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ON THE STABILITY FOR CUBIC FUNCTIONAL EQUATION OF MIXED TYPE

HARK-MAHN KIM, SHEON-YOUNG KANG, AND ICK-SOON CHANG*

Department of Mathematics, Chungnam National University, 220 Gung-dong Yuseong-gu, Taejon 305-764, Republic of Korea hmkim@math.cnu.ac.kr
National Institute for Mathematical Sciences, 385-16 Doryong-dong Yuseong-gu, Taejon 305-340, Republic of Korea skang@nims.re.kr
*Department of Mathematics, Mokwon University, 800 Doan-dong Seo-gu, Taejon 302-729, Republic of Korea ischang@mokwon.ac.kr

ABSTRACT. In this paper, we consider the general solution for a mixed type cubic functional equation

$$lf(\sum_{i=1}^{m-1} x_i + lx_m) + lf(\sum_{i=1}^{m-1} x_i - lx_m) + 2\sum_{i=1}^{m-1} f(lx_i) = 2lf(\sum_{i=1}^{m-1} x_i) + l^3 \sum_{i=1}^{m-1} [f(x_i + x_m) + f(x_i - x_m)],$$

where $l \geq 2$ and $m \geq 3$ are any integers and investigate the Hyers-Ulam-Rassias stability of this equation.

Key words: Stability; Cubic function; Fixed point alternative

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1. INTRODUCTION

The stability problem of functional equations has originally been formulated by S.M. Ulam [23] in 1940: *Under what condition does there exists a homomorphism near an approximate homomorphism?* In following year, D.H. Hyers [7] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [18]. Since then, a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to a number of functional equations have been developed as well.

In particular, one of the important functional equations studied is the following functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

*Corresponding author
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It is easy to see that the cubic function $f(x) = cx^3$ is a solution of the functional equation (1.1). In this case the equation (1.1) said to be a *cubic functional equation* and every solution of the equation (1.1) is called a *cubic function.* The cubic functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [11]. In fact, they proved that a function $f : X \to Y$ between real vector spaces is a solution of the functional equation (1.1) if and only if there exists a function $H : X^3 \to Y$ such that $f(x) = H(x, x, x)$ for all $x \in X$, and H is symmetric for each fixed one argument and additive for fixed two arguments. The function H is given by

$$H(x,y,z) = \frac{1}{24}\big[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)\big]$$

for all $x, y, z \in X$. In addition, they investigated the Hyers-Ulam-Rassias stability for the cubic functional equation. After then, Y.-S. Jung and I.-S. Chang [14] introduced different type of cubic functional equation,

$$\begin{aligned} &f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ &\quad = 2f(x+y) + 4[f(x+z) + f(x-z) + f(y+z) + f(y-z)], \end{aligned} \tag{1.2}$$

which is equivalent to (1.1) and they have established the Hyers-Ulam-Rassias stability of this functional equation. Recently, H.-Y. Chu and D.-S. Kang [5] extended the functional equation (1.2) to the n-dimensional cubic functional equation

$$\begin{aligned} &f(\sum_{i=1}^{m-1} x_i + 2x_m) + f(\sum_{i=1}^{m-1} x_i - 2x_m) + \sum_{i=1}^{m-1} f(2x_i) \\ &\quad = 2f(\sum_{i=1}^{m-1} x_i) + 4\sum_{i=1}^{m-1}[f(x_i + x_m) + f(x_i - x_m)] \end{aligned} \tag{1.3}$$

and they dealt with stability of the above functional equation.

In this paper, we now consider the mixed type cubic functional equation

$$\begin{aligned} &lf(\sum_{i=1}^{m-1} x_i + lx_m) + lf(\sum_{i=1}^{m-1} x_i - lx_m) + 2\sum_{i=1}^{m-1} f(lx_i) \\ &\quad = 2lf(\sum_{i=1}^{m-1} x_i) + l^3\sum_{i=1}^{m-1}[f(x_i + x_m) + f(x_i - x_m)], \end{aligned} \tag{1.4}$$

where $l \geq 2$ and $m \geq 3$ are any integers, that is to say, we obtain the general solution of this equation. Furthermore, we adopt the idea of Cădariu and Radu [3] and offer the Hyers-Ulam-Rassias stability for this equation. In 1996, G. Isac and Th.M. Rassias [10] were the first to provide applications of the generalized Hyers-Ulam stability theory of functional equations for the proof of new fixed point theorems.

2. THE REQUIRED RESULTS

We now recall the fundamental results of fixed point theory.

Theorem 2.1 ([2]). *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ be a strictly contractive mapping, that is,*

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$ and for some the Lipschitz constant $L < 1$. Then

(1) *the mapping T has a unique fixed point $x^* = Tx^*$;*

(2) *the fixed point x^* is globally attractive, that is,*

$$\lim_{n\to\infty} T^n x = x^*$$

for any starting point $x \in X$;

(3) *one has the following estimation inequalities:*

$$\begin{aligned} d(T^n x, x^*) &\leq L^n d(x, x^*), \\ d(T^n x, x^*) &\leq \frac{1}{1-L} d(T^n x, T^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Tx) \end{aligned}$$

for all $x \in X$ and all nonnegative integer n.

The following theorem play an important role in proving the stability problem.

Theorem 2.2 (The alternative of fixed point [15]). *Suppose that we are given a complete generalized metric space (Ω, d), i.e., one for which d may assume infinite values, and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $L < 1$. Then, for each given $x \in \Omega$, either*

(1) *$d(T^n x, T^{n+1} x) = \infty$ for all $n \geq 0$,*

or

(2) *there exists a nonnegative integer n_0 such that $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$.*

Actually, if (2) holds, then the followings are true:

- *the sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
- *y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega |\ d(T^{n_0} x, y) < \infty\}$;*
- *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [9] for an extensive theory of fixed points with a large variety of applications.

First of all, we will find out the general solutions of functional equation (1.4). Now we will start with $m = 3$.

Lemma 2.3. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned} lf(x+y+lz) + lf(x+y-lz) + 2f(lx) + 2f(ly) \\ = 2lf(x+y) + l^3[f(x+z) + f(x-z) + f(y+z) + f(y-z)] \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, where $l \geq 2$ is any integers if and only if f is cubic.

Proof. Let a function $f : X \to Y$ satisfy the equation (2.1) for $l = 2$. Then f is cubic. We also see that

$$f(x+2z) + f(x-2z) + 6f(x) = 4f(x+z) + 4f(x-z),$$

which, by the proof of [12, Theorem 2.1], gives the equation

$$\begin{aligned} f(x+y+z) + f(x+y-z) + 2f(x) + 2f(y) \\ = 2f(x+y) + f(x+z) + f(x-z) + f(y+z) + f(y-z). \end{aligned}$$

Now make the induction assumption that (2.1) is true for any integer a with $2 < a \leq l$. Then we can rewrite the equation (2.1) as

$$\begin{aligned} f(x+y+az) + f(x+y-az) + 2a^2[f(x) + f(y)] \\ = 2f(x+y) + a^2[f(x+z) + f(x-z) + f(y+z) + f(y-z)]. \end{aligned} \tag{2.2}$$

Taking $x = 0, y = z$ and replacing x by $x + z$ in (2.2) equipped with $a = l$, separately, it yields $f((l+1)z) = (l+1)^3 f(z)$ and

$$\begin{aligned} f(x+y+(l+1)z) + f(x+y-(l-1)z) + 2l^2[f(x+z) + f(y)] \\ = 2f(x+y+z) + l^2[f(x+2z) + f(x) + f(y+z) + f(y-z)]. \end{aligned} \tag{2.3}$$

Combining the equation (2.3) and the equation with $x = -z$ in (2.3), we figure out

$$\begin{aligned} f(x+y+(l+1)z) + f(x+y-(l+1)z) + 2(l+1)^2[f(x) + f(y)] \\ = 2f(x+y) + (l+1)^2[f(x+z) + f(x-z) + f(y+z) + f(y-z)]. \end{aligned}$$

By multiplying by $l + 1$ in this equation, then we see that (2.1) is fulfilled for $l + 1$, which prove the validity of (2.1) for $l + 1$. Therefore the equation (2.1) implies that f is cubic.

Conversely, if there exists a function $H : X^3 \to Y$ such that $f(x) = H(x, x, x)$ for all $x \in X$, and H is symmetric for each fixed one argument and additive for fixed two arguments, we may easily show that f satisfies the equation (2.1). □

Using the Lemma 2.3, we can verify the following no difficulty.

Lemma 2.4. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.4) *if and only if f is cubic.*

3. THE STABILITY OF FUNCTIONAL EQUATION (1.4)

In recent years, L. Cădariu and V. Radu [3] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such an elegant idea, they could present a short and simple proof for the stability of that equation [2, 16].

From now on, let X be a real vector space and Y be a real Banach space, respectively, unless we give any specific reference. As a matter of convenience, for a given mapping $f : X \to Y$, we set

$$Df(x_1, x_2, \ldots, x_m) := lf(\sum_{i=1}^{m-1} x_i + lx_m) + lf(\sum_{i=1}^{m-1} x_i - lx_m) + 2\sum_{i=1}^{m-1} f(lx_i)$$
$$-2lf(\sum_{i=1}^{m-1} x_i) - l^3 \sum_{i=1}^{m-1}[f(x_i + x_m) + f(x_i - x_m)],$$

for all $x_1, x_2, \ldots, x_m \in X$, where $l \geq 2$ and $m \geq 3$ are any integers.

Based on the idea of Cădariu and Radu, we now construct a stability of the functional equation (1.4) as follow.

Theorem 3.1. *Suppose that a function $f : X \to Y$ satisfies the condition $f(0) = 0$ and the inequality*

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \varphi(x_1, x_2, \ldots, x_m) \tag{3.1}$$

for all $x_1, x_2, \ldots, x_m \in X$, where $\varphi : X^m \to [0, \infty)$ is a given function. If there exists $L < 1$ such that the function

$$x \mapsto \psi(x) = \varphi\big(0, \underbrace{\frac{x}{l}, \ldots, \frac{x}{l}}_{m-2}, 0\big)$$

has the property

$$\psi(x) \leq L \cdot \lambda_j^3 \cdot \psi(\frac{x}{\lambda_j}) \tag{3.2}$$

for all $x \in X$, and if φ has the function with

$$\lim_{n\to\infty} \frac{\varphi(\lambda_j^n x_1, \lambda_j^n x_2, \ldots, \lambda_j^n x_m)}{\lambda_j^{3n}} = 0 \tag{3.3}$$

for all $x_1, x_2, \ldots, x_m \in X$, where $\lambda_j = l$ if $j = 0$ and $\lambda_j = \frac{1}{l}$ if $j = 1$, then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\|f(x) - C(x)\| \leq \frac{L^{1-j}}{2(m-2)(1-L)}\psi(x) \tag{3.4}$$

for all $x \in X$.

Proof. We consider the set

$$\Omega := \{g : X \to Y |\ g(0) = 0\}$$

and the generalized metric on Ω,

$$d(g,h) = d_\psi(g,h) = \inf\{K \in (0,\infty)|\ \|g(x) - h(x)\| \le K\psi(x),\ \text{for all } x \in X\}.$$

One can easily check that (Ω, d) is complete.

Next, let $T : \Omega \to \Omega$ be a function defined by

$$Tg(x) := \frac{1}{\lambda_j^3}\, g(\lambda_j x)$$

for all $x \in X$ with $\lambda_j = l^{1-2j}$.

We first prove that T is a strictly contractive on Ω: Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) < K \qquad &\Longrightarrow \|g(x) - h(x)\| \le K\psi(x),\ x \in X \\ &\Longrightarrow \Big\|\frac{1}{\lambda_j^3}\, g(\lambda_j x) - \frac{1}{\lambda_j^3}\, h(\lambda_j x)\Big\| \le \frac{1}{\lambda_j^3}\, K\psi(\lambda_j x),\ x \in X \\ &\Longrightarrow \big\|Tg(x) - Th(x)\big\| \le LK\psi(x),\ x \in X \\ &\Longrightarrow d(Tg, Th) \le LK. \end{aligned}$$

Hence we see that for all $g, h \in \Omega$,

$$d(Tg, Th) \le Ld(g,h).$$

We now want to show that $d(f, Tf) < \infty$: If we put $x_1 = 0, x_i = x$ $(i = 2, \ldots, m-1)$ and $x_m = 0$ in (3.1) and use (3.2) with the case $j = 0$, then we arrive at

$$\|f(lx) - l^3 f(x)\| \le \frac{1}{2(m-2)}\varphi(0, \underbrace{x, \ldots, x}_{m-2}, 0), \tag{3.5}$$

which is reduced to

$$\Big\|f(x) - \frac{1}{l^3} f(lx)\Big\| \le \frac{1}{2(m-2)l^3}\psi(lx) \le \frac{L}{2(m-2)}\psi(x)$$

for all $x \in X$, viz.,

$$d(f, Tf) \le \frac{L}{2(m-2)} = \frac{L^1}{2(m-2)} < \infty.$$

If we substitute $x := \frac{x}{l}$ in (3.5) and use (3.2) with the case $j = 1$, then we find that

$$\Big\|f(x) - l^3 f\big(\frac{x}{l}\big)\Big\| \le \frac{1}{2(m-2)}\psi(x)$$

for all $x \in X$, viz.,

$$d(f, Tf) \le \frac{1}{2(m-2)} = \frac{L^0}{2(m-2)} < \infty.$$

Thus we conclude that

$$d(f,Tf) \le \frac{L^{1-j}}{2(m-2)} < \infty.$$

Therefore, by the fixed point alternative, we can prove that there is a unique cubic function $C : X \to Y$ such that the inequality (3.4): Now, from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x) = \lim_{n\to\infty} \frac{1}{\lambda_j^{3n}} f(\lambda_j^n x) \tag{3.6}$$

for all $x \in X$, since $\lim_{n\to\infty} d(T^n f, C) = 0$.

Again, using the fixed point alternative, we can get

$$d(f,C) \le \frac{1}{1-L} d(f,Tf) \le \frac{L^{1-j}}{2(m-2)(1-L)},$$

which yields the inequality (3.4).

In order to show that the function $C : X \to Y$ is cubic, let us replace $\lambda_j^n x_i$ instead of x_i in (3.1) and divide by λ_j^{3n}. Then we have by (3.3) and (3.6)

$$\begin{aligned} \|DC(x_1, x_2, \ldots, x_m)\| &= \lim_{n\to\infty} \frac{1}{\lambda_j^{3n}} \|Df(\lambda_j^n x_1, \lambda_i^n x_2, \ldots, \lambda_j^n x_m)\| \\ &\le \lim_{n\to\infty} \frac{\varphi(\lambda_j^n x_1, \lambda_i^n x_2, \ldots, \lambda_j^n x_m)}{\lambda_j^{3n}} = 0 \end{aligned}$$

for all $x_1, x_2, \ldots, x_m \in X$, viz., C satisfies the functional equation (1.4). Thus Lemma 2.4 guarantees that C is cubic.

To prove the uniqueness of the such cubic function, let us assume that there exists another cubic function $C_1 : A \to A$ satisfying the inequality (3.4). Since C_1 is a cubic,

$$C_1(x) = \frac{1}{\lambda_j^3} C_1(\lambda_j x) = (TC_1)(x)$$

and so C_1 is a fixed point of T. In view of (3.4) and the definition of d, we deduce that

$$d(f, C_1) \le \frac{L^{1-j}}{2(1-L)} < \infty,$$

viz., $C_1 \in \Delta = \{g \in X |\ d(f,g) < \infty\}$. By the fixed point alternative, we find that $C = C_1$, which proves that C is unique. This ends the proof of the theorem. □

Here and now, we will use the direct method to prove the stability for the functional equation (1.4).

Theorem 3.2. *Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0,\infty)$ such that*

$$\sum_{i=0}^{\infty} \frac{1}{l^{3i}} \phi(l^i x_1, l^i x_2, \dots, l^i x_m)$$

converges and

$$\|Df(x_1, x_2, \dots, x_m)\| \le \phi(x_1, x_2, \dots, x_m) \tag{3.7}$$

for all $x_1, x_2, \dots, x_m \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\|f(x) - C(x)\| \le \frac{1}{2(m-2)l^3} \sum_{i=0}^{\infty} \frac{1}{l^{3i}} \widetilde{\phi}(l^i x) \tag{3.8}$$

for all $x \in X$, where $\widetilde{\phi}$ is given by $\widetilde{\phi}(x) = \phi(0, \underbrace{x, \dots, x}_{m-2}, 0)$ for all $x \in X$.

Proof. Putting $x_1 = x_m = 0,\ x_2 = \cdots = x_{m-1} = x$ in (3.7) and dividing by l^3, we have

$$\left\| f(x) - \frac{1}{l^3} f(lx) \right\| \le \frac{1}{2(m-2)l^3} \widetilde{\phi}(x) \tag{3.9}$$

for all $x \in X$. By replacing x by lx in (3.9) and dividing by l^3 and then summing the resulting inequality with (3.9), we get

$$\left\| f(x) - \left(\frac{1}{l^3}\right)^2 f(l^2 x) \right\| \le \frac{1}{2(m-2)l^3} \widetilde{\phi}(x) + \frac{1}{2(m-2)} \left(\frac{1}{l^3}\right)^2 \widetilde{\phi}(lx). \tag{3.10}$$

An induction implies that

$$\left\| f(x) - \frac{1}{l^{3s}} f(l^s x) \right\| \le \frac{1}{2(m-2)l^3} \sum_{i=0}^{s-1} \frac{1}{l^{3i}} \widetilde{\phi}(l^i x). \tag{3.11}$$

To prove convergence of the sequence $\{\frac{f(l^s x)}{l^{3s}}\}$, we divide inequality (3.11) by l^{3n} and also replace x by $l^n x$ to find that for $s > n > 0$,

$$\begin{aligned} \left\| \frac{1}{l^{3n}} f(l^n x) - \frac{1}{l^{3(s+n)}} f(l^s l^n x) \right\| &= \frac{1}{l^{3n}} \left\| f(l^n x) - \frac{1}{l^{3s}} f(l^s l^n x) \right\| \\ &\le \frac{1}{2(m-2)l^{3(n+1)}} \sum_{i=0}^{s-1} \frac{1}{l^{3i}} \widetilde{\phi}(l^{n+i} x). \end{aligned} \tag{3.12}$$

Since the right-hand side of the inequality goes to 0 as $n \to \infty$, a sequence $\{\frac{f(l^s x)}{l^{3s}}\}$ is Cauchy. Therefore, we may define a function $C : X \to Y$ by

$$C(x) := \lim_{s \to \infty} \frac{f(l^s x)}{l^{3s}}$$

for all $x \in X$. By letting $s \to \infty$ in (3.11), we arrive at the formula (3.8).

We now show that C satisfies the functional equation (1.4): Let us replace x_i by $l^s x_i$ $(i = 1, 2, \ldots, m)$ in (3.7) and divide by l^{3s}. Then it follows that

$$\begin{aligned} DC(x_1, x_2, \ldots, x_m) &= \lim_{s\to\infty} \frac{1}{l^{3s}} \| Df(l^s x_1, l^s x_2, \ldots, l^s x_m)\| \\ &\le \lim_{s\to\infty} \frac{1}{l^{3s}} \phi(l^s x_1, l^s x_2, \ldots, l^s x_m) = 0. \end{aligned}$$

Hence we obtain the desired result. Thus the Lemma 2.3 implies that C is cubic.

It only remains to prove the claim that C is unique: Let us assume that there exists a cubic function C_1 which satisfies (1.4) and the inequality (3.8). It is clear that $C(l^s x) = l^{3s} C(x)$ and $C_1(l^s x) = l^{3s} C_1(x)$ for all $x \in X$ and $s \in \mathbb{N}$. Hence it follows from (3.8) that

$$\begin{aligned} \|C(x) - C_1(x)\| &= \frac{1}{l^{3s}} \|C(l^s x) - C_1(l^s x)\| \\ &\le \frac{1}{l^{3s}} \Big[\|C(l^s x) - f(l^s x)\| + \|f(l^s x) - C_1(l^s x)\| \Big] \\ &\le \frac{1}{(m-2) l^{3(s+1)}} \sum_{i=0}^{\infty} \widetilde{\phi}(l^{s+i} x). \end{aligned}$$

By letting $s \to \infty$, we have $C(x) = C_1(x)$, which ends the proof of the theorem. $\square$

Using the crucial inequality (3.9) and following the same approach as in Theorem 3.2, we obtain the next theorem.

Theorem 3.3. *Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} l^{3(i-1)} \phi\left(\frac{x_1}{l^i}, \frac{x_2}{l^i}, \ldots, \frac{x_m}{l^i}\right)$$

converges and satisfies the inequality (3.7) *for all $x_1, x_2, \ldots, x_m \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfying the inequality*

$$\|f(x) - C(x)\| \le \frac{1}{2(m-2)} \sum_{i=1}^{\infty} l^{3(i-1)} \widetilde{\phi}\left(\frac{x}{l^i}\right)$$

for all $x \in X$, where $\widetilde{\phi}$ is given as in Theorem 3.2.

4. THE APPLICATIONS

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [18] of the functional equation (1.4). Of course, by using Theorem 3.2 and Theorem 3.3, we also prove the following corollary, but we remark that Theorem 3.1 is more simpler.

Corollary 4.1. *Let X and Y be a normed space and a Banach space, respectively. Let $p \geq 0$ be given with $p \neq 3$. Assume that $\varepsilon \geq 0$ are fixed. Suppose that a function $f : X \to Y$ satisfies the condition $f(0) = 0$ and the inequality*

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_m\|^p)$$

for all $x_1, x_2, \ldots, x_m \in X$. Then there exists a unique cubic function $C : X \to Y$ such that the inequality

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{2|\ l^p - l^3|}\|x\|^p \tag{4.1}$$

for all $x \in X$.

Proof. Consider a mapping φ defined by

$$\varphi(x_1, x_2, \ldots, x_m) := \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_m\|^p)$$

for all $x_1, x_2, \ldots, x_m \in X$. Then it follows that

$$\frac{\varphi(\lambda_j^n x_1, \lambda_j^n x_2, \ldots, \lambda_j^n x_m)}{\lambda_j^{3n}} = (\lambda_j^n)^{p-3}\varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_m\|^p) \to 0$$

as $n \to \infty$, where $p < 3$ if $j = 0$ and $p > 3$ if $j = 1$, viz., (3.3) is seen to be true.

Since the inequality

$$\frac{1}{\lambda_j^3}\psi(\lambda_j x) = (m-2)\frac{\lambda_j^{p-3}}{l^p}\varepsilon\|x\|^p \leq \lambda_j^{p-3}\psi(x),$$

where $p < 3$ if $j = 0$ and $p > 3$ if $j = 1$, we see that the inequality (3.2) holds with either $L = l^{p-3}$ or $L = \frac{1}{l^{p-3}}$. Now the inequality (3.4) yields the property (4.1), which complete the proof of the corollary. □

The following corollary is the Hyers-Ulam stability [7] of the functional equation (1.4).

Corollary 4.2. *Let X and Y be a normed space and a Banach space, respectively. Assume that $\theta \geq 0$ is fixed. Suppose that a function $f : X \to Y$ satisfies the conditon $f(0) = 0$ and the inequality*

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta$$

for all $x_1, x_2, \ldots, x_m \in X$. Then there exists a unique cubic function $C : X \to Y$ such that the inequality

$$\|f(x) - C(x)\| \leq \frac{1}{2m(l^3 - 1)}\theta$$

for all $x \in X$

Proof. Putting $p := 0$ and $\varepsilon := \frac{\theta}{m}$ in the corollary 4.1, we arrive at the assertion of the corollary. □

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