Dynamic Systems and Applications 17 (2008) 595-608

MONOTONE SOLUTIONS OF TWO-DIMENSIONAL NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

MARIELLA CECCHI, ZUZANA DOŠLÁ, AND MAURO MARINI

Depart. of Electronics and Telecommunications, University of Florence
Via S. Marta 3, 50139 Firenze, Italy
Depart. of Mathematics and Statistics, Masaryk University, Janáčkovo nám. 2a,
60200 Brno, Czech Republic
Depart. of Electronics and Telecommunications, University of Florence
Via S. Marta 3, 50139 Firenze, Italy

ABSTRACT. The existence of nonoscillatory solutions with different asymptotic properties for a two-dimensional nonlinear functional differential system is studied. Some discrepancies in the coexistence of nonoscillatory solutions between the general nonlinear system and the Emden-Fowler system or the half-linear equation are pointed out. The roles of deviating arguments are also discussed.

AMS (MOS) Subject Classification. Primary 34C10, Secondary 34C15

1. INTRODUCTION

Consider the nonlinear differential system

$$\begin{cases} x' = a(t)f(y(r(t))) \\ y' = -b(t)g(x(s(t))) \end{cases} \tag{1.1}$$

where a, b, r, s are positive continuous functions on $[1, \infty)$, $r(\infty) = s(\infty) = \infty$, and f, g are nondecreasing continuous functions on $\mathbb{R}$ satisfying $uf(u) > 0, ug(u) > 0$ for $u \neq 0$ and

$$-f(u) = f(-u), \ -g(u) = g(-u) \qquad \text{for } u \in \mathbb{R}. \tag{1.2}$$

A continuously differentiable vector function (x, y) defined on $[t_0, \infty)$, $t_0 \geq 1$, is said to be a solution of (1.1) on $[t_0, \infty)$ if there exist two continuous functions x_0, y_0, defined on $(-\infty, t_0]$, such that (1.1) is satisfied on $[t_0, \infty)$, where

$$x(t) = x_0(t), \quad y(t) = y_0(t) \ \text{ for } t \leq t_0.$$

Throughout this paper we shall consider only the solutions of (1.1) which exist on some ray $[T, \infty)$, where $T \geq 1$ may depend on the particular solution. For the continuability problem we refer to [15, Proposition A]. As usually, a component x $[y]$

Received September 14, 2007

of a solution (x, y) of (1.1), defined on some neighborhood of infinity, is said to be *nonoscillatory* if $x(t) \neq 0$ [$y(t) \neq 0$] for any large t, and oscillatory otherwise. Clearly, x is nonoscillatory if and only if y is nonoscillatory too. So, a solution (x, y) of (1.1) is said to be *oscillatory* or *nonoscillatory* according to both components are oscillatory or nonoscillatory, respectively.

Particular cases of (1.1) are the Emden-Fowler system

$$\begin{cases} x' = a(t)|y(t)|^{1/\alpha}\text{sgn } y(t) \\ y' = -b(t)|x(t)|^{\beta} \text{ sgn } x(t) \end{cases} \tag{1.3}$$

where $\alpha > 0, \beta > 0$, $\alpha \neq \beta$, the nonlinear equation with p-Laplacian operator

$$(A(t)|x'(t)|^{\alpha}\text{sgn}\, x'(t))' + b(t)g(x(s(t))) = 0, \tag{1.4}$$

where $A(t) = a^{-1/\alpha}(t)$, and the half-linear equation

$$(A(t)|x'(t)|^{\alpha}\text{sgn}\, x'(t))' + b(t)|x(t)|^{\alpha}\text{sgn } x(t) = 0. \tag{1.5}$$

System (1.1) with $s(t) = r(t) = t$ and its particular cases (1.3), (1.4), (1.5), have been widely investigated, see, e.g., the papers [2, 8, 10, 15] for oscillation problems, [3, 5, 7, 11, 12, 14] for nonoscillation ones and [6, 8] for both. We refer also to the monographs [4, 13], in which a detailed study of (1.3) and (1.5), respectively, is presented and to [1, 9], in which a detailed analysis on the above topics, jointly with some interesting open problems, are given.

Put

$$I_a = \int_1^{\infty} a(\tau)d\tau, \qquad I_b = \int_1^{\infty} b(\tau)d\tau.$$

Here two cases are considered, namely

$$\text{I)} \quad I_a = \infty,\ I_b < \infty; \qquad \text{II)} \quad I_a < \infty,\ I_b = \infty.$$

In both cases, nonoscillatory solutions of (1.1) can be classified as subdominant, intermediate or dominant solutions, according to their asymptotic behavior (see below for the definition). As it is claimed in [1, page 241], the existence of intermediate solutions for (1.1) is a difficult problem, even in the special case where $s(t) = r(t) = t$. Moreover, their possible coexistence with different types of nonoscillatory solutions is a well-known problem (see, e.g., [6, page 213]), which has been completely resolved for (1.5) in [3].

The aim of this paper is to study the existence of intermediate, subdominant and dominant solutions of (1.1). We show some discrepancies in the coexistence of these solutions for (1.1) and (1.3), which are caused by the growth of nonlinearities f, g. Sections 3, 4 deal with the case I). In Section 5, by means of a duality property, the obtained results are extended to the case II). Our results improve or generalize analogous ones in [5, 7, 8, 12]. The role of the deviating arguments r, s are also discussed and several examples illustrate the obtained results.

2. PRELIMINARIES

When (1.1) is nonoscillatory, for sake of simplicity, we will restrict our attention only to solutions (x, y) of (1.1) for which x is eventually positive. We will denote such solutions as solutions of class $\mathbb{M}^+$ or $\mathbb{M}^-$, according to y is eventually positive or eventually negative. The remaining cases can be easily treated using analogous arguments.

If $(x, y) \in \mathbb{M}^+$, then x is positive increasing and y is positive decreasing for large t; if $(x, y) \in \mathbb{M}^-$, then x is positive decreasing and y is negative decreasing for large t. It is easy to show that, if $I_a = \infty$, then $\mathbb{M}^- = \emptyset$. Similarly, if $I_b = \infty$, then $\mathbb{M}^+ = \emptyset$. So, if $(x, y) \in \mathbb{M}^+$ and x is bounded, then $\lim_{t\to\infty} y(t) = 0$. Similarly, if $(x, y) \in \mathbb{M}^-$ and $\lim_{t\to\infty} x(t) > 0$, then $\lim_{t\to\infty} y(t) = -\infty$. Thus, in case I) solutions in $\mathbb{M}^+$can be *a-priori* divided into the subclasses:

$$\mathbb{M}^+_{\infty,\ell} = \{(x, y) \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \ell_y,\ 0 < \ell_y < \infty\},$$
$$\mathbb{M}^+_{\infty,0} = \{(x, y) \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = 0\},$$
$$\mathbb{M}^+_{\ell,0} = \{(x, y) \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \ell_x, \lim_{t\to\infty} y(t) = 0,\ 0 < \ell_x < \infty\},$$

and in case II) solutions in $\mathbb{M}^-$ into the subclasses:

$$\mathbb{M}^-_{\ell,-\infty} = \{(x, y) \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = \ell_x, \lim_{t\to\infty} y(t) = -\infty,\ 0 < \ell_x < \infty\},$$
$$\mathbb{M}^-_{0,-\infty} = \{(x, y) \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = -\infty\},$$
$$\mathbb{M}^-_{0,-\ell} = \{(x, y) \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = -\ell_y,\ 0 < \ell_y < \infty\}.$$

Notice that this classification is similar to the one given in [12], when $r(t) \equiv s(t) \equiv t$.

Following [6], solutions in $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\infty,0}$, $\mathbb{M}^+_{\ell,0}$ are called *dominant solutions, intermediate solutions* and *subdominant solutions*, respectively. This terminology is due to the fact that, when g is unbounded and $(x_1, y_1) \in \mathbb{M}^+_{\infty,\ell}$, $(x_2, y_2) \in \mathbb{M}^+_{\infty,0}$, $(x_3, y_3) \in \mathbb{M}^+_{\ell,0}$, then $x_1(t) > x_2(t) > x_3(t)$ and $y_1(t) > y_2(t) > y_3(t)$ for any large t. The same terminology of intermediate solutions is used for solutions in $\mathbb{M}^-_{0,-\infty}$, for a similar reason.

An important role in the existence of nonoscillatory solutions, is played by the following integrals depending on the parameters. In case I) denote

$$J_\mu = \int_1^\infty a(\tau) f\left(\mu \int_{r_1(\tau)}^\infty b(\sigma)d\sigma\right) d\tau, \quad K_\lambda = \int_1^\infty b(\tau) g\left(\lambda \int_1^{s_1(\tau)} a(\sigma)d\sigma\right) d\tau,$$

and in case II)

$$W_\lambda = \int_1^\infty a(\tau) f\left(\lambda \int_1^{r_1(\tau)} b(\sigma)d\sigma\right) d\tau, \quad Z_\mu = \int_1^\infty b(\tau) g\left(\mu \int_{s_1(\tau)}^\infty a(\sigma)d\sigma\right) d\tau,$$

where

$$r_1(t) = \max\{1, r(t)\}, \quad s_1(t) = \max\{1, s(t)\}.$$

3. ON INTERMEDIATE SOLUTIONS

Theorem 3.1. *Assume $I_a = \infty$, $I_b < \infty$. If there exist two positive constants λ and μ, where $\mu < \lim_{u\to\infty} g(u)$, such that*

$$K_\lambda < \infty, \quad J_\mu = \infty,$$

then $\mathbb{M}^+_{\infty,0} \neq \emptyset$.

Proof. Let c be a positive constant such that

$$g(c) > \mu \tag{3.1}$$

and put

$$m = f^{-1}(\lambda)/2.$$

Choose $t_0 \geq 1$ large enough such that $r(t) \geq 1$, $s(t) \geq 1$ for $t \geq t_0$ and

$$g(c) \int_{t_0}^{\infty} b(\sigma) d\sigma \leq m, \quad c \leq \lambda \int_1^{t_0} a(\sigma) d\sigma, \tag{3.2}$$

$$\int_{t_0}^{\infty} b(\tau) g\left(\lambda \int_1^{s(\tau)} a(\sigma) d\sigma\right) d\tau \leq m. \tag{3.3}$$

Put $T_s = \inf\{t \geq t_0 : s(\tau) \geq t_0 \ \forall \tau \geq t\}$ and let $\overline{s}, \overline{r}$ be the functions

$$\overline{s}(t) = \begin{cases} t_0 & \text{for } t \in [t_0, T_s) \\ s(t) & \text{for } t \geq T_s \end{cases}, \qquad \overline{r}(t) = \max\{r(t), t_0\}. \tag{3.4}$$

Clearly, if $T_s > t_0$, then $s(T_s) = t_0$ and so $\overline{s}$ is continuous for $t \geq t_0$.

Denote with $C[t_0, \infty)$ the Fréchet space of all continuous functions on $[t_0, \infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$ and consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \left\{ v \in C[t_0, \infty) : \mu \int_t^{\infty} b(\sigma) d\sigma \leq v(t) \leq 2m, \ t \geq t_0 \right\}.$$

Observe that from (3.1) and (3.2) we have $\mu \int_t^\infty b(\sigma) d\sigma < 2m$ for $t \geq t_0$. Define in Ω the operator T given by

$$T(v)(t) = \int_t^{\infty} b(\tau) g\left(c + \int_{t_0}^{\overline{s}(\tau)} a(\sigma) f\left(v(\overline{r}(\sigma))\right) d\sigma\right) d\tau.$$

From (3.1) we have

$$T(v)(t) \geq g(c) \int_t^{\infty} b(\tau) d\tau \geq \mu \int_t^{\infty} b(\tau) d\tau.$$

Moreover, in virtue of (3.2), (3.3), we have

$$\begin{aligned} T(v)(t_0) &= g(c)\int_{t_0}^{T_s} b(\tau)d\tau + \int_{T_s}^{\infty} b(\tau)g\left(c+\int_{t_0}^{\overline{s}(\tau)} a(\sigma)f(v(\overline{r}(\sigma)))d\sigma\right)d\tau \\ &\le g(c)\int_{t_0}^{\infty} b(\tau)d\tau + \int_{T_s}^{\infty} b(\tau)g\left(c+\lambda\int_{t_0}^{\overline{s}(\tau)} a(\sigma)d\sigma\right)d\tau, \end{aligned}$$

thus

$$T(v)(t) \le g(c)\int_{t_0}^{\infty} b(\tau)d\tau + \int_{T_s}^{\infty} b(\tau)g\left(\lambda\int_{1}^{\overline{s}(\tau)} a(\sigma)d\sigma\right)d\tau \le 2m, \tag{3.5}$$

and so T maps Ω into itself. Let us show that $T(\Omega)$ is relatively compact, i.e. $T(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval I of $[t_0,\infty)$. Because $T(\Omega)\subset\Omega$, the elements of $T(\Omega)$ are equibounded with first derivatives equibounded on I and so the compactness follows. Now we show that T is continuous in $\Omega\subset C[t_0,\infty)$. Let $\{v_n\}$, $n\in\mathbb{N}$, be a sequence in Ω which uniformly converges on every compact interval of $[t_0,\infty)$ to $\bar{v}\in\Omega$. Because $T(\Omega)$ is relatively compact, the sequence $\{T(v_n)\}$ admits a subsequence $\{T(v_{n_j})\}$ converging, in the topology of $C[t_0,\infty)$, to $\bar{z}_v$. In view of (3.3) and (3.5), by applying the Lebesgue dominated convergence theorem, the sequence $\{T(v_{n_j})(t)\}$ pointwise converges to $T(\bar{v})(t)$. In view of the uniqueness of the limit, $T(\bar{v})=\bar{z}_v$ is the only cluster point of the compact sequence $\{T(v_n)\}$, that is the continuity of T in the topology of $C[t_0,\infty)$. Hence, by the Tychonov fixed point Theorem there exists a solution of the integral equation

$$y(t) = \int_{t}^{\infty} b(\tau)g\left(c+\int_{t_0}^{\overline{s}(\tau)} a(\sigma)f\left(y(\overline{r}(\sigma))\right)d\sigma\right)d\tau. \tag{3.6}$$

It is easy to verify that (x,y), where

$$x(t) = c+\int_{t_0}^{t} a(\sigma)f(y(\overline{r}(\sigma)))d\sigma, \tag{3.7}$$

is a solution of (1.1) for large t, say $t\ge T_1\ge T_s$. Since $y\in\Omega$, we have for $t\ge T_1$

$$x(t)-x(T_1) \ge \int_{T_1}^{t} a(\tau)f\left(\mu\int_{r(\tau)}^{\infty} b(\sigma)d\sigma\right)d\tau$$

and so, because $J_\mu=\infty$, we obtain $x(\infty)=\infty$. From (3.2) and (3.6) it results for $t\ge T_1$

$$y(t) \le \int_{t}^{\infty} b(\tau)g\left(c+\lambda\int_{t_0}^{s(\tau)} a(\sigma)d\sigma\right)d\tau \le \int_{t}^{\infty} b(\tau)g\left(\lambda\int_{1}^{s(\tau)} a(\sigma)d\sigma\right)d\tau$$

and so, in view of $K_\lambda<\infty$, we get $y(\infty)=0$. $\square$

Remark 3.2. As already claimed, when $s(t) = r(t) = t$, the existence of intermediate solutions for (1.1) is considered in [12, Theorem 2.4] and, for an equation including (1.4), in [5, Theorem 9], [8, Theorem 1.3]. Theorem 3.1 substantially extends [5, Theorem 9], [8, Theorem 1.3], [12, Theorem 2.4], because in these results it is assumed that $J_\mu = \infty$ for *any* $\mu > 0$. The Example 1 below illustrates this fact. Notice that the proof of [12, Theorem 2.4] is different, since a different operator T and a different set Ω is considered. Moreover, it is not complete, because it remains to verify that the set Ψ, considered in the proof, is convex, which seems difficult to prove.

The following examples illustrate the role of the nonlinearity f and the condition $\mu < \lim_{u\to\infty} g(u)$ in Theorem 3.1, respectively.

Example 3.3. Consider the system

$$x'(t) = e^t f(y(t)), \quad y'(t) = -2t^{-3} g(x(t)), \tag{3.8}$$

where f, g are nondecreasing continuous functions on $\mathbb{R}$ such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$ and

$$\begin{aligned} f(u) &= \exp(-1/(\sqrt{u}) && \text{if} \quad 0 < u < 1 \\ g(u) &= \log u && \text{if} \quad u > e - 1. \end{aligned}$$

We have

$$K_1 = 2\int_1^\infty \tau^{-3} g(e^\tau - e) d\tau < \infty, \quad J_1 = \int_1^\infty e^\tau e^{-\tau} d\tau = \infty,$$

and so, in view of Theorem 3.1, this system has intermediate solutions. Nevertheless the assumption $J_\mu = \infty$ does not hold for any $\mu > 0$. Indeed for $\mu = 1/4$ we have $J_{1/4} = \int_1^\infty e^{-\tau} d\tau < \infty$.

Example 3.4. Consider the system

$$x'(t) = t^{-2} e^t f(y(t)), \quad y'(t) = -2t^{-3} g(x(t)),$$

where f, g are nondecreasing continuous functions on $\mathbb{R}$ such that $uf(u) > 0, ug(u) > 0$ for $u \neq 0$ and

$$\begin{aligned} f(u) &= \exp(-1/\sqrt{u}) && \text{if} \quad 0 < u < 1, \\ g(u) &= 1 && \text{if} \quad u > 1. \end{aligned}$$

Such a system does not have intermediate solution. Indeed, if there exists $(x, y) \in \mathbb{M}^+_{\infty,0}$, we have $y(t) = t^{-2}$ for large t and so it results $x'(t) = t^{-2}$, which contradicts the unboundedness of x. Notice that $K_\lambda < \infty$ for $\lambda > 0$, $J_\mu < \infty$ for $0 < \mu \leq 1$ and $J_\mu = \infty$ for $\mu > 1 (= \lim_{u\to\infty} g(u))$.

4. COEXISTENCE RESULTS

A natural question, which arises, is to study whenever the existence of subdominant and dominant solutions depends on the limit value of their first and second component, respectively. The following holds.

Theorem 4.1. *Assume $I_a = \infty$, $I_b < \infty$.*

i_1) If there exists $\lambda > 0$ such that $K_\lambda = \infty$, then (1.1) *does not have solutions* (x, y) *satisfying*

$$x(\infty) = \infty, \quad y(\infty) = L, \quad L > f^{-1}(\lambda).$$

i_2) If there exists $\lambda > 0$ such that $K_\lambda < \infty$, then (1.1) *has solutions* (x, y) *satisfying*

$$x(\infty) = \infty, \quad y(\infty) = L, \quad 0 < L < f^{-1}(\lambda).$$

Proof. Claim i_1). By contradiction, assume there exists a solution (x, y) of (1.1) with $x(\infty) = \infty,\ y(\infty) = L > f^{-1}(\lambda)$. Since y is eventually positive decreasing, without loss of generality, we can assume $x(t) > 0$, $L < y(r(t)) < 2L$ on $[T, \infty)$, $T \geq 1$. From the first equation in (1.1) we get

$$x(t) = x(T) + \int_T^t a(\tau) f(y(r(\tau)))d\tau \geq f(L) \int_T^t a(\tau)d\tau.$$

Let $T_1 \geq T$ such that $s(t) \geq T$ for $t \geq T_1$. Hence, from the second equation in (1.1) we obtain for $t \geq T_1$

$$y(T_1) - L \geq \int_{T_1}^{\infty} b(\tau) g\left(f(L)\int_T^{s(\tau)} a(\sigma)d\sigma\right) d\tau. \tag{4.1}$$

Since $I_a = \infty$, fixed ε with $\lambda(f(L))^{-1} < \varepsilon < 1$, it results for large τ, say $\tau \geq T_2 \geq T_1$,

$$\int_T^{s(\tau)} a(\sigma)d\sigma > \varepsilon \int_1^{s(\tau)} a(\sigma)d\sigma$$

and so from (4.1) we get

$$\int_{T_2}^{\infty} b(\tau) g\left(\varepsilon f(L)\int_1^{s(\tau)} a(\sigma)d\sigma\right) d\tau < \infty.$$

Since $\varepsilon f(L) > \lambda$, we obtain a contradiction with $K_\lambda = \infty$.

Claim i_2). Fixed $c > 0$, choose $t_0 \geq 1$ satisfying (3.2), (3.3) with $m = 2^{-1}(f^{-1}(\lambda) - L)$ and $r(t) \geq 1$, $s(t) \geq 1$ for $t \geq t_0$. Let $\overline{r}, \overline{s}$ be the functions defined in (3.4). Now consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \left\{v \in C[t_0, \infty) : L \leq v(t) \leq f^{-1}(\lambda) \text{ for } t \geq t_0\right\}$$

and define in Ω the operator T as follows

$$T(v)(t) = L + \int_t^{\infty} b(\tau) g\left(c + \int_{t_0}^{\overline{s}(\tau)} a(\sigma) f(v(\overline{r}(\sigma)))d\sigma\right) d\tau$$

Reasoning as in the proof of Theorem 3.1, and applying the Tychonov fixed point Theorem, we obtain that there exists a solution of the integral equation

$$y(t) = L - \int_t^\infty b(\tau) g\left(c + \int_{t_0}^{\overline{s}(\tau)} a(\sigma) f(y(\overline{r}(\sigma)))d\sigma\right) d\tau \qquad (t \geq t_0).$$

It is easy to verify that (x, y), where x is given by (3.7), is a solution of (1.1) for large t, say $t \geq T_1 \geq T_s$. From (3.7), taking into account that $y \in \Omega$, we obtain $x(t) \geq f(L) \int_{t_0}^t a(\sigma)d\sigma$ for $t \geq T_1$ and so $x(\infty) = \infty$. Clearly $y(\infty) = L$ and the proof is complete. □

Theorem 4.2. *Assume $I_a = \infty$, $I_b < \infty$.*

i_1) If there exists $\mu > 0$ such that $J_\mu = \infty$, then (1.1) does not have solutions (x, y) satisfying

$$x(\infty) = L, \quad L > g^{-1}(\mu), \quad y(\infty) = 0.$$

i_2) If there exists $\mu > 0$ such that $J_\mu < \infty$, then (1.1) has solutions (x, y) satisfying

$$x(\infty) = L, \quad 0 < L \leq g^{-1}(\mu), \quad y(\infty) = 0.$$

Proof. Claim i_1). By contradiction, assume there exists a solution (x, y) of (1.1) with $x(\infty) = L > g^{-1}(\mu)$, $y(\infty) = 0$. Let L_ε such that $L > L_\varepsilon > g^{-1}(\mu)$. Since $(x, y) \in \mathbb{M}^+_{\ell,0}$, we can suppose, without loss of generality, $x(r(t)) > L_\varepsilon$, $y(t) > 0$ for any $t \geq T \geq 1$. From the second equation in (1.1) we get for $\sigma \geq T$

$$y(\sigma) = \int_\sigma^\infty b(\tau) g(x(s(\tau)))d\tau. \tag{4.2}$$

Let $T_1 \geq T$ such that $r(t) \geq T$ for $t \geq T_1$. Using (4.2), from the first equation in (1.1) we obtain for $t \geq T_1$

$$L - x(T_1) = \int_{T_1}^\infty a(\sigma) f\left(\int_{r(\sigma)}^\infty b(\tau) g(x(r(\tau)))d\tau\right) d\sigma$$
$$\geq \int_{T_1}^\infty a(\sigma) f\left(g(L_\varepsilon) \int_{r(\sigma)}^\infty b(\tau)d\tau\right) \geq \int_{T_1}^\infty a(\sigma) f\left(\mu \int_{r(\sigma)}^\infty b(\tau)d\tau\right),$$

which is a contradiction.

Claim i_2). The assertion follows by applying the Tychonov fixed point theorem to the operator T given by

$$T(u)(t) = L - \int_t^\infty a(\sigma) f\left(\int_{\overline{r}(\sigma)}^\infty b(\tau) g(u(\overline{s}(\tau)))d\tau\right) d\sigma$$

in the set $\Omega \subset C[t_0, \infty)$

$$\Omega = \left\{u \in C[t_0, \infty) : \frac{1}{2}L \leq u(t) \leq L \text{ for } t \geq t_0\right\}.$$

where $\overline{r}(t) = \max\{r(t), t_0\}$, $\overline{s}(t) = \max\{s(t), t_0\}$ and t_0 is large so that

$$\int_{t_0}^{\infty} a(\tau) f\left(g(L)\int_{r(\tau)}^{\infty} b(\sigma)d\sigma\right) d\tau \leq \frac{L}{2}.$$

The argument is similar to the one given in [12, Theorem 2.2], with minor changes. □

From Theorems 4.1, 4.2 we obtain the following.

Corollary 4.3. *Assume* $I_a = \infty$, $I_b < \infty$. *Then*

$$\mathbb{M}^+_{\infty,\ell} \neq \emptyset \iff K_\lambda < \infty \textit{ for some } \lambda > 0.$$
$$\mathbb{M}^+_{\ell,0} \neq \emptyset \iff J_\mu < \infty \textit{ for some } \mu > 0.$$

Remark 4.4. Corollary 4.3 can be proved directly by using a similar argument to the one given in [12, Theorems 2.2, 2.3], with minor changes.

For the Emden-Fowler system (1.3) the convergence or divergence of integrals K_λ, J_μ does not depend on the choice of the parameters λ, μ and so in case I), if $K_1 < \infty$ [$J_1 < \infty$], then (1.3) has a solution $(x, y) \in \mathbb{M}^+_{\infty,\ell}$ [$\mathbb{M}^+_{\ell,0}$] such that $y(\infty) = L$ [$x(\infty) = L$] for any $L > 0$.

From Theorem 3.1 and Corollary 4.3, we obtain the following coexistence result.

Corollary 4.5. *Assume* $I_a = \infty$, $I_b < \infty$. *If there exist three positive constants* λ, μ *and* ν, $\mu < \nu < \lim_{u\to\infty} g(u)$, *such that*

$$K_\lambda < \infty, \quad J_\mu < \infty, \quad J_\nu = \infty,$$

then all subclasses in $\mathbb{M}^+$ *are nonempty, i.e.*

$$\mathbb{M}^+_{\ell,0} \neq \emptyset, \quad \mathbb{M}^+_{\infty,0} \neq \emptyset, \quad \mathbb{M}^+_{\infty,\ell} \neq \emptyset.$$

Remark 4.6. As already claimed, the coexistence of all types of nonoscillatory solutions is impossible for the half-linear equation (1.5). Example 3.3 illustrates that this is possible for system (1.1), since $K_1 < \infty$, $J_{1/4} < \infty$ and $J_1 = \infty$.

Another possible discrepancy between (1.1) and (1.5), concerning the nonoscillatory solutions, is a consequence of the following result.

Theorem 4.7. *Assume* $I_a = \infty$, $I_b < \infty$. *If there exists* $\mu > 0$ *such that*

$$\int_1^{\infty} b(\tau) g\left(\int_T^{s_1(\tau)} a(\sigma) f\left(\mu \int_{r_1(\sigma)}^{\infty} b(\xi)d\xi\right) d\sigma\right) d\tau = \infty, \tag{4.3}$$

for any $T \geq 1$, *then* $\mathbb{M}^+_{\infty,0} = \mathbb{M}^+_{\infty,\ell} = \emptyset$.

Proof. First, observe that assumption (4.3) yields $g(\infty) = \infty$. Let $(x, y) \in \mathbb{M}^+_{\infty,0}$. Let $T \geq 1$ be such that x is positive increasing, y is positive decreasing for $t \geq T$ and $r(t) \geq 1$, $s(t) \geq 1$ for $t \geq T$. Using the l'Hopital rule, we obtain that there exists a positive constant d such that $y(t) > d \int_t^\infty b(\xi) d\xi$ for $t \geq T \geq 1$. Then eventually we have

$$y(r(t)) > d \int_{r(t)}^\infty b(\xi) d\xi. \tag{4.4}$$

Without loss of generality, suppose that (4.4) holds for any $t \geq T$. Let $T_1 \geq T$ such that $s(t) \geq T$ on $[T_1, \infty)$. From the first equation in (1.1) we obtain for $t \geq T$

$$x(t) \geq \int_T^t a(\sigma) f\left(y(r(\sigma))\right) d\sigma \geq \int_T^t a(\sigma) f\left(d \int_{r(\sigma)}^\infty b(\xi) d\xi\right) d\sigma$$

and so for $t \geq T_1$

$$g\left(x(s(t))\right) \geq g\left(\int_T^{s(t)} a(\sigma) f\left(d \int_{r(\sigma)}^\infty b(\xi) d\xi\right) d\sigma\right).$$

Hence

$$-y'(t) = b(t) g(x(s(t))) \geq b(t) g\left(\int_T^{s(t)} a(\sigma) f\left(d \int_{r(\sigma)}^\infty b(\xi) d\xi\right) d\sigma\right). \tag{4.5}$$

Integrating this inequality on (t, ∞), $t \geq T_1$, we obtain a contradiction and so $\mathbb{M}^+_{\infty,0} = \emptyset$. Let us show that $\mathbb{M}^+_{\infty,\ell} = \emptyset$. Since $I_b < \infty$, from (4.3) we obtain that g is unbounded and $J_\mu = \infty$. Since intermediate solutions do not exist, by Theorem 3.1 we obtain $K_\lambda = \infty$ for any $\lambda > 0$. Thus, applying Corollary 4.3 we get $\mathbb{M}^+_{\infty,\ell} = \emptyset$. $\square$

The condition (4.3) has to be verified for any $T \geq 1$. Two stronger conditions, which do not depend on the choice of T, are given by the following:

Corollary 4.8. *Assume $I_a = \infty$, $I_b < \infty$. Let one of the following conditions hold:*

i_1) there exist $0 < \lambda < 1$ and $\mu > 0$ such that

$$\int_1^\infty b(\tau) g\left(\lambda \int_1^{s_1(\tau)} a(\sigma) f\left(\mu \int_{r_1(\sigma)}^\infty b(\xi) d\xi\right) d\sigma\right) d\tau = \infty; \tag{4.6}$$

i_2) $g(uv) \leq g(u) g(v)$ for any $u > 1, v > 1$ and for some $\mu > 0$

$$\int_1^\infty b(\tau) g\left(\int_1^{s_1(\tau)} a(\sigma) f\left(\mu \int_{r_1(\sigma)}^\infty b(\xi) d\xi\right) d\sigma\right) d\tau = \infty. \tag{4.7}$$

Then $\mathbb{M}^+_{\infty,0} = \mathbb{M}^+_{\infty,\ell} = \emptyset$.

Proof. Fixed $T \geq 1$, set

$$J(T, t) = \int_T^t a(\sigma) f\left(\mu \int_{r_1(\sigma)}^\infty b(\xi) d\xi\right) d\sigma.$$

Claim i_1). Since $I_b < \infty$, the function g is unbounded and $J(T, \infty) = \infty$. Hence there exists $\lambda \in (0, 1)$ such that $\lambda J(1, s_1(\tau)) \leq J(T, s_1(\tau))$ for large τ. So, from the monotonicity of g, (4.3) holds and the assertion follows from Theorem 4.7.

Claim i_2). Choose $\tau \geq T$ large so that $J(1, T)[J(1, s_1(\tau))]^{-1} < 1$. Hence the assertion follows from Theorem 4.7 and

$$J(1, s_1(\tau)) = J(T, s_1(\tau)) \left(1 + \frac{J(1, T)}{J(T, s_1(\tau))}\right) \leq 2J(T, s_1(\tau)).$$

$\square$

Remark 4.9. Theorem 4.7 is not significant for the Emden-Fowler system (1.3) with $\alpha \neq \beta$. Indeed for (1.3), the integrals J_μ, K_λ reads as

$$J = \int_1^\infty a(\tau) \left(\mu \int_\tau^\infty b(\sigma) d\sigma\right)^{1/\alpha} d\tau, \quad K = \int_1^\infty b(\tau) \left(\lambda \int_1^\tau a(\sigma) d\sigma\right)^\beta d\tau,$$

respectively, and so, when the case I) holds, (4.3) implies $J = K = \infty$. But, in such a case, it is known that all solutions of (1.3) are oscillatory (see, e.g., [13, Theorems 11.3, 11.4]). When $\alpha = \beta$, the following example illustrates a possible discrepancy between the coexistence of solutions of (1.1) (with $r(t) = s(t) = t$) and (1.5).

Example 4.10. Consider the system

$$x'(t) = e^t f(y(t)), \quad y'(t) = -2t^{-3}|x(t))|x(t),$$

where f is a nondecreasing continuous function on $\mathbb{R}$ such that $uf(u) > 0$ for $u \neq 0$ and

$$f(u) = \exp(-1/\sqrt{u}) \qquad \text{if } 0 < u < 1.$$

Then (4.3) is satisfied for $\mu = 1$. Since $J_{1/4} < \infty$, in view of Corollaries 4.3, 4.8 we have $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^-_{\infty,0} = \emptyset$, $\mathbb{M}^-_{\infty,\ell} = \emptyset$. Note that such situation never occurs for (1.5), because solutions in $\mathbb{M}^+_{\ell,0}$ always coexist with solutions either in $\mathbb{M}^+_{\infty,0}$ or in $\mathbb{M}^+_{\infty,\ell}$ ([3]).

We close this section by illustrating the role of the deviating arguments to the asymptotic properties of solutions of (1.1).

Example 4.11. Consider the system

$$x'(t) = |y(t)|^\alpha \operatorname{sgn} y(t), \quad y'(t) = -\frac{1}{t^3} g(x(s(t))) \tag{4.8}$$

where $\alpha \in (0, 1/2]$ and g is a nondecreasing continuous function on $\mathbb{R}$ such that $ug(u) > 0$ for $u \neq 0$ and

$$g(u) = \begin{cases} eu & \text{if } u \in [0, 1) \\ e^u & \text{if } u \geq 1 \end{cases}$$

If $s(t) = 1 + \log t$, then Theorem 3.1 is applicable with $\lambda = \mu = 1$, and (4.8) has intermediate solutions. If $s(t) = t$, then $K_\lambda = \infty$ for any $\lambda > 0$ and all solutions of this system are oscillatory, as follows from [10] (see also [1, Theorem 7.1.2]).

Example 4.12. Consider the system

$$x' = \frac{1}{t} y(r(t)), \quad y' = -\frac{1}{t^2} x(t).$$

If $r(t) = 1 + \log t$, assumptions in Theorem 3.1 are verified with $\lambda = \mu = 1$ and

$$x \in \mathbb{M}^+_{\infty,0} \neq \emptyset, \quad x \in \mathbb{M}^+_{\infty,\ell} \neq \emptyset, \quad x \in \mathbb{M}^+_{\ell,0} = \emptyset.$$

If $r(t) = t$, from Corollary 4.3 we have $\mathbb{M}^+_{\infty,\ell} \neq \emptyset, \mathbb{M}^+_{\ell,0} \neq \emptyset$. Moreover, intermediate solutions do not exist, because the system is linear.

5. THE CASE II)

In this Section we show how it is possible to extend all the above results to the case II), i.e. when $I_a < \infty, I_b = \infty$. In view of (1.2), if (x, y) is a solution of (1.1), then $(y, -x)$ is a solution of the system

$$\begin{cases} z' = b(t) g(w(s(t)) \\ w' = -a(t) f(z(r(t))) \end{cases} \tag{5.1}$$

and vice-versa, if (z, w) is a solution of (5.1), then $(w, -z)$ is a solution of (1.1). Observe that (5.1) comes out from (1.1) by interchanging the roles of the functions a, f, r with b, g, s, respectively, and vice-versa. Moreover, the integrals J_μ, K_λ become Z_μ and W_λ, respectively. So, such a *duality property* permits us to extend all the above results to case $I_a < \infty, I_b = \infty$ in a very simple way.

Nevertheless, we point out that the assumption (1.2) is not necessary for our results, because the same results can be also proved in a direct way by adding suitable assumptions on nonlinearities on $(-\infty, 0)$.

We recall that in case II), all nonoscillatory solutions of (1.1) belong to $\mathbb{M}^-$. Here we report only those results which are significative as regards the possible coexistence in the subclasses of $\mathbb{M}^-$ or the comparison with known results. The remaining extensions are left to the reader.

Theorem 5.1. *Assume $I_a < \infty$, $I_b = \infty$. If there exist two positive constants λ, μ, $\mu < \lim_{u\to\infty} f(u)$, such that*

$$W_\lambda < \infty, \quad Z_\mu = \infty,$$

then $\mathbb{M}^-_{0,-\infty} \neq \emptyset$.

Proof. The assertion follows from Theorem 3.1, by applying the duality property. □

Remark 5.2. When $s(t) = r(t) = t$, the existence of solutions in $\mathbb{M}^-_{0,-\infty}$ is considered in [12, Theorem 3.7]. Theorem 5.1 substantially extends such a result, because Theorem 3.7 requires $Z_\mu = \infty$ for any $\mu > 0$ and, implicitly, also the boundedness of f.

Concerning the coexistence of nonoscillatory solutions, the following holds.

Corollary 5.3. *Assume $I_a < \infty$, $I_b = \infty$. If there exist three positive constants λ, μ and ν, $\nu < \mu < \lim_{u\to\infty} f(u)$, such that*

$$Z_\nu < \infty, \quad W_\lambda < \infty, \quad Z_\mu = \infty,$$

then all subclasses in $\mathbb{M}^-$ are nonempty, i.e.

$$\mathbb{M}^-_{0,-\ell} \neq \emptyset, \quad \mathbb{M}^-_{0,-\infty} \neq \emptyset, \quad \mathbb{M}^-_{\ell,-\infty} \neq \emptyset.$$

Proof. The assertion follows from Theorem 5.1, applying the duality property to Corollary 4.3. $\square$

Notice that Corollary 5.3 gives a possibility of coexistence of all three types of nonoscillatory solutions of (1.1), which is impossible for (1.5), as already observed in Remark 4.6.

Applying the duality property to Theorem 4.7, we obtain the following.

Theorem 5.4. *Assume $I_a < \infty$, $I_b = \infty$. If there exists $\lambda > 0$ such that*

$$\int_1^\infty a(\tau) f\left(\int_T^{r_1(\tau)} b(\sigma) g\left(\lambda \int_{s_1(\sigma)}^\infty a(\xi)d\xi\right) d\sigma\right) d\tau = \infty \tag{5.2}$$

for any $T \geq 1$, then $\mathbb{M}^-_{0,-\infty} = \mathbb{M}^-_{\ell,-\infty} = \emptyset$.

Remark 5.5. Theorem 5.4 extends [12, Theorem 3.8], where a necessary condition for the existence of solutions in $\mathbb{M}^-_{0,-\infty}$, when $r(t) = s(t) = t$, is given. However such a result is not well formulated, because (3.8) in [12] should be (5.2) (with $r_1(\tau) = \tau$, $s_1(\sigma) = \sigma$), as it follows from the proof of Theorem 3.8.

Acknowledgment. The research of the second author was supported by the Research Project 0021622409 of the Ministery of Education of the Czech Republic and by the Grant 201/08/046 of the Grant Agency of the Czech Republic.

REFERENCES

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Acad. Publ. G, Dordrecht, 2003.

[2] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, On oscillatory solutions of quasilinear differential equation, *J. Math. Anal. Appl.*, 320:108–120, 2006.

[3] M. Cecchi, Z. Došlá, M. Marini, On intermediate solutions and the Wronskian for half-linear differential equations, *J. Math. Anal. Appl.*, 336:905–918, 2007.

[4] O. Došlý, P. Řehák, *Half-linear Differential Equations*, North-Holland, Mathematics Studies 202, Elsevier Sci. B.V., Amsterdam, 2005.

[5] X. Fan, W.T. Li, C. Zhong, A classification schemes for positive solutions of second order nonlinear iterative differential equations, *Elect. J. Diff. Equat.*, 25:1–14, 2000.

[6] H. Hoshino, R. Imabayashi, T. Kusano, T. Tanigawa, On second-order half-linear oscillations, *Adv. Math. Sci. Appl.*, 8:199–216, 1998.

[7] T. H. Hwang, H. J. Li, C. C. Yeh, Asymptotic behavior of nonoscillatory solutions of second-order differential equations, *Computers Math. Appl.*, 50:271–280, 2005.

[8] W. Jingfa, Oscillation and nonoscillation theorems for a class of second order quasilinear functional differential equations, *Hiroshima Math. J.*, 27:449–466, 1997.

[9] I. T. Kiguradze, A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Acad. Publ. G., Dordrecht, 1993.

[10] I. G. E. Kordonis, Ch. G. Philos, On the oscillation of nonlinear two-dimensional differential systems, *Proc. Amer. Math. Soc.*, 126:1661-1667, 1998.

[11] M. K. Kwong, J. S. W. Wong, A nonoscillation theorem for sublinear Emden-Fowler equations, *Nonlinear Anal., T.M.A.*, 64:1641–1646, 2006.

[12] W. T. Li, S. S. Cheng, Limit behaviours of non-oscillatory solutions of a pair of coupled nonlinear differential equations, *Proc. Edinb. Math. Soc.*, 43:457–473, 2000.

[13] J. D. Mirzov, *Asymptotic properties of solutions of the systems of nonlinear nonautonomous ordinary differential equations*, (Russian), Maikop, Adygeja Publ. 1993. English translation: Folia, Mathematics 14, Masaryk University Brno 2004.

[14] J. Sugie, M. Onitsuka, A non-oscillation theorem for nonlinear differential equations with p-Laplacian, *Proc. Roy. Edinburgh Sect. A.*, 136:633–647, 2006.

[15] N. Yamaoka, Oscillation criteria for second-order damped nonlinear differential equations with p-Laplacian, *J. Math. Anal. Appl.*, 325:932–948, 2007.