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# FURTHER RESULTS ON THE EXISTENCE OF CONTINUOUS SELECTIONS OF SOLUTION SETS OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

E.O. AYOOLA

Department of Mathematics, University of Ibadan, Ibadan, Nigeria
eoayoola@ictp.it

**ABSTRACT.** We prove that the map that associates to the initial value the set of solutions to the Lipschitzian Quantum Stochastic Differential Inclusion (QSDI) admits a selection which is continuous from the locally convex space of stochastic processes to the space of adapted and weakly absolutely continuous solutions. As a corollary, the reachable set multifunction admits a continuous selection. In the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus, these results are achieved subject to some compactness conditions on the set of initial values and on some coefficients of the inclusion.

**Key Words:** Continuous selections, Lipschitzian quantum stochastic differential inclusions, Reachable sets, solution sets

**AMS (MOS) Subject Classification:** 81S25, 60H10

## 1. INTRODUCTION

This work is concerned with further investigations of the existence and applications of continuous selections of solution sets of quantum stochastic differential inclusions (QSDI). In the context of classical differential inclusions defined in finite dimensional Euclidean spaces, such investigations have attracted considerable attention in the literature. Some well known results on continuous selections and their applications in the finite dimensional Euclidean settings can be found in [1, 2, 14, 15, 18, 20, 22]. As in [8, 18, 20, 22], selection results have been used among other things for the interpolation of a given finite set of trajectories of classical differential inclusions.

However, in the non commutative quantum setting, investigations of the existence of continuous selections and their applications have not received a comparable attention in the literature. In the framework of the Hudson and Parthasarathy [17, 19] formulations of quantum stochastic calculus, we established in our previous work [4], some continuous selections of solution sets of quantum stochastic differential inclusion (QSDI) defined on the set of the matrix elements of initial points with values

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in the set of matrix elements of solutions. However, on this occasion and in the same framework of quantum stochastic calculus, we establish the existence of a selection map continuous from a compact set of initial values contained in the space of quantum stochastic processes into the locally convex space of adapted weakly absolutely continuous quantum stochastic processes. In addition, as a corollary, we deduce that the reachable set multifunction admits a continuous selection. This work, therefore, complements our results in [4] where the set of the matrix elements of solutions and the reachable set respectively admit continuous selections and some continuous representations.

The proof of our main results here adapts the techniques employed in Cellina [1] in a way that is suitable for the analysis of QSDI where the solutions live in certain locally convex spaces. Our main tools in the construction of the selection are some suitable use of Liapunov's theorem on the range of vector measures (see [1, 15, 16]) and Ekhaguere's existence result [11] for the solutions of QSDI (2.3). The result is a generalization of Filippov's extension of Gronwall's inequalities to solutions of QSDI (2.3).

The plan for the rest of the paper is as follows: In section 2, we present some fundamental results, notations and assumptions. The main results of the paper are reported in Section 3.

## 2. PRELIMINARY RESULTS AND ASSUMPTIONS

In what follows, we adopt the notations, formulation and the frameworks as reported in [3, 4, 11, 12, 13]. Detailed definitions of various spaces that appear below can be found in [11]. In the sequel, $\gamma$ is a fixed Hilbert space, $\mathbb{D}$ is an inner product space with $\mathcal{R}$ as its completion, and $\Gamma(L^2_\gamma(\mathbb{R}_+))$ is the Boson Fock Space determined by the function space $L^2_\gamma(\mathbb{R}_+)$. The set $\mathbb{E}$ is the subset of the Fock space generated by the exponential vectors. If $\mathcal{N}$ is a topological space, then we denote by $clos(\mathcal{N})$ (resp. $comp(\mathcal{N})$), the family of all nonempty closed subsets of $\mathcal{N}$ (resp. compact members of $clos(\mathcal{N})$).

In our formulations, quantum stochastic processes are $\tilde{\mathcal{A}}$-valued maps on $[t_0, T]$. The space $\tilde{\mathcal{A}}$ is the completion of the linear space

$$\mathcal{A} = L^+_W(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$

endowed with the locally convex operator topology generated by the family of seminorms $\{x \to \|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|, \quad \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Here, $\mathcal{A}$ consists of linear operators from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ with the property that the domain of the adjoint operator contains $\mathbb{D}\underline{\otimes}\mathbb{E}$. We adopt the notation and the definitions of Hausdorff topology on $clos(\tilde{\mathcal{A}})$ as explained in [11]. The Hausdorff topology is determined by some family of pseudo-metrics. On the set $\mathbb{C}$ of complex numbers, we employ the metric topology on $clos(\mathbb{C})$ induced by the Hausdorff metric $\rho$. Thus for $A, B, \in clos(\mathbb{C})$, $\rho(A,B)$ is the Hausdorff distance between the sets and for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\mathcal{N}, \mathcal{M} \in clos(\tilde{\mathcal{A}})$, $\rho_{\eta\xi}(\mathcal{N}, \mathcal{M})$ denotes pseudo-metrics as in [11, 12, 13].

A quantum stochastic process $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ will be said to be weakly continuous on the interval $I = [t_0, T]$ if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $t \to \Phi_{\eta\xi}(t)$ is continuous. Here, $\Phi_{\eta\xi}(t) := \langle \eta, \Phi(t)\xi\rangle$. We shall denote by $C[I, \tilde{\mathcal{A}}]$ the set of all weakly continuous quantum stochastic processes on $[t_0, T]$ and for each $\Phi \in C[I, \tilde{\mathcal{A}}]$, we set

$$\|\Phi_{\eta\xi}\|_c := \sup_I |\Phi_{\eta\xi}(t)| = \sup_I \|\Phi(t)\|_{\eta\xi}. \tag{2.1}$$

By employing the symbol $Ad(\tilde{\mathcal{A}})_{wc}$ to denote the set of all adapted weakly continuous stochastic processes, then we have the following set inclusion

$$Ad(\tilde{\mathcal{A}})_{wac} \subseteq Ad(\tilde{\mathcal{A}})_{wc} \subseteq C[I, \tilde{\mathcal{A}}],$$

since all weakly absolutely continuous stochastic processes are weakly continuous.

As in [11], we denote by $wac(\tilde{\mathcal{A}})$, the completion of $Ad(\tilde{\mathcal{A}})_{wac}$ in the topology generated by the family of seminorms

$$|\Phi|_{\eta\xi} = \|\Phi(t_0)\|_{\eta\xi} + \int_{t_0}^{T} |\frac{d}{ds}\langle \eta, \Phi(s)\xi\rangle| ds \tag{2.2}$$

for each $\Phi \in Ad(\tilde{\mathcal{A}})_{wac}$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The existence of the continuous selections which we study in this paper concerns solution and the reachable sets of quantum stochastic differential inclusions in the integral form given by:

$$\begin{aligned} X(t) \in a + \int_0^t & (E(s, X(s)) d\wedge_\pi (s) + F(s, X(s)) dA_f(s) + G(s, X(s)) dA_g^+(s) \\ & + H(s, X(s)) ds), \; t \in [t_0, T], \end{aligned} \tag{2.3}$$

where the coefficients $E, F, G, H$ are continuous and lie in the space $L^2_{loc}([t_0, T] \times \tilde{\mathcal{A}}))_{mvs}$, $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$. Here, $B(\gamma)$ is the space of bounded endomorphisms of $\gamma$ and $(t_0, a) \in [t_0, T] \times \tilde{\mathcal{A}}$ is a fixed point.

For any pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$, $c, d \in \mathbb{D}$, as in our previous works in [3, 4, 5, 6, 7], we shall in what follows, employ the equivalent form of (2.3) as established in [11] given by the nonclassical ordinary differential inclusion:

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi), \qquad X(t_0) = a, \;\; t \in [t_0, T]. \tag{2.4}$$

The multivalued map $P$ appearing in (2.4) is of the form

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi\rangle$$

where the map $P_{\alpha\beta} : [t_0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ is given by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : [t_0, T] \to \mathbb{C}$ are defined by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma,\ \nu_\beta(t) = \langle f(t), \beta(t)\rangle_\gamma,$$

$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\gamma,\ t \in [t_0, T]$$

for all $(t, x) \in [0, T] \times \tilde{\mathcal{A}}$ and the coefficients $E, F, G, H$ belong to the space $L^2_{loc}([t_0, T] \times \tilde{\mathcal{A}})_{mvs}$ of multivalued stochastic processes with closed values.

As explained in [11], the map $P$ cannot in general be written in the form:

$$P(t, x)(\eta, \xi) = \tilde{P}(t, \langle \eta, x\xi\rangle)$$

for some complex valued multifunction $\tilde{P}$ defined on $[t_0, T] \times \mathbb{C}$, for $t \in [t_0, T],\ x \in \tilde{\mathcal{A}},\ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Under the condition of compactness of the values of the map $(t, x) \to P(t, x)(\eta, \xi)$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we prove that the map which associates to the initial point $a \in \tilde{\mathcal{A}}$, the set of solutions $S^{(T)}(a)$ to (2.4) admits a continuous selection from the space $\tilde{\mathcal{A}}$ to the completion (denoted by $wac(\tilde{\mathcal{A}})$) of the locally convex space of adapted weakly absolutely continuous stochastic processes indexed by elements of the interval $[t_0, T]$. In particular, we show that the map $a \to R^{(T)}(a)$ admits a continuous selection, where $R^{(T)}(a)$ is the reachable set at $t = T$ of the QSDI (2.3).

To establish our main results, we need the notion of partition of unity subordinate to any covering of a compact subset of $\tilde{\mathcal{A}}$ corresponding to an arbitrary pair of vectors in $\mathbb{E}$, the subspace of the Fock space generated by the exponential vectors. In what follows, unless otherwise indicated, we consider quantum stochastic processes defined on a simple Fock space. That is we shall take the initial space $\mathcal{R} = \mathbb{C}$ so that $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)) \equiv \Gamma(L^2_\gamma(\mathbb{R}_+))$ and $\mathbb{D}\underline{\otimes}\mathbb{E} \equiv \mathbb{E}$.

**Definition 2.1.** . Let $A$ be a compact subset of the locally convex space $\tilde{\mathcal{A}}$ and let $\{\Omega_i\}_{i\in J}$ be an open covering for $A$ with a finite sub covering $\{\Omega_i,\ \ i = 1, 2, \ldots, m\}$. A family of functions $\{\Pi_{\eta\xi,i}(\cdot)\},\ \ i = 1, 2, \ldots, m$ corresponding to an arbitrary pair of elements $\eta,\ \xi\ \in \mathbb{E}$ defined on $A$ is called a Lipschitzian partition of unity subordinate to the finite subcovering if:

(1) The map $\Pi_{\eta\xi,i}(\cdot)$ is Lipschitzian for all $i = 1, 2, \ldots, m$. That is there exist constants $L_{\eta\xi} > 0$ such that for any pair $a,\ a'\ \in A$,

$$|\Pi_{\eta\xi,i}(a) - \Pi_{\eta\xi,i}(a')| \le L_{\eta\xi}\|a - a'\|_{\eta\xi}.$$

(2) $\Pi_{\eta\xi,i}(a) > 0$ for $a \in \Omega_i \bigcap A$ and $\Pi_{\eta\xi,i}(a) = 0$ for $a \in A\backslash\Omega_i$.

(3) For each $a \in A$, $\sum_{i=1}^m \Pi_{\eta\xi,i}(a) = 1$.

**Lemma 2.2.** . *Let $A$ be a compact subset of the space $\tilde{\mathcal{A}}$. Then, there exists a family of Lipschitzian partitions of unity subordinate to any finite subcovering of an open covering for the set $A$.*

*Proof.* We outline the proof as follows: Let $\{\Omega_i\}, i = 1, 2, \ldots, m$ be a finite open subcovering of an open covering $\{\Omega_i\}_{i\in J}$ of $A$. First we claim that the map $q_{\eta\xi} : \tilde{\mathcal{A}} \to \mathbb{R}_+$ defined by

$$q_{\eta\xi}(x) = \mathbf{d}_{\eta\xi}(x, Q), \quad Q \in clos(\tilde{\mathcal{A}}),$$

satisfies for any pair $x_1, x_2 \in \tilde{\mathcal{A}}$,

$$|q_{\eta\xi}(x_1) - q_{\eta\xi}(x_2)| \leq \|x_1 - x_2\|_{\eta\xi}. \tag{2.5}$$

Inequality (2.5) can be established as follows: Let $\epsilon > 0$ be given. Since $\mathbf{d}_{\eta\xi}(x, Q) = \inf_{y\in Q} \|x - y\|_{\eta\xi}$, then there exists $y_1 \in Q$ satisfying

$$\|x_1 - y_1\|_{\eta\xi} \leq \mathbf{d}_{\eta\xi}(x_1, Q) + \epsilon.$$

Hence,

$$\begin{aligned} \mathbf{d}_{\eta\xi}(x_2, Q) &\leq \|x_2 - y_1\|_{\eta\xi} \\ &\leq \|x_2 - x_1\|_{\eta\xi} + \|x_1 - y_1\|_{\eta\xi} \\ &\leq \|x_2 - x_1\|_{\eta\xi} + \mathbf{d}_{\eta\xi}(x_1, Q) + \epsilon. \end{aligned}$$

Interchanging $x_1$ and $x_2$, we have

$$|\mathbf{d}_{\eta\xi}(x_1, Q) - \mathbf{d}_{\eta\xi}(x_2, Q)| \leq \|x_1 - x_2\|_{\eta\xi} + \epsilon.$$

Inequality (2.5) follows since $\epsilon$ is arbitrary.

For $i = 1, 2, \ldots, m$, define the family of functions $q_{\eta\xi,i} : A \to \mathbb{R}_+$ by

$$q_{\eta\xi,i}(a) = \mathbf{d}_{\eta\xi}(a, A\backslash\Omega_i)$$

and functions $\Pi_{\eta\xi,i} : A \to \mathbb{R}_+$ defined by

$$\Pi_{\eta\xi,i}(a) = \frac{q_{\eta\xi,i}(a)}{\sum_{j=1}^{m} q_{\eta\xi,j}(a)} \tag{2.6}$$

For at least one $j \in \{1, 2, \ldots, m\}$, $a \in \Omega_j$. Hence, $\sum_{j=1}^{m} q_{\eta\xi,j}(a) > 0$. Also, by the definition of the seminorm $\|\cdot\|_{\eta\xi}$ and the properties of the exponential vectors $\eta, \xi \in \mathbb{E}$, the value $\|x\|_{\eta\xi}$ can never be zero when $x$ is not a zero process. This follows from the fact that for any pair of exponential vectors $\eta, \xi \in \mathbb{E}$ such that $\eta = e(\alpha)$, $\xi = e(\beta)$, $\alpha$, $\beta \in L^2_\gamma(\mathbb{R}_+)$, we have $\langle e(\alpha), e(\beta)\rangle = e^{\langle\alpha,\beta\rangle}$ (see [6] for some details). Consequently, (2.6) is well defined. The rest of the proof follows a similar argument as in the proof of Lemma 2.1 in [4]. This shows that $\{\Pi_{\eta\xi,i}(\cdot)\}_{i=1}^{m}$ is a family of Lipschitzian partition of unity subordinate to the covering. ∎

In the proof of our main results, we shall make use of the following maps that are associated with the family $\Pi_{\eta\xi,i}(\cdot))$ given by (2.6). Define the maps

$$\sigma_{\eta\xi}(i,a) = \sum_{1\leq j\leq i} \Pi_{\eta\xi,j}(a), \quad a \in A, \ i \in \{1,2,\dots,m\}. \tag{2.7}$$

**Definition 2.3**: Let $\epsilon > 0$ be fixed. Then the common modulus of continuity $\Theta_{\eta\xi}(\epsilon)$ depending on the pair $\eta,\xi \in \mathbb{E}$, of the map $a \to \sigma_{\eta\xi}(i,a)$ is defined by:

$$\Theta_{\eta\xi}(\epsilon) = \sup\{|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| : a,a' \in A, \ \|a-a'\|_{\eta\xi} \leq \epsilon, \ i = 1,2,\dots,m\}. \tag{2.8}$$

*Remarks*: As in the case of the modulus of continuity of real valued functions defined on the real line, (see [21, p. 2], for example), the modulus of continuity $\Theta_{\eta\xi}(\epsilon)$ defined by (2.8) satisfies the following inequalities as consequences of the definition. That is,

$$\Theta_{\eta\xi}(\epsilon) \leq \Theta_{\eta\xi}(\epsilon'), \quad \text{whenever} \ \ \epsilon \leq \epsilon'$$

and

$$\Theta_{\eta\xi}(\lambda\epsilon) \leq (1+\lambda)\Theta_{\eta\xi}(\epsilon), \quad \text{for any positive number } \lambda. \tag{2.9}$$

These follow directly from (2.8).

In what follows, we shall employ the space of complex valued sesquilinear forms on $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ denoted by $Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ and assume that the multivalued map $(t,x) \to P(t,x)(\eta,\xi)$ appearing in Equation (2.4) satisfies the following conditions:

$\mathcal{S}$(a). $P : \Omega \subseteq [t_0,T] \times \tilde{\mathcal{A}} \to 2^{Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})}$ defined on an open subset $\Omega \subseteq [t_0,T] \times \tilde{\mathcal{A}}$ bounded on $\Omega$ by constants $M_{\eta\xi}$ that depend on $\eta,\xi$, i.e

$$|P(t,x)(\eta,\xi)| \leq M_{\eta\xi}, \quad (t,x) \in \Omega, \ \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}.$$

$\mathcal{S}$(b). The map $t \to P(t,x)(\eta,\xi)$ is measurable for fixed $x \in \tilde{\mathcal{A}}$ and for all $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$\mathcal{S}$(c). The map $(t,x) \to P(t,x)(\eta,\xi)$ is Lipschitzian with Lipschitz function $K_{\eta\xi}(t)$ lying in $L^1_{loc}([t_0,T])$, i.e. for $x,y \in \tilde{\mathcal{A}}$

$$\rho(P(t,x)(\eta,\xi), \ P(t,y)(\eta,\xi)) \leq K_{\eta\xi}(t)\|x-y\|_{\eta\xi}.$$

$\mathcal{S}$(d). The set $P(t,x)(\eta,\xi)$ is compact in $\mathbb{C}$, the field of complex numbers, for all $(t,x) \in \Omega, \ \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$\mathcal{S}$(e). There exists a compact set $A \subseteq \tilde{\mathcal{A}}$ such that $\forall \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the set

$$\{(t, a+v(t-t_0) : a \in A, \ v \in \tilde{\mathcal{A}} \text{ such that } \ \|v\|_{\eta\xi} \leq M_{\eta\xi}, \ t \in [t_0,T]\} \subseteq \Omega.$$

Moreover, we set

$$Y_{\eta\xi}(t) = \int_{t_0}^{t} K_{\eta\xi}(s)ds. \tag{2.10}$$

We shall assume that the interval $I = [t_0,T]$ satisfies the following:

$$\Lambda_{\eta\xi} = 3(e^{Y_{\eta\xi} - Y_{\eta\xi}(s)} - 1) < 1; \quad \forall\, \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \tag{2.11}$$

where

$$Y_{\eta\xi} = \int_{t_0}^{T} K_{\eta\xi}(s)ds.$$

In what follows, we set

$$\Gamma_{\eta\xi} = \int_{t_0}^{T} e^{Y_{\eta\xi} - Y_{\eta\xi}(s)}ds.$$

## 3. ESTABLISHMENT OF THE SELECTION MAP

By a solution of QSDI (2.3) we mean a quantum stochastic process $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (2.3). We denote by $S^{(T)}(a)$, the set of solutions of Lipschitzian QSDI (2.3). It has been established in [11] that under the conditions $\mathcal{S}(a) - \mathcal{S}(e)$, this set is not empty. Similar existence result under a general Lipschitz condition has recently been established in [3]. Our main result below shows that there exists a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$ such that for each $a \in A$, $\tilde{\Phi}(a) \in S^{(T)}(a) \subseteq wac(\tilde{\mathcal{A}})$.

**Theorem 3.1.** *Suppose that the map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies the assumptions* $\mathcal{S}$(a)–$\mathcal{S}$(e). *Then there exists a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$ such that for every $a \in A$, $\tilde{\Phi}(a)$ is a solution to the QSDI* (2.4).

*Proof.* The proof shall be presented in six parts in what follows. The pair of elements $\eta, \xi \in \mathbb{E}$ are arbitrary unless otherwise indicated. We note here that it would be enough for us to establish the existence of the continuous selection by establishing appropriate estimates in the seminorms that generate the topology of the spaces $\tilde{\mathcal{A}}$ and $wac(\tilde{\mathcal{A}})$. A justification for this can be found in [23, p. 5].

**Part A:** We claim that there exists two sequences of adapted stochastic processes $\Phi^n(a),\ \Psi^n(a) : [t_0, T] \to \tilde{\mathcal{A}}$ such that

(i) $\Psi^n(a) \in S^{(T)}(a)$; $\Phi^n(a)$ is adapted weakly absolutely continuous such that $\Phi^n(a)(t_0) = a$. Setting $\Phi^n_{\eta\xi}(a)(t) := \langle \eta, (\Phi^n(a)(t))\xi \rangle$, then,

(ii) $\|\Phi^n_{\eta\xi}(a) - \Psi^n_{\eta\xi}(a)\|_c = \sup_I |\langle \eta, (\Phi^n(a)(t))\xi \rangle - \langle \eta, (\Psi^n(a)(t))\xi \rangle| \leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi}$.

(iii) For every $\epsilon > 0$, there exists $\delta(\epsilon) = \delta(\epsilon, n, \eta, \xi) > 0$ and a function $R^n_{\eta\xi}(a, \epsilon) : I \to \mathbb{R}_+$ satisfying

$$\int_I R^n_{\eta\xi}(a, \epsilon)(s)ds \leq 2M_{\eta\xi}\epsilon \tag{3.1}$$

such that

$$|\frac{d}{dt}\langle \eta, (\Phi^n(a)(t))\xi \rangle - \frac{d}{dt}\langle \eta, (\Phi^n(a')(t))\xi \rangle| \leq R^n_{\eta\xi}(a, \epsilon)(t)$$

whenever $\|a - a'\|_{\eta\xi} \leq \delta(\epsilon)$.

(iv) $|\frac{d}{dt}\langle \eta, (\Phi^n(a)(t))\xi \rangle - \frac{d}{dt}\langle \eta, (\Psi^n(a)(t))\xi \rangle| \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)},\ n \geq 2.$

(v) $|\Phi^n(a) - \Phi^{n-1}(a)|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi},\ \ n \geq 3.$

**Part B:** We apply mathematical induction as follows: Set $\Phi^1(a) = a$. Then trivially, $\Phi^1(a)$ lies in $Ad(\tilde{\mathcal{A}})_{wac}$. Also by the boundedness of the map $P$,

$$d\left(\frac{d}{dt}\langle\eta,(\Phi^1(a)(t))\xi\rangle,\ P(t,\Phi^1(t))(\eta,\xi)\right) = d(0,P(t,a)(\eta,\xi)) \leq M_{\eta\xi}.$$

By the existence results of Ekhaguere [11], there exists $\Psi^1(a) \in S^{(T)}(a)$ such that $\forall t \in [t_0,T]$,

$$\|\Phi^1(a)(t) - \Psi^1(a)(t)\|_{\eta\xi} \leq \int_{t_0}^{t} e^{(Y_{\eta\xi}(t)-Y_{\eta\xi}(s))} M_{\eta\xi} ds \leq M_{\eta\xi}\Gamma_{\eta\xi}.$$

The above shows that $\Phi^1,\ \Psi^1$ satisfy items (i), (ii) in Part A, with n=1. Item (iii) also holds by putting $R^1_{\eta\xi}(a,\epsilon) = 0$ for n=1.
Assume that we have defined $\Phi^\nu(a)$ and $\Psi^\nu(a)$ satisfying items (i) – (iii), for $\nu = 1,2,\ldots,n-1$. We claim that we can define $\Phi^n(a)$ and $\Psi^n(a)$ satisfying items (i) – (iv) for $n \geq 2$.

**Part C:** For notational simplification, we will denote $\Phi^{n-1}$ by $\Phi$ and $\Psi^{n-1}$ by $\Psi$. The map $\Phi_{\eta\xi} : A \to C[I,\mathbb{C}]$, $a \to \Phi_{\eta\xi}(a)$ is uniformly continuous on account of our assumption in Part A above. This can be shown as follows:
Let $r > 0$ be a real number satisfying $r \leq \delta(\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi})$, where $\delta$ is defined in Part A, item (iii) above. Then $a',\ a''$ lying in the set $B[a,r] = \{x \in \tilde{\mathcal{A}} : \|x-a\|_{\kappa\vartheta} \leq r,\ \forall\kappa,\vartheta \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ implies that $\|a-a'\|_{\eta\xi} \leq r \leq \delta$ and $\|a-a''\|_{\eta\xi} \leq r \leq \delta$.

By item (iii), Part A,

$$|\frac{d}{dt}\langle\eta,(\Phi(a)(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a'(t))\xi\rangle| \leq R^{n-1}_{\eta\xi}(a,\epsilon)(t)$$

and

$$|\frac{d}{dt}\langle\eta,(\Phi(a)(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a''(t))\xi\rangle| \leq R^{n-1}_{\eta\xi}(a,\epsilon)(t)$$

so that

$$|\frac{d}{dt}\langle\eta,(\Phi(a')(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a''(t))\xi\rangle| \leq 2R^{n-1}_{\eta\xi}(a,\epsilon)(t). \tag{3.2}$$

But by the absolute continuity of the map $t \to (\langle\eta,\Phi(a')(t)\xi\rangle - \langle\eta,\Phi(a'')(t)\xi\rangle)$, we have

$$\begin{aligned} &|\langle\eta,(\Phi(a')(t))\xi\rangle - \langle\eta,(\Phi(a''(t))\xi\rangle| \\ &\qquad = |\int_I \frac{d}{ds}\left(\langle\eta,(\Phi(a')(s))\xi\rangle - \langle\eta,(\Phi(a''(s))\xi\rangle\right) ds| \end{aligned} \tag{3.3}$$

Hence from (3.3) and using (3.1)

$$|\langle\eta,\Phi(a')(t)\xi\rangle - \langle\eta,\Phi(a'')(t)\xi\rangle| \leq 2\int_I R^{n-1}_{\eta\xi}(a,\epsilon)(s)ds \leq 4M_{\eta\xi}\epsilon.$$

If $r \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}$, then $\|a'-a''\|_{\eta\xi} \leq 2r \leq \frac{2}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}$, implies that

$$\|\Phi(a')(t) - \Phi(a'')(t)\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi},$$

where $\epsilon$ is small enough so that

$$\epsilon \leq \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1} \leq \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

Consequently, we have for $a',\ a'' \in B[a,r]$

$$\|\Phi_{\eta\xi}(a') - \Phi_{\eta\xi}(a'')\|_c \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

Our claim of uniform continuity of the map $a \to \Phi_{\eta\xi}(a)$ follows.

Let $\{B(a_i, r), i = 1, 2, \ldots, m\}$ be a finite open cover of the compact set $A$, $a_i \in A\ \forall i$ and $\Pi_{\eta\xi,i} : A \to \mathbb{R}_+$, a partition of unity subordinate to the cover. Here

$$B(a,r) = \{x \in \tilde{\mathcal{A}} : \|x - a\|_{\kappa\vartheta} < r, \forall \kappa, \vartheta \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

and

$$\sum_{i=1}^{m} \Pi_{\eta\xi,i}(a) = 1,\ \ \Pi_{\eta\xi,i}(a) > 0,\ \ \forall a \in A \bigcap B(a_i, r).$$

The existence of such family of Lipschitzian partition of unity follows from Lemma 2.2.

Next, we define

$$\sigma_{\eta\xi}(j,a) = \sum_{1\leq i\leq j} \Pi_{\eta\xi,i}(a) \ \text{ and }\ \Psi_i(t) = \Psi(a_i)(t).$$

Let $\delta > 0$ be such that $\frac{T-t_0}{\delta} = m'$, an integer and $\delta < \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$.

The subintervals

$$J(j) = [t_0 + (j-1)\delta, t_0 + j\delta),\ j = 1, 2, \ldots, m'$$

form a partition of the interval $I = [t_0, T]$. Corresponding to an arbitrary pair of elements $\eta, \xi \in \mathbb{E}$, we consider the family of complex valued maps on $[t_0, T])$ defined by:

$$D_{\eta\xi,i,j}(t) = \frac{d}{dt}\langle \eta, \Psi_i(t)\xi\rangle I_{J(j)}(t), \quad i = 1, 2 \ldots, m;\ j = 1, 2, \ldots, m', \tag{3.4}$$

where $I_{J(j)}$ is the characteristic function on the set $J(j)$. For $\alpha \in [0,1]$, let $\{B(\alpha)\}$ be a nested family of measurable subsets of the interval $[t_0, T]$ such that $B(0) = \emptyset,\ B(1) = [t_0, T]$ satisfying

$$\int_{B(\alpha)} D_{\eta\xi,i,j}(t)dt = \alpha \int_{t_0}^{T} D_{\eta\xi,i,j}(t)dt,\ \ \mu(B(\alpha)) = \alpha(T - t_0). \tag{3.5}$$

Such a family exists by a Corollary to Liapunov's theorem (see [1, 15]).

Since $\Psi_i \in S^{(T)}(a_i)$ then as shown in [11], there exists processes $V_i : I \to \tilde{\mathcal{A}}$ lying in $L^1_{loc}(\tilde{\mathcal{A}})$ such that $\Psi_i(t) = a_i + \int_{t_0}^{t} V_i(s)ds$ and

$$\frac{d}{dt}\langle \eta, \Psi_i(t)\xi\rangle = \langle \eta, V_i(t)\xi\rangle.$$

It follows from (3.4) that

$$D_{\eta\xi,i,j}(t) = \langle \eta, V_i(t)I_{J(j)}(t)\xi\rangle, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, m'.$$

Hence by (3.5) and putting

$$V_{i,j}(t) = V_i(t)I_{J(j)}(t), \quad i = 1, 2, \dots, m,\ j = 1, 2, \dots, m',$$

we have

$$\int_{B(\alpha)} V_{i,j}(t)dt = \alpha \int_{t_0}^{T} V_{i,j}(t)dt. \tag{3.6}$$

Next we define the stochastic process $\Phi^n(a) : [t_0, T] \to \tilde{\mathcal{A}}$ by

$$\Phi^n(a)(t) = a + \sum_i \int_{t_0}^{t} V_i(s)I_{B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a))}(s)ds, \tag{3.7}$$

with its matrix element given by

$$\langle \eta, (\Phi^n(a)(t))\xi\rangle = \langle \eta, a\xi\rangle + \sum_i \int_{t_0}^{t} \langle \eta, (V_i(s))\xi\rangle I_{B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a))}(s)ds.$$

We remark that the process $\Phi^n(a)$ given by (3.7) lies in $wac(\tilde{\mathcal{A}})$ since each $V_i \in L^1_{loc}(\tilde{\mathcal{A}})$ and in addition, $\Phi^n(a)$ is an adapted and weakly absolutely continuous process.

To show that $\Phi^n(a)$ satisfies item (iii) of Part A, we note that as in the proof of the only Theorem in [1], $\frac{d}{dt}\langle \eta, \Phi^n(a)\xi\rangle$ and $\frac{d}{dt}\langle \eta, \Phi^n(a')\xi\rangle$ differ only on the subset $E' \subset [t_0, T]$ given by

$$E' = \bigcup_{i=1}^{m} \{(B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a)))\triangle(B(\sigma_{\eta\xi}(i,a'))\backslash B(\sigma_{\eta\xi}(i-1,a')))\}$$

and that

$$E' \subset \bigcup_{i=1}^{m} \{B(\sigma_{\eta\xi}(i,a))\triangle B(\sigma_{\eta\xi}(i,a'))\}, \tag{3.8}$$

where for any two subsets $S, B$ of $[t_0, T]$, $S\triangle B := (S \cup B)\backslash(S \cap B)$.

As in [1], we fix $\epsilon > 0$ and let $\Theta_{\eta\xi} = \Theta_{\eta\xi}(\epsilon)$ be the common modulus of continuity of the map $a \to \sigma_{\eta\xi}(i, a)$, given by (2.8). Then, whenever $\|a - a'\|_{\eta\xi} < \Theta_{\eta\xi}(\frac{\epsilon}{2m})$, the superset in (3.8) is contained in the set

$$E''(a, \epsilon) = \bigcup_{i=1}^{m} \{B(\sigma_{\eta\xi}(i,a) + \frac{\epsilon}{2m})\backslash B(\sigma_{\eta\xi}(i,a) - \frac{\epsilon}{2m})\} \tag{3.9}$$

and the total measure of $E''(a, \epsilon)$ is bounded by $\epsilon$ or

$$\int_I I_{E''(a,\epsilon)} < \epsilon. \tag{3.10}$$

The foregoing assertion follows from the fact that if

$$\|a - a'\|_{\eta\xi} < \Theta_{\eta\xi}(\frac{\epsilon}{2m}),$$

then

$$|\sigma_{\eta\xi}(i, a) - \sigma_{\eta\xi}(i, a')| \le \Theta_{\eta\xi}\left(\Theta_{\eta\xi}(\frac{\epsilon}{2m})\right) \le \Theta_{\eta\xi}\left(\Theta_{\eta\xi}(\epsilon)\right).$$

Since $\Theta_{\eta\xi}(\epsilon)$ is positive and finite, we can write $\Theta_{\eta\xi}(\epsilon) = \lambda_{\eta\xi}\epsilon$ for some $\lambda_{\eta\xi} > 0$. Then, by (2.9),

$$|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| \leq (1+\lambda_{\eta\xi})\lambda_{\eta\xi}\epsilon = \frac{\epsilon}{2m}.$$

Thus,

$$|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| \leq \frac{\epsilon}{2m}, \tag{3.11}$$

for some positive number $\lambda_{\eta\xi}$ satisfying the algebraic equation

$$\lambda_{\eta\xi}^2 + \lambda_{\eta\xi} - \frac{1}{2m} = 0.$$

The claim follows by employing (3.11) and the property of the nested family of sets $\{B(\cdot)\}$.

Consequently we have

$$|\frac{d}{dt}\langle\eta, \Phi^n(a)(t)\xi\rangle - \frac{d}{dt}\langle\eta, \Phi^n(a')(t)\xi\rangle| \leq 2M_{\eta\xi}I_{E''(a,\epsilon)}(t)$$

so that item (iii) in Part A follows with

$$\delta(\epsilon) = \Theta_{\eta\xi}(\frac{\epsilon}{2m}) \text{ and } R^n_{\eta\xi}(a,\epsilon)(t) = 2M_{\eta\xi}I_{E''(a,\epsilon)}(t).$$

**Part D:** We estimate here the pseudo-distance of $\Phi^n(a)$ from the set of solution $S^{(T)}(a)$. To this end, let $t \in [t_0 + r\delta, t_0 + (r+1)\delta))$. At the point $t = t_0 + r\delta$, the integral in (3.7) can be written as

$$\begin{aligned}
& \sum_i \int_{t_0}^{t_0+r\delta} V_i(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} ds \\
= & \sum_i \sum_{l\leq r} \int V_i(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} I_{J(l)}(s) ds \\
= & \sum_i \sum_{l\leq r} \int V_{i,l}(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))}(s) ds \\
= & \sum_i \sum_{l\leq r} \int_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} V_{i,l}(s) ds \\
= & \sum_{l\leq r} \sum_i \Pi_{\eta\xi,i}(a) \int_I V_{i,l}(s) ds, \\
= & \sum_i \sum_{l\leq r} \Pi_{\eta\xi,i}(a)\{\Psi_i(t_0 + l\delta) - \Psi_i(t_0 + (l-1)\delta)\} \\
= & \sum_i \Pi_{\eta\xi,i}(a)\{\Psi_i(t_0 + r\delta) - \Psi_i(t_0)\}.
\end{aligned}$$

This follows from (3.6) and the definition of $\sigma_{\eta\xi}(\cdot,\cdot)$.

Hence, we have

$$\Phi^n(a)(t_0 + r\delta) - a = \sum_i \Pi_{\eta\xi,i}(a)(\Psi_i(t_0 + r\delta) - a_i)$$

For any $j \in \{1, 2, \ldots, m\}$, we can write

$$\begin{aligned} \|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} &\leq \|\Phi^n(a)(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi} \\ &+ \|\Phi^n(a)(t_0 + r\delta) - \Phi^n(a)(t)\|_{\eta\xi} + \|\Psi_j(t) - \Psi_j(t_0 + r\delta)\|_{\eta\xi} \end{aligned} \tag{3.12}$$

Since

$$|\frac{d}{dt} < \eta, \Phi^n(a)(t)\xi > | \leq M_{\eta\xi}$$

and

$$|\frac{d}{dt} < \eta, \Psi_j(t)\xi > | \leq M_{\eta\xi},$$

by our choice of $\delta$, the sum of the last two terms in (3.12) is bounded by $\frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$. Hence, from (3.12)

$$\begin{aligned} \|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} &\leq \|a - \sum_i \Pi_{\eta\xi,i}(a)a_i\|_{\eta\xi} \\ &+ \| \sum_i \Pi_{\eta\xi,i}(a) \left(\Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\right) \|_{\eta\xi} + \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}. \end{aligned} \tag{3.13}$$

By our choice of $r$ in Part C, whenever $\Pi_{\eta\xi,i}(a) > 0$, then

$$\|a - a_i\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

This estimate also holds for the first term at the right hand side of (3.13). Furthermore,

$$\begin{aligned} &\|\Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi} \\ &\quad \leq \|\Psi_i(t_0 + r\delta) - \Phi(a_i)(t_0 + r\delta)\|_{\eta\xi} + \|\Phi(a_i)(t_0 + r\delta) - \Phi(a_j)(t_0 + r\delta)\|_{\eta\xi} \\ &\quad + \|\Phi(a_j)(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi}. \end{aligned} \tag{3.14}$$

When both $\Pi_{\eta\xi,i}(a) > 0$ and $\Pi_{\eta\xi,j}(a) > 0$ and by the choice of $r$, the second term on the right of (3.14) satisfies

$$\|\Phi(a_i)(t_0 + r\delta) - \Phi(a_j)(t_0 + r\delta)\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}, \tag{3.15}$$

so that by item (ii) in Part A and the recursive assumption, we finally have

$$\|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}, \tag{3.16}$$

Equation (3.16) holds for every $j$ such that $\Pi_{\eta\xi,j}(a) > 0$. By the definition of $\Phi^n(a)(t)$ given by (3.7), at any point $t$ except on a set of measure zero in $I$,

$$\frac{d}{dt}\langle\eta, \Phi^n(a)(t)\xi\rangle = \frac{d}{dt}\langle\eta, \Psi_j(t)\xi\rangle$$

for some $j$ such that $\Pi_{\eta\xi,j}(a) > 0$.

Since $\Psi_j \in S^{(T)}(a_j)$, then

$$\frac{d}{dt}\langle\eta, \Psi_j(t)\xi\rangle \in P(t, \Psi_j(t))(\eta, \xi)$$

and therefore we have

$$d\left(\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle,\ P(t,\Phi^n(a)(t))(\eta,\xi)\right) \leq \rho\left(P(t,\Psi_j(t))(\eta,\xi),\ P(t,\Phi^n(a)(t))(\eta,\xi)\right)$$

$$\leq K_{\eta\xi}(t)\|\Psi_j(t)-\Phi^n(a)(t)\|_{\eta\xi}$$

$$\leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t) \tag{3.17}$$

on account of (3.16) and the fact that the map $(t,x)\to P(t,x)(\eta,\xi)$ is Lipschitzian. We notice that estimate (3.17) is independent of $j$ and therefore holds on $I=[t_0,T]$. Again by the existence result in [11], there exists a stochastic process $\Psi^n(a)\in S^{(T)}(a)$ such that

$$\|\Psi^n(a)(t)-\Phi^n(a)(t)\|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}(e^{Y_{\eta\xi}(t)}-1)\leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}, \tag{3.18}$$

and

$$|\frac{d}{dt}\langle\eta,\Psi^n(a)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle| \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}. \tag{3.19}$$

Inequalities (3.18) and (3.19) prove items (ii) and (iv) in Part A for all $n\geq 2$.

**Part E:** It is now left for us to show that if items (i)–(iv) hold up to $n-1$, then item (v) holds for $n$. We use the same notations as before to fix any $t$ and let $j$ be such that

$$\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle=\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle$$

so that $\Pi_{\eta\xi,j}(a)>0$. Then we have

$$|\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi^{n-1}(a)(t)\xi\rangle| = |\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a)(t)\xi\rangle|$$

$$\leq |\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a_j)(t)\xi\rangle|$$

$$+|\frac{d}{dt}\langle\eta,\Phi(a_j)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a)(t)\xi\rangle| \tag{3.20}$$

By item (iv), the first term in (3.20) is bounded by $3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}$ while by the choice of $r$, and applying item (iii), the second term in (3.20) is bounded by the functions $R_{\eta\xi}^{n-1}(a,\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}):I\to\mathbb{R}_+$ satisfying the conditions of item (iii). These bounds do not depend on $j$ and so hold on the whole of interval $I$.

Since

$$\int_I R_{\eta\xi}^{n-1}(a,\epsilon)(t)dt<2M_{\eta\xi}\epsilon,$$

we have

$$|\Phi^n(a)-\Phi^{n-1}(a)|_{\eta\xi} = \int_I |\frac{d}{dt}\langle\eta,(\Phi^n(a)(t)-\Phi^{n-1}(a)(t))\xi\rangle|dt$$

$$\leq 3\int_I M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}dt+\int_I R_{\eta\xi}^{n-1}(a,\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1})(t)dt$$

$$\leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}+2M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1},$$

proving item (v).

**Part F:** By item (iii) in Part A, we have

$$\begin{aligned}|\Phi^n(a)-\Phi^n(a')|_{\eta\xi} &= \|a-a'\|_{\eta\xi}+\int_{t_0}^{T}\frac{d}{dt}|\langle\eta,\Phi^n(a)(t)\xi\rangle-\langle\eta,\Phi^n(a')(t)\xi\rangle|dt\\ &\le \delta(\epsilon)+2M_{\eta\xi}\epsilon.\end{aligned}$$

This shows that each map $\Phi^n : A \to wac(\tilde{\mathcal{A}})$ is uniformly continuous. Since $\Lambda_{\eta\xi} < 1$ for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, item (v) shows that the sequence $\{\Phi^n(a)\}$ is Cauchy. Since $wac(\tilde{\mathcal{A}})$ is complete, the sequence converges to a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$.

By construction, the sequence $\{\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle\}$ converges in $L^1[I]$ to $\frac{d}{dt}\langle\eta,\tilde{\Phi}(a)(t)\xi\rangle$. Hence, a subsequence converges to $\frac{d}{dt}\langle\eta,\tilde{\Phi}(a)(t)\xi\rangle$ pointwise almost everywhere.

By item (iv),

$$d\left(\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle,\ P(t,\Phi^n(a)(t))(\eta,\xi)\right)\to 0 \quad \text{as} \quad n\to\infty.$$

Since the images $P(t,x)(\eta,\xi)$ are compact in the field of complex numbers, and therefore closed and since the map $(t,x)\to P(t,x)(\eta,\xi)$ is continuous, then we have:

$$\frac{d}{dt}\langle\eta,\tilde{\Phi}(a)(t)\xi\rangle \in P(t,\tilde{\Phi}(a)(t))(\eta,\xi)$$

showing that

$$\tilde{\Phi}(a)\in S^{(T)}(a)\subseteq wac(\tilde{\mathcal{A}}).$$

□

The next result is a direct consequence of Theorem 3.1 concerning the reachable sets of QSDI (2.3) at the time $t = T$ defined by:

$$R^{(T)}(a)=\{\Psi(a)(T):\Psi(a)\in S^{(T)}(a)\}\subseteq\tilde{\mathcal{A}}. \tag{3.21}$$

**Corollary 3.2.** The multivalued map $R^{(T)} : A \to 2^{\tilde{\mathcal{A}}}$ admits a continuous selection where $R^{(T)}(a)$ is given by (3.21).

*Proof.* We define a continuous map $h : wac(\tilde{\mathcal{A}}) \to \tilde{\mathcal{A}}$ by

$$h(\Phi(\cdot))=\Phi(T),\ \Phi(\cdot)\in wac(\tilde{\mathcal{A}}).$$

Thus, by Theorem (3.1), the map $h(\tilde{\Phi}(a)(\cdot)) = \tilde{\Phi}(a)(T)$ is continuous for each $a \in A$ and $\tilde{\Phi}(a)(T) \in R^{(T)}(a)$.

The conclusion of the corollary follows. □

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