{Ker\mathcal{B}}, \quad \mathcal{R} \equiv \left(\mathcal{B}|_{KerL}\right)^{-1}, \quad F : I \longrightarrow 2^E \setminus \emptyset,$$

and $G : E \longrightarrow Z$, is the state-output map and $C(Z, Y)$ is the space of continuous maps from the Banach space Z to the Banach space Y representing potential feedback control laws mapping the output space into the space where controls take their values from. For detailed derivation leading to the above formulation see [2, p 59–66; 3, p 214–220].

We use the abstract model (6)–(7) to study nonlinear output feedback boundary control problems. This generalizes our recent paper [1] where we used linear output feedback boundary control laws.

Suppose $-A$ is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, in E; F is a graph measurable multifunction and is assumed to have integrable selections $\{w\}, w(t) \in F(t), a.e.\ t \in I$. Let $\mathcal{S}_F$ denote the corresponding set of measurable selections (precise assumptions to follow shortly). Using the variation of constants formula, the evolution inclusion (6)–(7) can be written as the following integral inclusion in E,

$$x(t) \in S(t)x_0 + \int_0^t S(t-s)F(s)ds + \int_0^t AS(t-s)\mathcal{R}f(G(x(s)))ds, t \in I.$$

By a solution of this inclusion, we mean a solution of the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)w(s)ds + \int_0^t AS(t-s)\mathcal{R}f(G(x(s)))ds, t \in I, \tag{8}$$

for each selection $w \in \mathcal{S}_F$. A solution of this integral equation is known as the mild solution of (6)–(7) with the multifunction F replaced by a selection $w \in \mathcal{S}_F$.

The Set of Uncertainty $\mathcal{S}_F$: Let $wkc(E)$ denote the class of nonempty weakly compact convex subsets of E and $F : I \longrightarrow wkc(E)$ be graph measurable and integrably bounded. Let $\mathcal{S}_F$ denote the set of all integrable selections of the multifunction F. Then it is known [8, Theorem 3.34, p 187] that the set $\mathcal{S}_F \subset L_1(I,E)$ is weakly compact and convex. Therefore, by Eberlein-Smulian theorem [8, Theorem A.3.62, p 914], it is also weakly sequentially compact. We choose $\mathcal{S}_F$ to be the set of uncertainty.

Admissible Control Laws $\mathcal{F}_{ad}$: Recall that Z and Y are two Banach spaces with the former representing the space of outputs and the later the space where the controls take their values from. Let $C(Z,Y) \subset Y^Z$ denote the space of continuous maps from Z to Y furnished with topology of pointwise convergence on Z in the weak topology of Y. We denote this topology by τ_{pcw} and the space by $(C(Z,Y), \tau_{pcw})$. For a given positive number K, let $\mathcal{F}_K \subset C(Z,Y)$ denote the class of functions satisfying the following properties:

$$\begin{aligned}
&(1) : \| f(z) \|_Y \leq K(1+ \| z \|_Z) \ \forall \ z \in Z \\
&(2) : \text{for every } r \in (0,\infty), \exists \ K_r \geq 0 \text{ such that} \\
&\| f(z_1) - f(z_2) \|_Y \leq K_r \| z_1 - z_2 \|_Z, z_1, z_2 \in B_r(Z).
\end{aligned}$$

For each $z \in Z$, let P_z denote the evaluation map: $P_z(f) = f(z), f \in C(Z,Y)$. For admissible control laws we choose the set $\mathcal{F}_{ad} \subset \mathcal{F}_K$ satisfying the following two properties:

(P1): $\mathcal{F}_{ad}$ is point wise closed in $(C(Z,Y), \tau_{pcw})$

(P2): for each $z \in Z$, the set $\{P_z(f) : f \in \mathcal{F}_{ad}\} \subset Y$ is relatively weakly compact.

Clearly Y furnished with the weak topology is a Hausdorff space. Thus it follows from [11, Theorem 42.3, p 278] that this set is compact in the topology of point wise (weak) convergence.

Remark 3.1 For any pair of Banach spaces $\{Z,Y\}$, the set $\mathcal{F}_K$ as defined above is τ_{pcw} closed but not necessarily compact. This is easily verified by use of Hahn-Banach theorem. However, if Y is a reflexive Banach space, one can easily verify that the set $\mathcal{F}_K$ itself satisfies both the properties (P1) and (P2). In this case we can choose $\mathcal{F}_{ad} = \mathcal{F}_K$.

Here we quote a result from [2] which we use throughout the paper.

Theorem 3.2 Suppose the following assumptions hold: (A1): $-A$ is the generator of an analytic semigroup $S(t)$, $t \geq 0$, in E, (A2): There exist constants $c > 0$ and $\beta \in (0,1]$ such that

$$\| AS(t)\mathcal{R} \|_{\mathcal{L}(Y,E)} \leq \frac{c}{t^{1-\beta}}, t > 0,$$

(A3): there exists a $p \in (1/\beta, \infty)$ such that $\mathcal{S}_F \subset L_p(I,E)$, (A4): $G : E \longrightarrow Z$ is locally Lipschitz having at most linear growth. Then for each $x_0 \in E$, $w \in \mathcal{S}_F$ and $f \in \mathcal{F}_K$, the integral equation (8) has a unique solution $x \in C(I,E)$ and hence the system (6)–(7) has a nonempty set of mild solutions.

Proof. See [2, Theorem 3.1, p 64; Theorem 3.2, p 69].

Problem Statement: Let $\mathcal{F}_{ad} \subset \mathcal{F}_K$ denote the class of admissible (output) feedback operators. Suppose $x_0 \in E$ is fixed, and let $x(w,f) \equiv x(\cdot, w, f)$ denote the mild solution of the evolution equation

$$\dot{x} + Ax = w + A\mathcal{R}f(G(x)), x(0) = x_0, t \in I, \tag{9}$$

or more precisely (equivalent) the solution of the integral equation (8) corresponding to $(w,f) \in \mathcal{S}_F \times \mathcal{F}_{ad}$. Define

$$J(w,f) \equiv \int_I \ell(t, G(x(t,w,f)))dt + \Phi(G(x(T,w,f))), \tag{10}$$

where the first term gives the running cost and the second gives the terminal cost with $\ell : I \times Z \longrightarrow R$ and $\Phi : Z \longrightarrow R$ being Borel measurable. Precise assumptions on ℓ and Φ are given in Lemma 4.2 below. The problem is to find a feedback control law $f^o \in \mathcal{F}_{ad}$ such that

$$J^o(f^o) \leq J^o(f) \equiv \max\{J(w,f), w \in \mathcal{S}_F\}$$

for all $f \in \mathcal{F}_{ad}$. This is the basic problem and it is equivalent to the minmax problem: find a pair $(w^o, f^o) \in \mathcal{S}_F \times \mathcal{F}_{ad}$ such that

$$J(w^o, f^o) = \min_{f \in \mathcal{F}_{ad}} \max_{w \in \mathcal{S}_F} J(w,f). \tag{11}$$

Clearly, this is a games problem (games against natural uncertainty), and naturally we are interested in the saddle point of the problem (if one exists).

Definition 3.3 A point $(w^o, f^o) \in \mathcal{S}_F \times \mathcal{F}_{ad}$ is said to be a saddle point of the functional J if it satisfies the following inequalities

$$J(w, f^o) \leq J(w^o, f^o) \leq J(w^o, f) \;\forall\; (w,f) \in \mathcal{S}_F \times \mathcal{F}_{ad}. \tag{12}$$

A celebrated result on games theory due to Von Neumann-Ky Fan-Sion (1928, 1952, 1958) can be found in [9, Theorem 9.1, p 458]. This result requires the Banach spaces involved to be reflexive and the functional $J(w,f)$ to be quasi-concave in w and quasi-convex in f. Unfortunately these conditions are not satisfied for the problem considered here. Clearly the space $(C(Z,Y), \tau_{pcw})$ is not reflexive, E is not assumed to be reflexive and further the concavity and convexity conditions do not hold. In view of this we must attack the problem directly.

4. EXISTENCE OF OPTIMAL STRATEGIES: SADDLE POINTS

In this section we prove the existence of a saddle point for the functional given by (10) and hence existence of an optimal feedback control law. For this we need the continuity of the map

$$(w,f) \longrightarrow x(w,f).$$

Let τ^w denote the weak topology on $L_p(I,E)$ and τ_{pcw} denote the topology of point wise (weak) convergence in $(C(Z,Y), \tau_{pcw})$ as introduced earlier (see admissible control laws). Let τ_u denote the standard topology of uniform convergence in $C(I,E)$ and τ_p the point wise convergence topology. We prove, under certain assumptions, that the map $(w,f) \longrightarrow x(w,f)$ is continuous with respect to the topologies $\tau^w \times \tau_{pcw}$ on $\mathcal{S}_F \times \mathcal{F}_{ad}$ and τ_p on $C(I,E)$.

Lemma 4.1 Suppose the assumptions of Theorem 3.2 hold. Further assume that the semigroup $S(t), t > 0$, is compact. Then the map $(w,f) \longrightarrow x(w,f)$ is continuous with respect to the topologies $\tau^w \times \tau_{pcw}$ on $\mathcal{S}_F \times \mathcal{F}_{ad}$ and τ_p on $C(I,E)$.

Proof Let (w^n, f^n) be a sequence from $\mathcal{S}_F \times \mathcal{F}_{ad}$ with

$$w^n \xrightarrow{\tau^w} w^o, \;\; f^n \xrightarrow{\tau_{pcw}} f^o.$$

Since $\mathcal{S}_F \times \mathcal{F}_{ad}$ is $\tau^w \times \tau_{pcw}$ compact $(w^o, f^o) \in \mathcal{S}_F \times \mathcal{F}_{ad}$. By Theorem 3.1, equation (9) has unique mild solutions corresponding to the pairs (w^n, f^n) and (w^o, f^o) respectively. Let $x^n \equiv x(w^n, f^n) \in C(I, E), x^o \equiv x(w^o, f^o) \in C(I, E)$ denote these solutions. Define

$$v^n \equiv w^n - w^o, \text{ and } e^n = x^n - x^o.$$

Using the variation of constants formula, the reader can easily verify that

$$e^n(t) = \int_0^t S(t-s)v^n(s)ds + \int_0^t AS(t-s)\mathcal{R}\{f^n(G(x^o(s))) - f^o(G(x^o(s)))\}ds$$

$$+ \int_0^t AS(t-s)\mathcal{R}\{f^n(G(x^n(s))) - f^n(G(x^o(s)))\}ds \tag{13}$$

$$\equiv E_1^n(t) + E_2^n(t) + \int_0^t AS(t-s)\mathcal{R}\{f^n(G(x^n(s))) - f^n(G(x^o(s)))\}ds \tag{14}$$

where for convenience we have denoted the first two components by E_1^n and E_2^n respectively. Since weakly convergent sequences are bounded in norm and by assumption both G as well as the elements of the set $\mathcal{F}_{ad} \subset \mathcal{F}_K$ have at most linear growth, it is clear that the sequence x^n including x^o are contained in a bounded subset of $C(I, E)$. Hence there exists a finite positive number b, possibly dependent on the growth rate of G and that of the family $\mathcal{F}_K$, such that $\{x^n(t), x^o(t)\} \in B_b(E)$ for all $t \in I$ where $B_b(E)$ denotes the ball of radius b in E. Thus it follows from the local Lipschitz hypothesis on G and the family $\mathcal{F}_K$ that there exists a positive number $K_b < \infty$, independent of n, such that for all $n \in N$,

$$|f^n(G(x^n(t))) - f^n(G(x^o(t)))|_Y \le K_b|x^n(t) - x^o(t)|, t \in I.$$

Using this fact and the assumptions (A2)-(A4) of Theorem 3.1, it follows from (14) that

$$|e^n(t)|_E \le |E_1^n(t)|_E + |E_2^n(t)|_E + \int_0^t (cK_b)/(t-s)^{1-\beta}|e^n(s)|_E ds, t \in I. \tag{15}$$

Since $v^n \xrightarrow{\tau^w} 0$ and the semigroup $S(t), t > 0$, is compact, $E_1^n(t) \xrightarrow{s} 0$ in E uniformly on I. Considering E_2^n, we prove that it converges (strongly) in E to zero for each $t \in I$. Define

$$g_n(t) \equiv [f^n(G(x^o(t))) - f^o(G(x^o(t)))], t \in I.$$

Since $\{f^n, f^o\} \subset \mathcal{F}_{ad} \subset \mathcal{F}_K$ and G is a continuous map from E to Z, and $x^o \in C(I, E)$, the function $t \longrightarrow g_n(t)$ is continuous. Further, since $f^n \xrightarrow{\tau_{pcw}} f^o$ it is clear that $g_n(t) \xrightarrow{w} 0$ in Y for each $t \in I$. By assumption, $S(t)$, $t > 0$, is a compact semigroup and $\mathcal{R}$ is a bounded operator from Y to E and so the composition $(S\mathcal{R})(t)$, $t > 0$, is also a family of compact operators from Y to E. Thus

$$\eta_n(t) \equiv \int_0^t S(t-s)\mathcal{R}g_n(s)ds \xrightarrow{s} 0 \text{ in } E$$

for each $t \in I$. It follows from assumption (A2) and (A3) of Theorem 3.1 with $1/\beta < p < \infty$, that the integral operator H given by

$$(Hg)(t) \equiv \int_0^t AS(t-r)\mathcal{R}g(r)dr, \quad t \in I$$

maps $L_p(I, Y)$ to E for each $t \in I$. Since A is a closed operator and $g_n \in L_p(I, Y)$, this means that, for each $t \in I$, $\eta_n(t) \in D(A)$ and that

$$A\eta_n(t) = (Hg_n)(t) \xrightarrow{s} 0 \text{ in } E.$$

Thus we have proved that

$$E_2^n(t) \xrightarrow{s} 0 \text{ in } E$$

for each $t \in I$, and hence

$$E_n(t) \equiv |E_1^n(t)|_E + |E_2^n(t)|_E \longrightarrow 0 \text{ for each } t \in I.$$

For convenience of notation we set $\varphi_n(t) \equiv |e^n(t)|_E$ and rewrite inequality (15) as follows

$$\varphi_n(t) \leq E_n(t) + \int_0^t M/(t-s)^{1-\beta}\varphi_n(s)ds, \quad t \in I, \tag{16}$$

with $M = (cK_b)$. Clearly $t \longrightarrow \varphi_n(t)$ is continuous and nonnegative. The function $E_n(t)$ is also nonnegative, bounded measurable, and converges point wise to zero. We must show that $\varphi_n(t) \longrightarrow 0$ point wise in I. By repeated substitution of (16) into itself it is easy to verify that

$$\varphi_n(t) \leq E_n(t) + (\mathcal{K}E_n)(t), \qquad t \in I, \tag{17}$$

where the operator $\mathcal{K}$ is the Volterra integral operator, called the resolvent operator,

$$(\mathcal{K}h)(t) = \int_0^t K(t-s)h(s)ds, \qquad t \in I$$

with the kernel K given by

$$K(t) = \sum_{m=1}^{\infty} K_m(t),$$
$$K_m(t) \equiv \big\{(M\Gamma(\beta))^m/\Gamma(m\beta)\big\}(1/t^{1-m\beta}), \quad m \geq 1,$$

where Γ denotes the standard gamma function. These are obtained by repeated iteration of the basic kernel $K_1(t) \equiv M/t^{1-\beta}, t \in I$, which appears in our integral inequality (16). For example, the $(m+1)$-th iterated kernel is given by

$$K_{m+1}(t-s) \equiv \int_s^t K_m(t-r)K_1(r-s)dr, \quad t > s.$$

By simple computation using standard gamma functions one can easily verify that

$$K_{m+1}(t-s) = \left(\frac{(M\Gamma(\beta))^{m+1}}{\Gamma((m+1)\beta)}\right)1/(t-s)^{1-(m+1)\beta}, \quad t > s.$$

Using the properties of Γ functions, it is easy to verify that, for each $\varepsilon > 0$, the infinite series giving the kernel K converges uniformly on $I_\varepsilon \equiv [\varepsilon, T]$ for every finite $T > 0$. Further $K \in L_1^+[0,T]$ for any finite $T \geq 0$. Thus the integral operator $\mathcal{K}$ maps $L_p(I)$ to $L_p(I)$ for every $p \in [1,\infty]$. Let $B(I,R) \subset L_\infty(I,R)$, furnished with the sup norm topology, denote the Banach space of bounded measurable functions on I with values in R. Clearly $\mathcal{K}$ is also a bounded linear operator in $B(I,R)$. Since $E_n \in B(I,R)$ and it converges to zero point wise on I, it follows from the inequality (17) that $\varphi_n(t) \to 0$ point wise on I. Thus $e^n(t) \xrightarrow{s} 0$ in E for each $t \in I$ proving that $x^n(t) \xrightarrow{s} x^o(t)$ in E point wise on I. This proves the continuity as stated in the lemma. •

Lemma 4.2 Suppose the assumptions of Lemma 4.1 hold and that the integrand $\ell : I \times Z \longrightarrow \bar{R}$ is measurable in the first argument and continuous in the second and there exist $h \in L_1^+(I)$, $r \geq 1$ and $c_1 \geq 0$ such that

$$|\ell(t,z)| \leq h(t) + c_1|z|_Z^r, \ \forall\ (t,z) \in I \times Z. \tag{18}$$

The functional $\Phi : Z \longrightarrow \bar{R}$ is continuous and there exist constants $c_2 \geq 0, c_3 \geq 0$ such that

$$|\Phi(z)| \leq c_2 + c_3|z|_Z^r \ \forall\ z \in Z. \tag{19}$$

Then the objective functional $(w,f) \longrightarrow J(w,f)$ given by (10) is jointly continuous from $\mathcal{S}_F \times \mathcal{F}_{ad}$ to R with respect to the product topology $\tau^w \times \tau_{pcw}$.

Proof By Lemma 4.1, $(w, f) \longrightarrow x(w, f)$ is continuous from $\mathcal{S}_F \times \mathcal{F}_{ad}$ to $C(I, E)$ with respect to the topologies $\tau^w \times \tau_{pcw}$ and τ_p respectively. Since $G : E \longrightarrow Z$ is continuous, it is clear that

$$(w, f) \longrightarrow G(x(t, w, f))$$

is continuous for all $t \in I$. For convenience of notation let τ^{ww} denote the product topology $\tau^w \times \tau_{pcw}$ and suppose

$$(w^n, f^n) \xrightarrow{\tau^{ww}} (w^o, f^o).$$

Then by Lemma 4.1,

$$x^n(t) \equiv x(t, w^n, f^n) \xrightarrow{s} x(t, w^o, f^o) = x^o(t) \text{ in } E$$

for each $t \in I$. Thus by continuity of G from E to Z, and continuity of ℓ in its second argument on Z, and continuity of Φ on Z, we conclude that

$$\ell(t, G(x^n(t))) \longrightarrow \ell(t, G(x^o(t))) \ a.e\ t \in I, \tag{20}$$

$$\Phi(G(x^n(T))) \longrightarrow \Phi(G(x^o(T))). \tag{21}$$

It follows from (18) and the growth assumptions on G and ℓ, that there exists a constant $\tilde{c}_1 > 0$ such that

$$|\ell(t, G(x^n(t)))| \leq h(t) + \tilde{c}_1(1 + |x^n(t)|_E^r), \quad t \in I. \tag{22}$$

Since $f^n, f^o \in \mathcal{F}_{ad} \subset \mathcal{F}_K$ and G is assumed to have at most linear growth there exists a constant $b > 0$ such that

$$\sup\{\| x^n \|_{C(I,E)}, \| x^o \|_{C(I,E)}\} \leq b.$$

So $\ell^n(\cdot) \equiv \ell(\cdot, G(x^n(\cdot))) \in L_1(I)$ and it is dominated by the integrable function given by $\tilde{\ell}(t) \equiv h(t) + \tilde{c}_1(1 + b^r)$. Hence by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_I \ell(t, G(x^n(t)))dt = \int_I \lim_{n\to\infty} \ell(t, G(x^n(t)))dt = \int_I \ell(t, G(x^o(t)))dt. \tag{23}$$

By continuity of Φ and G it is clear that

$$\lim_{n\to\infty} \Phi(G(x^n(T))) = \Phi(G(x^o(T))). \tag{24}$$

Thus it follows from the expression (10) defining the functional J that

$$J(w^n, f^n) \longrightarrow J(w^o, f^o).$$

This proves the joint continuity as stated in the lemma. •

Now we are prepared to prove the existence of a saddle point for the functional J as stated in the problem statement in section 3. The basic steps of proof are similar to those of our paper [1, Theorem 3.3, p 67]. Recall that $\mathcal{S}_F$ denotes the set of uncertainty and $\mathcal{F}_{ad}$ the admissible control laws.

Theorem 4.3 Consider the objective functional J as defined by (10) and let the assumptions of Lemma 4.2 hold. Suppose the uncertainty set $\mathcal{S}_F \subset L_p(I, E)$ is compact in the weak topology τ^w, and the admissible control laws $\mathcal{F}_{ad} \subset (C(Z, Y), \tau_{pcw})$ is compact in the topology τ_{pcw}. Then J has a saddle point.

Proof Define the map $W : \mathcal{F}_{ad} \longrightarrow \mathcal{S}_F$ by

$$arg\{\max_{w\in\mathcal{S}_F} J(w, f)\} \equiv W(f). \tag{25}$$

Since for each $f \in \mathcal{F}_{ad}$, $w \longrightarrow J(w, f)$ is weakly continuous and $\mathcal{S}_F$ is weakly compact, $J(\cdot, f)$ attains its maximum on $\mathcal{S}_F$ and so $W(f)$ is well defined. Thus $J(W(f), f) \geq J(w, f)$ for all $w \in \mathcal{S}_F$ and for all $f \in \mathcal{F}_{ad}$. Hence for any sequence $f^n \in \mathcal{F}_{ad}$ we have $J(W(f^n), f^n) \geq$

$J(w, f^n)$, $\forall\ w \in \mathcal{S}_F$. We show that $W : \mathcal{F}_{ad} \longrightarrow \mathcal{S}_F$ is continuous with respect to the given topologies. Denote the sequence $W(f^n)$ by w^n giving

$$J(W(f^n), f^n) = J(w^n, f^n) \geq J(w, f^n), \forall w \in \mathcal{S}_F. \tag{26}$$

By definition of the map W, we have $\{w^n\} \subset \mathcal{S}_F$. Since $\mathcal{S}_F \times \mathcal{F}_{ad}$ is compact with respect to the topology $\tau^{ww} \equiv \tau^w \times \tau_{pcw}$, there exists a subsequence of the sequence $\{w^n, f^n\}$, relabeled as the original sequence, and an element $\{w^o, f^o\} \in \mathcal{S}_F \times \mathcal{F}_{ad}$ such that

$$\{w^n, f^n\} \xrightarrow{\tau^{ww}} \{w^o, f^o\}.$$

By Lemma 4.2, J is jointly continuous and thus letting $n \to \infty$, it follows from the expression (26) that

$$J(w^o, f^o) \geq J(w, f^o), \forall w \in \mathcal{S}_F. \tag{27}$$

Since by definition, $W(f^o) \in \mathcal{S}_F$, it follows from the above inequality that

$$J(w^o, f^o) \geq J(W(f^o), f^o). \tag{28}$$

On the other hand since $w^o \in \mathcal{S}_F$, and $W(f^o)$ is a maximizer of $w \longrightarrow J(w, f^o)$ over $\mathcal{S}_F$, we also have

$$J(W(f^o), f^o) \geq J(w^o, f^o). \tag{29}$$

From these we conclude that

$$J(w^o, f^o) = J(W(f^o), f^o) \tag{30}$$

proving that

$$W(f^n) \xrightarrow{\tau^w} W(f^o)$$

as $f^n \xrightarrow{\tau_{pcw}} f^o$. Similarly, for a fixed but any $w \in \mathcal{S}_F$, define the map $\Pi : \mathcal{S}_F \longrightarrow \mathcal{F}_{ad}$ by

$$arg\{\min_{f \in \mathcal{F}_{ad}} J(w, f)\} \equiv \Pi(w), \tag{31}$$

giving

$$J(w, \Pi(w)) \leq J(w, f)\ \forall\ f \in \mathcal{F}_{ad}. \tag{32}$$

Following similar arguments again one can verify that, for any sequence $w^n \in \mathcal{S}_F$ converging weakly to w^o, we have

$$\Pi(w^n) \xrightarrow{\tau_{pcw}} \Pi(w^o) \text{ in } \mathcal{F}_{ad}.$$

Now we consider the maps

$$\mathcal{F}_{ad} \ni f \longrightarrow J(W(f), f) \in \bar{R} \text{ } and \text{ } \mathcal{S}_F \ni w \longrightarrow J(w, \Pi(w)) \in \bar{R}. \tag{33}$$

Choose $\{f^n\} \subset \mathcal{F}_{ad}$ a minimizing sequence for the first and $\{w^n\} \subset \mathcal{S}_F$ a maximizing sequence for the second function. Since these sets are compact, there exist $f^o \in \mathcal{F}_{ad}$ and $w^o \in \mathcal{S}_F$ such that, along a subsequence if necessary, $f^n \xrightarrow{\tau_{pcw}} f^o$ and $w^n \xrightarrow{\tau^w} w^o$ and

$$J(W(f^n), f^n) \longrightarrow J(W(f^o), f^o) \equiv M_{min} \tag{34}$$

$$J(w^n, \Pi(w^n)) \longrightarrow J(w^o, \Pi(w^o)) \equiv M_{max}. \tag{35}$$

Clearly

$$J(w, f^n) \leq J(W(f^n), f^n)\ \forall\ n \in N, w \in \mathcal{S}_F.$$

Hence in the limit, by continuity of J and W, it follows from the above inequality and (34) that

$$J(w, f^o) \leq J(W(f^o), f^o) \leq J(W(f^o), f)\ \forall\ w \in \mathcal{S}_F, f \in \mathcal{F}_{ad}. \tag{36}$$

Thus the pair $(W(f^o), f^o)$ is a saddle point of the functional J. By similar arguments one can conclude that the pair $(w^o, \Pi(w^o))$ is also a saddle point for the functional J. We show that the values corresponding to these saddle points are the same, that is, $M_{min} = M_{max}$. For convenience of notation set

$$(W(f^o), f^o) \equiv (w_1^o, f_1^o) \text{ and } (w^o, \Pi(w^o)) \equiv (w_2^o, f_2^o).$$

Then by definition of saddle points it is easy to see that

$$\begin{aligned} J(w_2^o, f_1^o) \le J(w_1^o, f_1^o) \le J(w_1^o, f_2^o) \\ J(w_1^o, f_2^o) \le J(w_2^o, f_2^o) \le J(w_2^o, f_1^o)\ . \end{aligned}$$

This leads to the following inequality,

$$J(w_1^o, f_2^o) \le J(w_2^o, f_2^o) \le J(w_2^o, f_1^o) \le J(w_1^o, f_1^o) \le J(w_1^o, f_2^o),$$

and hence the equality

$$J(w_1^o, f_2^o) = J(w_2^o, f_2^o) = J(w_2^o, f_1^o) = J(w_1^o, f_1^o) = J(w_1^o, f_2^o)\ . \tag{37}$$

This proves that

$$J(w_1^o, f_1^o) = J(w_2^o, f_2^o) = M_{min} = M_{max}.$$

Thus we have proved that J has a saddle point and that

$$\min_{f \in \mathcal{F}_{ad}} \max_{w \in \mathcal{S}_F} J(w, f) = \max_{w \in \mathcal{S}_F} \min_{f \in \mathcal{F}_{ad}} J(w, f). \tag{38}$$

This completes the proof. •

Remark 4.4 Since the functional J may not be strictly concave and convex in the first and second argument respectively, there may be multiple saddle points.

5. EXTENSION TO STATE DEPENDENT UNCERTAINTY

In section 4, we considered the multifunction F, representing the uncertainty, to be independent of state. Here we present a class of systems with state dependent uncertainty for which the technique of proof of the results presented in section 3 remain the same.

Let Ξ be a reflexive Banach space with dual Ξ^* and $B_a(\Xi^*)$ the closed ball in Ξ^* of radius $a > 0$ around the origin. Let $L_\infty(I, \Xi^*)$ denote the space of essentially bounded measurable functions with values in Ξ^* and $\mathcal{S}_{B_a(\Xi^*)}$ denote the set of all measurable selections of the (constant) multifunction $B_a(\Xi^*)$, that is, measurable functions $\{w\}$ with values $w(t) \in B_a(\Xi^*)$ for all $t \in I$. Clearly, it follows from well known results on the theory of measurable selections [8, Theorem 2.14, p 158; Lemma 3.2, p 175] that the set $\mathcal{S}_{B_a(\Xi^*)}$ is a nonempty subset of $B(I, \Xi^*) \subset L_\infty(I, \Xi^*)$. Let $H : I \times E \longrightarrow \mathcal{L}(\Xi^*, E)$ be an operator valued function and define the multifunction F given by

$$F(t, x) \equiv \{H(t, x)w(t) : w \in \mathcal{S}_{B_a(\Xi^*)}\}, t \in I, x \in E.$$

The system (6) is now replaced by the following model,

$$\dot{x} + Ax \in F(t, x) + A\mathcal{R}f(G(x)), x(0) = x_0, t \in I. \tag{39}$$

We need the following assumptions for H:

(H1): H is Borel measurable (in the uniform operator topology) and there exists an $h \in L_p^+(I)$ such that

$$\| H(t, \xi) \|_{\mathcal{L}(\Xi^*, E)} \le h(t)\big(1 + |\xi|_E\big),\ \forall\ \xi \in E.$$

(H2): For every $r > 0$, there exists a nonnegative number K_r such that

$$\| H(t, \xi) - H(t, \eta) \|_{\mathcal{L}(\Xi^*, E)} \le K_r |\xi - \eta|_E,\ \forall\ \xi, \eta \in B_r(E), t \in I,$$

where $B_r(E)$ denotes the ball of radius r around the origin in E.

Theorem 5.1 Suppose the assumptions (A1),(A2),(A4) and those for G and f of theorem 3.1 hold and let H satisfy the hypotheses (H1) and (H2). Then, for each $x_0 \in E$, and $f \in \mathcal{F}_{ad}$, system (39) has a nonempty set of mild solutions $X \equiv \{x\} \subset C(I,E)$.

Proof. The proof is similar to that of Theorem 3.2. Let $x_0 \in E$ and $f \in \mathcal{F}_{ad}$ be given. Since $\mathcal{S}_{B_a(\Xi^*)} \neq \emptyset$, we can choose any $w \in \mathcal{S}_{B_a(\Xi^*)}$ and consider the following integral equation,

$$\begin{aligned} x(t) = S(t)x_0 &+ \int_0^t S(t-s)H(s,x(s))w(s)ds \\ &+ \int_0^t AS(t-s)\mathcal{R}f(G(x(s)))ds, \qquad t \in I. \end{aligned} \tag{40}$$

By our assumptions, the operators $\{H, f, G\}$ have at most linear growth and they are locally Lipschitz. Hence this equation has a unique solution $x(w) \in C(I,E)$. The proof is based on the technique given in [2, Theorem 3.1,p64; Theorem 3.2,p69] dealing with the singularity of the third integrand and Banach fixed point theorem. Thus for each selection w, the corresponding evolution equation,

$$\dot{x} + Ax = H(t,x)w + A\mathcal{R}f(G(x)), x(0) = x_0, t \in I,$$

has a unique mild solution $x(w) \in C(I,E)$. Hence the differential inclusion (39) has a nonempty set of mild solutions $X \equiv \{x(w), w \in \mathcal{S}_{B_a(\Xi^*)}\} \subset C(I,E)$. This completes the outline of the proof. •

Using the above theorem, we can prove similar continuity results of Lemma 4.1 and 4.2 for this system. On the basis of these results we obtain the following result for state dependent uncertainty.

Theorem 5.2 Consider the system (39) with the objective functional given by (10) and suppose the assumptions of Theorem 5.1 and those of Lemma 4.1 and 4.2 hold and that $\mathcal{F}_{ad}$ satisfies the associated assumption in Theorem 4.3. Then J has a saddle point.

Proof. The proof being similar, we present a brief outline. Since Ξ^* is a dual space and $B_a(\Xi^*)$ is a closed bounded convex set, by Alaoglu theorem it is weak star compact and hence $\mathcal{S}_{B_a(\Xi^*)}$ is a weak star compact convex subset of $L_\infty(I,\Xi^*)$. In fact, $\mathcal{S}_{B_a(\Xi^*)}$ is a subset of $B(I,\Xi^*)$ which is a subset of $L_\infty(I,\Xi^*)$. By using the hypothesis (H1) one can verify that, for any $x \in C(I,E)$,

$$H(\cdot,x(\cdot))w^n(\cdot) \xrightarrow{w} H(\cdot,x(\cdot))w^o(\cdot)$$

in $L_p(I,E)$ for any sequence $\{w^n\} \subset \mathcal{S}_{B_a(\Xi^*)}$ that converges in the weak star topology to w^o. Now, in the present case, the term E_1^n of equation (14) takes the form,

$$\begin{aligned} E_1^n(t) &\equiv \int_0^t S(t-s)H(s,x^o(s))\big(w^n(s)-w^o(s)\big)ds \\ &+ \int_0^t S(t-s)\Big(H(s,x^n(s))-H(s,x^o(s))\Big)w^n(s)ds \\ &\equiv E_{1,1}^n(t) + E_{1,2}^n(t), t \in I, \end{aligned} \tag{41}$$

while the rest of the terms remain unchanged. By virtue of compactness of the semigroup $S(t), t > 0$, and the weak convergence mentioned above, $E_{1,1}^n(t) \xrightarrow{s} 0$ in E for every $t \in I$. The term $E_{1,2}^n$ can be added to the last term of equation (14) without affecting the singularity of the integral operator and so treated equally. Replacing the weak topology τ^w by the weak star topology τ^{w^*}, this information is sufficient to follow similar arguments as in Lemma 4.1 to prove the continuity. From this result also follows the continuity result of Lemma 4.2

with $\mathcal{S}_F$ replaced by $\mathcal{S}_{B_a(\Xi^*)}$ and the product topology $\tau^w \times \tau_{pcw}$ replaced by $\tau^{w^*} \times \tau_{pcw}$. The rest of the proof is identical to that of Theorem 4.3 once the statement: ($\mathcal{S}_F \subset L_p(I,E)$ is compact in the weak topology): is replaced by the statement :($\mathcal{S}_{B_a(\Xi^*)} \subset L_\infty(I,\Xi^*)$ is compact in the weak star topology). This completes our brief outline of the proof. •

Remark 5.3. Note that the above result also holds for any w^* measurable multifunction Γ, in place of $B_a(\Xi^*)$, with values $\Gamma(t)$, $t \in I$, which are nonempty w^* compact convex subsets of Ξ^*.

Remark 5.4 In case of distributed control and boundary uncertainty, the abstract model takes the form

$$\dot{x} + Ax \in Bu + A\mathcal{R}F, x(0) = x_0, \qquad t \in I,$$

where B is a bounded linear operator from a suitable Banach space U to E, and F is a mutifunction $F : I \longrightarrow 2^Y \setminus \emptyset$. Such problems can be treated in similar manner.

An Open Problem. In a recent paper [1], we proved also the necessary conditions of optimality [1, Theorem 4.1, p 70] under the assumption that the set of admissible feedback operators is a subset of the space of linear operators $\mathcal{L}_s(Z,Y)$ furnished with the strong operator topology. Here we have not attempted to present necessary conditions of optimality. This will require additional regularity on the admissible set $\mathcal{F}_{ad} \subset (C(Z,Y), \tau_{pcw})$. We leave this as an open problem.

Future Directions: (D1): We have used a special class of state dependent perturbations in the system model (39). It would be interesting to consider more general multi functions $F(t,x)$, $t \in I$.

(D2): Let $\Sigma \equiv \sigma(I)$ denote the sigma algebra of subsets of the interval I. Replace the multifunction $t \ni I \longrightarrow F(t)$ of equation (6) by a multimeasure $\Sigma \ni \sigma \longrightarrow M(\sigma)$. An E-valued countably additive bounded vector measure ν is said to be a selection of the multimeasure M if for every $\sigma \in \Sigma$, $\nu(\sigma) \in M(\sigma)$. Again, let $\mathcal{S}_M \subset \mathcal{M}_{cabv}(\Sigma, E)$ denote the set of all such selections of the multimeasure M where $\mathcal{M}_{cabv}(\Sigma, E)$ denotes the space of countably additive E-valued bounded vector measures having bounded variation. In this case the system (6)–(7) takes the form

$$dx + Axdt \in M(dt) + A\mathcal{R}u\, dt, x(0) = x_0, \tag{42}$$

$$u \equiv f(G(x)), \quad f \in \mathcal{F}_{ad}, \qquad t \in I. \tag{43}$$

This model includes impulsive as well as smooth perturbations (uncertainties). Again, for any selection $\nu \in \mathcal{S}_M$, the mild solution is given by the solution of the integral equation,

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\nu(ds) + \int_0^t AS(t-s)\mathcal{R}f(G(x(s))ds, t \in I, \tag{44}$$

which is perturbed by the E-valued vector measure ν. For each $\nu \in \mathcal{M}_{cabv}(\Sigma, E)$, we can prove that this equation has a unique solution $x \in B(I,E) \supset C(I,E)$. The solution is in $C(I,E)$ only if the measure ν is nonatomic. If the set of perturbing measures $\mathcal{S}_M$ is nonatomic and weakly compact, our results hold also for this class of systems. However, if they are atomic, compactness of the semigroup may not be sufficient to prove the necessary continuity results stated in Lemma 4.1.

An alternative model is given by

$$dx + Axdt = B(t)\nu(dt) + A\mathcal{R}udt, x(0) = x_0, \tag{45}$$

$$\nu \in \mathcal{M}_d, u \equiv f(G(x)), f \in \mathcal{F}_{ad}, t \in I, \tag{46}$$

where $B \in C(I, \mathcal{L}(Q,E))$ with Q being a Banach space and $\nu \in \mathcal{M}_d \subset \mathcal{M}_{cabv}(\Sigma, Q)$. The uncertainty is generated by the family of vector measures $\mathcal{M}_d$. Assuming that Q

is a reflexive Banach space, one can exploit the Bartle-Dunford-Schwartz relative weak compactness criterion [12, Theorem 5, p 105] to derive similar results as presented in this paper. In general, for the system (45)–(46), our results apply for measures with or without atoms and also for noncompact semigroup $S(t)$, $t \geq 0$, provided the operator valued function B and the set $\mathcal{M}_d$ satisfy the following properties. The set $\mathcal{M}_d$ is weakly compact and the operator valued function B is such that, for every $\sigma \in \Sigma$ and every weakly convergent sequence $\{\nu_n\}$ with weak limit ν, we have

$$\mu_n(\sigma) \equiv \int_\sigma B(s)\nu_n(ds) \xrightarrow{s} \int_\sigma B(s)\nu(ds) \equiv \mu(\sigma)$$

in E.

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