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LIMIT CYCLES COMING FROM THE PERTURBATION OF 2-DIMENSIONAL CENTERS OF VECTOR FIELDS IN $\mathbb{R}^3$

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ABSTRACT. In this paper we study the limit cycles of polynomial vector fields in $\mathbb{R}^3$ which bifurcates from three different kinds of two dimensional centers (non-degenerate and degenerate). The study is down using the averaging theory.

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1. INTRODUCTIONS AND STATEMENT OF THE MAIN RESULTS

One of the main problems in the theory of differential systems is the study of their periodic orbits, their existence, their number and their stability. In this paper the study of the existence of periodic orbits of a differential system is reduced using the averaging theory to study the zeroes of a system of functions. One of the main problems for applying the averaging theory is to transform the differential system that we want to study into the normal form for applying the averaging method. When this method cannot be applied, sometimes there are other ways to reduce the problem of studying the existence of periodic orbits to study the zeroes of a system of functions. In general these methods are called alternative methods (see for instance Section 2.4 of Chow and Hale [4]), one of these particular alternative methods is the well known Liapunov–Schmidt method.

As usual a limit cycle of a differential equation is a periodic orbit isolated in the set of all periodic orbits of the differential equation. In this paper we shall study the limit cycles which bifurcate from the periodic orbits of three kinds of different 2–dimensional centers contained in a differential system of $\mathbb{R}^3$ when we perturb it. These kinds of bifurcations have been studied extensively for 2–dimensional systems (see for instance the book [6] and the references quoted there), but for 3–dimensional systems there are very few results, see for instance [1, 2, 7, 8].

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First we shall study the perturbation of a 2–dimensional linear center inside $\mathbb{R}^3$, and after the perturbation of two degenerate 2–dimensional centers also inside $\mathbb{R}^3$.

We consider the following system

$$
\begin{aligned}
\dot{x} &= -y + \varepsilon P(x, y, z), \\
\dot{y} &= \quad x + \varepsilon Q(x, y, z) + \varepsilon \cos t, \\
\dot{z} &= \ az + \varepsilon R(x, y, z),
\end{aligned} \tag{1.1}
$$

where $a \neq 0$ and

$$
P = \sum_{i+j+k=0}^{n} a_{i,j,k} x^i y^j z^k, \quad Q = \sum_{i+j+k=0}^{n} b_{i,j,k} x^i y^j z^k \quad R = \sum_{i+j+k=0}^{n} c_{i,j,k} x^i y^j z^k.
$$

System (1.1) has been studied in [5] when $a = 0$ and without the perturbation due to $\varepsilon \cos t$. When a and ε are zero the unperturbed system has all $\mathbb{R}^3$ except the z–axis filled by periodic orbits. For $a \neq 0$ and $\varepsilon = 0$ the unperturbed system (1.1) only has the plane $z = 0$ except the origin filled of periodic orbits. The center on the plane $z = 0$ is called *nondegenerate* when the eigenvalues of its linear part are of the form $\pm bi$ with $b \neq 0$. When we have a center having zero eigenvalues we say that it is *degenerate*.

Theorem 1.1. *The linear differential system* (1.1) *with* $\varepsilon = 0$ *restricted to the plane* $z = 0$ *has a global center at the origin (i.e. all the orbits contained in* $z = 0$ *with the exception of the origin are periodic). Then for convenient polynomials* P, Q *and* R, *system* (1.1) *with* $\varepsilon \neq 0$ *sufficiently small has at least* $m \in \{1, 2, \ldots, 2[(n-1)/2] + 1\}$ *limit cycles bifurcating from the periodic orbits of the linear center contained in* $z = 0$ *when* $\varepsilon = 0$, *where* $[\cdot]$ *denotes the integer part function. Moreover the existence or not of these limit cycles only depends on the coefficients* $a_{i,j,0}$ *and* $b_{i,j,0}$ *with* $i + j$ *odd.*

The proof of Theorem 1.1 is given in Section 2.

Now we consider the polynomial differential system in $\mathbb{R}^3$ given by

$$
\begin{aligned}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon P(x, y, z), \\
\dot{y} &= \quad x(x^2 - y^2) + \varepsilon Q(x, y, z), \\
\dot{z} &= \quad z(x^2 + y^2) + \varepsilon R(x, y, z).
\end{aligned} \tag{1.2}
$$

The unperturbed system (1.2) with $\varepsilon = 0$ has a degenerate center at the origin of the plane $z = 0$ (for more details see Section 3), a main difference with the unperturbed system (1.1) with $\varepsilon = 0$ whose center is non–degenerate.

Theorem 1.2. *The homogeneous polynomial differential system* (1.2) *with* $\varepsilon = 0$ *restricted to the plane* $z = 0$ *has a global center at the origin. Let* P, Q *and* R *be polynomials of degree at most* n. *Then for convenient polynomials* P, Q *and* R, *system* (1.2) *with* $\varepsilon \neq 0$ *sufficiently small has at least* $m \in \{1, 2, \ldots, [(n-1)/2]\}$ *limit cycles bifurcating from the periodic orbits of the center contained in* $z = 0$ *when* $\varepsilon = 0$.

The proof of Theorem 1.2 is given in Section 3. We remark that the perturbation of the degenerate center at the plane $z = 0$ inside the class of planar vector fields has been studied in [7].

Finally we consider the polynomial differential system in $\mathbb{R}^3$ defined by

$$\begin{aligned} \dot{x} &= -y(x^2+y^2) + \varepsilon P(x,y,z), \\ \dot{y} &= \quad x(x^2+y^2) + \varepsilon Q(x,y,z), \\ \dot{z} &= \quad z(x^2+y^2) + \varepsilon R(x,y,z). \end{aligned} \tag{1.3}$$

Theorem 1.3. *The homogeneous polynomial differential system* (1.3) *with* $\varepsilon = 0$ *restricted to the plane* $z = 0$ *has a global center at the origin. For convenient polynomials* P, Q *and* R *of degree at most* n, *system* (1.3) *with* $\varepsilon \neq 0$ *sufficiently small has at least* $m \in \{1, 2, \ldots, [(n-1)/2]\}$ *limit cycles bifurcating from the periodic orbits of the center contained in* $z = 0$ *when* $\varepsilon = 0$.

2. PROOF OF THEOREM 1.1

The origin $(0,0,0)$ is the unique singular point of system (1.1) when $\varepsilon = 0$. The eigenvalues of the linearized system at this singular point are $\pm i$ and a. So it has a linear center on the plane $z = 0$. Outside this plane all the orbits tends to it in forward time if $a < 0$, or in backward time if $a > 0$.

If we apply the notation introduced in the Appendix to system (1.1) we have that $\mathbf{x} = (x,y,z)$, $F_0(\mathbf{x},t) = (-y, x, az)^T$, $F_1(\mathbf{x},t) = (P, Q + \cos t, R)^T$ and $F_2(\mathbf{x},t) = \mathbf{0}$. Let $\mathbf{x}(t; x_0, y_0, z_0, \varepsilon)$ be the solution of system (1.1) such that $\mathbf{x}(0; x_0, y_0, z_0, \varepsilon) = (x_0, y_0, z_0)$. The periodic solution $\mathbf{x}(t; x_0, y_0, 0, 0) = (x(t), y(t), z(t))^T$ of the unperturbed system (1.1) with $\varepsilon = 0$ is

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = y_0 \cos t + x_0 \sin t, \quad z(t) = 0. \tag{2.1}$$

Note that all the periodic orbits of the linear center have period 2π.

For our system the V and the α of Theorem 5.1 of the Appendix are $V = \{(x,y,0) : 0 < x^2 + y^2 < \rho\}$ for some arbitrary $\rho > 0$ and $\alpha = (x_0, y_0) \in V$.

The fundamental solution matrix $M(t)$ of the variational equation of the unperturbed system $(1.1)_{\varepsilon=0}$ with respect to the periodic orbits (2.1) satisfying that $M(0)$ is the identity matrix is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{at} \end{pmatrix}.$$

We remark that it is independent of the initial condition $(x_0, y_0, 0)$. Moreover an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi a} \end{pmatrix}.$$

In short we have shown that all the assumptions of Theorem 5.1 of the Appendix hold. Hence we shall study the zeros $\alpha = (x_0, y_0) \in V$ of the two components of the function $\mathcal{F}(\alpha)$ given in (5.4). More precisely we have $\mathcal{F}(\alpha) = (\mathcal{F}_1(\alpha), \mathcal{F}_2(\alpha))$ where

$$\begin{aligned} \mathcal{F}_1(\alpha) &= \int_0^{2\pi} (\cos t\ P(\mathbf{x}(t; X_0)) + \sin t\ (Q(\mathbf{x}(t; X_0)) + \cos t))\, dt \\ &= \int_0^{2\pi} \cos t \left(\sum_{i+j=0}^{n} a_{i,j,0} X(t)^i Y(t)^j \right) dt \\ &\quad + \int_0^{2\pi} \sin t \left(\sum_{i+j=0}^{n} b_{i,j,0} X(t)^i Y(t)^j \right) dt, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_2(\alpha) &= \int_0^{2\pi} (-\sin t\ P(\mathbf{x}(t; X_0)) + \cos t\ (Q(\mathbf{x}(t; X_0)) + \cos t))\, dt \\ &= \pi - \int_0^{2\pi} \sin t \left(\sum_{i+j=0}^{n} a_{i,j,0} X(t)^i Y(t)^j \right) dt \\ &\quad + \int_0^{2\pi} \cos t \left(\sum_{i+j=0}^{n} b_{i,j,0} X(t)^i Y(t)^j \right) dt, \end{aligned}$$

with $X_0 = (x_0, y_0, 0, 0)$, $X(t) = x_0 \cos t - y_0 \sin t$ and $Y(t) = x_0 \sin t + y_0 \cos t$.

To simplify the computation of these two previous integrals we do the change of variables $(x_0, y_0) \longmapsto (r, s)$ given by

$$x_0 = r\cos s, \quad y_0 = -r\sin s, \tag{2.2}$$

where $r > 0$ and $s \in [0, 2\pi)$. In order to estimate the number of the periodic solutions of system (1.1), according with Theorem 5.1 we must study the solutions of the system $\mathcal{F}_1(x_0, y_0) = \mathcal{F}_2(x_0, y_0) = 0$.

From the change (2.2) and the expressions (2.1) a monomial $x^i y^j$ which appears in $P(x, y, z)$ and $Q(x, y, z)$ becomes $(-1)^j r^{i+j} \cos^i(t - s) \sin^j(t - s)$. Hence we obtain the following expressions

$$\begin{aligned} \mathcal{F}_1(r, s) &= \int_0^{2\pi} \cos t \sum_{i+j=0}^{n} (-1)^j a_{i,j,0}\, r^{i+j} \cos^i(t - s) \sin^j(t - s) dt + \\ &\quad \int_0^{2\pi} \sin t \sum_{i+j=0}^{n} (-1)^j b_{i,j,0}\, r^{i+j} \cos^i(t - s) \sin^j(t - s) dt, \end{aligned}$$

$$\begin{aligned}\mathcal{F}_2(r,s) &= \pi - \int_0^{2\pi} \sin t \sum_{i+j=0}^{n} (-1)^j a_{i,j,0}\, r^{i+j} \cos^i(t-s)\sin^j(t-s)dt + \\ &\int_0^{2\pi} \cos t \sum_{i+j=0}^{n} (-1)^j b_{i,j,0}\, r^{i+j} \cos^i(t-s)\sin^j(t-s)dt.\end{aligned}$$

Taking $u = t - s$ the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ can be written as

$$\begin{aligned}\mathcal{F}_1(r,s) &= \cos s(I_{a1} + I_{b2}) - \sin s(I_{a2} - I_{b1}), \\ \mathcal{F}_2(r,s) &= -\sin s(I_{a1} + I_{b2}) - \cos s(I_{a2} - I_{b1}) + \pi,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}I_{a1} = I_{a1}(r) &= \sum_{i+j=0}^{n} (-1)^j r^{i+j} a_{i,j,0} \int_0^{2\pi} \cos^{i+1} u \, \sin^j u \, du, \\ I_{a2} = I_{a2}(r) &= \sum_{i+j=0}^{n} (-1)^j r^{i+j} a_{i,j,0} \int_0^{2\pi} \cos^{i} u \, \sin^{j+1} u \, du, \\ I_{b1} = I_{b1}(r) &= \sum_{i+j=0}^{n} (-1)^j r^{i+j} b_{i,j,0} \int_0^{2\pi} \cos^{i+1} u \, \sin^j u \, du, \\ I_{b2} = I_{b2}(r) &= \sum_{i+j=0}^{n} (-1)^j r^{i+j} b_{i,j,0} \int_0^{2\pi} \cos^{i} u \, \sin^{j+1} u \, du.\end{aligned}$$

Set

$$I_1(r) = I_{a1}(r) + I_{b2}(r), \qquad I_2(r) = I_{a2}(r) - I_{b1}(r).$$

Using symmetries the integral $\int_0^{2\pi} \cos^p u \, \sin^q u \, du$ is not zero if and only if p and q are even. So $I_1(r)$ and $I_2(r)$ are polynomials in r having all their monomials of odd degree. Moreover if n is even the degree in the variable r of the polynomials $I_1(r)$ and $I_2(r)$ is $n-1$, and if n is odd that degree is n. So their degree always is odd and equal to $2[(n-1)/2]+1$. Of course we are playing with the fact that the coefficients of those polynomials can be chosen arbitrarily.

It is clear that the system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ given by (2.3) is equivalent to the system

$$\begin{pmatrix} I_1(r) \\ I_2(r) \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ -\sin s & -\cos s \end{pmatrix} \begin{pmatrix} 0 \\ -\pi \end{pmatrix} = \pi \begin{pmatrix} \sin s \\ \cos s \end{pmatrix}. \tag{2.4}$$

We claim that system (2.4) has at most $2[(n-1)/2]+1$ solutions providing different limit cycles of system (1.1), and that this number is reached.

For proving the claim first we observe that system (2.4) is equivalent to the system

$$I_1^2(r) + I_2^2(r) = \pi^2, \quad \frac{I_1(r)}{I_2(r)} = \tan s. \tag{2.5}$$

Since the first equation of system (2.5) is a polynomial equation in the variable r^2 of degree $2[(n-1)/2]+1$ playing with the fact that the coefficients of the polynomials $I_1(r)$ and $I_2(r)$ are arbitrary, it follows that it has at most $2[(n-1)/2]+1$ zeros in

$(0, \infty)$, and we can choose the coefficients $a_{i,j,0}$ and $b_{i,j,0}$ such that it has exactly m simple zeros $r_i > 0$ with $m \in \{1, 2, \dots, 2[(n-1)/2] + 1\}$.

There are two solutions s_i and $s_i + \pi$ in $[0, 2\pi)$ of the second equation for each zero $r_i > 0$ of the first equation of (2.5). But these two solutions only provides two different initial conditions of the same periodic orbit. In short applying Theorem 5.1 we would get at most $2[(n-1)/2] + 1$ limit cycles for system (1.1) if the jacobian $\det\big(\partial(\mathcal{F}_1, \mathcal{F}_2)/\partial(r, s)\big) \neq 0$ at $(r, s) = (r_i, s_i)$.

Playing with the coefficients $a_{i,j,0}$ and $b_{i,j,0}$ we get

$$I_1(r_i)I_1'(r_i) + I_2(r_i)I_2'(r_i) \neq 0, \tag{2.6}$$

for every solution (r_i, s_i) of system (2.5). Hence it is easy to check that

$$\left|\frac{\partial(\mathcal{F}_1(r,s), \mathcal{F}_2(r,s))}{\partial(r,s)}\right|_{(r,s)=(r_i,s_i)} = I_1(r_i)I_1'(r_i) + I_2(r_i)I_2'(r_i) \neq 0.$$

In short the claim is proved and consequently Theorem 1.1.

3. PROOF OF THEOREM 1.2

System (1.2) restricted to $z = 0$ and with $\varepsilon = 0$ becomes the homogeneous polynomial differential system

$$\dot{x} = -y(3x^2 + y^2), \qquad \dot{y} = x(x^2 - y^2),$$

of degree 3 that has the non–rational first integral

$$H(x, y) = (x^2 + y^2) \exp\left(-\frac{2x^2}{x^2 + y^2}\right),$$

as it is easy to check.

Doing the transformation $(x, y, z) \to (r\cos\theta, r\sin\theta, z)$ system (1.2) becomes

$$\begin{aligned}
\dot{r} &= -r^3 \sin 2\theta + \varepsilon\big(\cos\theta P + \sin\theta Q\big),\\
\dot{\theta} &= \quad r^2 + \varepsilon\frac{1}{r}\big(\cos\theta Q - \sin\theta P\big),\\
\dot{z} &= \quad zr^2 + \varepsilon R,
\end{aligned}$$

where $P = P(r\cos\theta, r\sin\theta, z)$, $Q = Q(r\cos\theta, r\sin\theta, z)$ and $R = R(r\cos\theta, r\sin\theta, z)$. This system is equivalent to

$$\begin{aligned}
\frac{dr}{d\theta} &= -r\sin 2\theta + \varepsilon F_1 + O(\varepsilon^2),\\
\frac{dz}{d\theta} &= \quad z + \varepsilon G_1 + O(\varepsilon^2),
\end{aligned} \tag{3.1}$$

where

$$F_1 = F_1(r\cos\theta, r\sin\theta, z) = \frac{1}{2r^2}\big((\cos\theta + \cos 3\theta)P + (3\sin\theta + \sin 3\theta)Q\big),$$

$$G_1 = G_1(r\cos\theta, r\sin\theta, z) = \frac{1}{r^3}\big(rR - z(\cos\theta Q - \sin\theta P)\big).$$

System $(3.1)|_{\varepsilon=0}$ restricted in $z=0$ is a global center around the origin with periodic orbits

$$r = r_0 e^{-\sin^2\theta}, \quad z = 0, \tag{3.2}$$

satisfying $r(0, r_0) = r_0$ and all of them with period 2π. The fundamental solution matrix $M(\theta)$ of the variational equation of the unperturbed system $(3.1)_{\varepsilon=0}$ with respect to the periodic orbits (3.2) satisfying that $M(0)$ is the identity matrix is

$$M(\theta) = \begin{pmatrix} e^{-\sin^2\theta} & 0 \\ 0 & e^{\theta} \end{pmatrix}.$$

We remark that it is independent of the initial condition $(r_0, 0)$. Moreover an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 \\ 0 & 1-e^{-2\pi} \end{pmatrix}.$$

Hence from Theorem 5.1 we must estimate the zeros of the function

$$\mathcal{F}(r_0) = 2\int_0^{2\pi} e^{\sin^2\theta} F_1(r_0 e^{-\sin^2\theta}\cos\theta, r_0 e^{-\sin^2\theta}\sin\theta, 0)d\theta.$$

Writing

$$P(x,y,0) = \sum_{l=0}^{n} P_l(x,y),\ Q(x,y,0) = \sum_{l=0}^{n} Q_l(x,y),$$

where P_l is a homogeneous polynomial of degree l. Then

$$\mathcal{F}(r_0) = \sum_{k=0}^{n} r_0^{k-2} I_k,$$

where

$$\begin{aligned} I_k \;&=\; \int_0^{2\pi} e^{(3-k)\sin^2\theta}\left[(\cos\theta + \cos 3\theta)P_k(\cos\theta,\sin\theta)\right. \\ &\qquad\qquad \left. +(3\sin\theta + \sin 3\theta)Q_k(\cos\theta,\sin\theta)\right] d\theta, \end{aligned}$$

is a function of the coefficients of P_k and Q_k.

By symmetry the integral $I_k \equiv 0$ if k is even. So, we have

$$\mathcal{F}(r_0) = \frac{1}{r_0}\sum_{v=0}^{[(n-1)/2]} r_0^{2v} I_{2v+1}. \tag{3.3}$$

This implies that $\mathcal{F}(r_0)$ can have at most $[(n-1)/2]$ positive real roots. Consequently, using Theorem 5.1 we can get at most $[(n-1)/2]$ limit cycles of system (1.2) bifurcating from the periodic orbits of system (1.2) with $\epsilon = 0$ in the plane $z = 0$.

We shall prove that the function $\mathcal{F}(r_0)$ can have $m \in \{1, 2, \ldots, [(n-1)/2]\}$ positive real zeros for suitable choice of the coefficients of P and Q with the maximum degree n. For example set

$$P(x, y, z) = 0, \qquad Q(x, y, z) = \sum_{v=0}^{[(n-1)/2]} b_{2v+1} y^{2v+1}.$$

Now the function $\mathcal{F}(r_0)$ in (3.3) has the I_{2v+1} as follows

$$I_{2v+1} = b_{2v+1} \int_0^{2\pi} e^{2(1-v)\sin^2\theta} (3\sin\theta + \sin 3\theta) \sin^{2v+1}\theta d\theta = b_{2v+1} K_{2v+1}.$$

With the help of Mathematica we get that

$$K_{2v+1} = J_{2v+1} / \left(2^{2v}\Gamma(2+v)\Gamma(3+v)\pi^{-1}\right),$$

where

$$\begin{aligned} J_{2v+1} &= 3(2+v)\Gamma(3+2v) M\left[\frac{3}{2}+v, 2+v, 2-2v\right] \\ &\quad -\Gamma(4+2v) M\left[\frac{5}{2}+v, 3+v, 2-2v\right], \end{aligned}$$

and Γ is the Gamma function, and M is the Kummer confluent hypergeometric function defined by

$$M[a, b, z] = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a)\Gamma(b+k)} \frac{z^k}{k!}.$$

Using the properties of the Gamma function and the integral representation of the Kummer function, i.e.,

$$M[a, b, z] = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{z-1} (1-t)^{b-a-1} dt,$$

we can prove that

$$J_{2v+1} = \frac{\Gamma(3+2v)\Gamma(2+v)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}+v)} (2+v) \int_0^1 e^{(2-2v)t} t^{1-2v} (1-t)^{-\frac{1}{2}} dt.$$

This means that I_{2v+1} is always the product of b_{2v+1} with a positive number depending on v. Hence in the polynomial (3.3) we always can choose the coefficients b_{2v+1} conveniently in order that the polynomial can have $1, 2, \ldots, [(n-1)/2]$ positive roots (by the Descartes rule). This completes the proof of Theorem 1.2.

4. **PROOF OF THEOREM 1.3**

System (1.3) under the transformation $(x, y, z) \to (r\cos\theta, r\sin\theta, z)$ becomes

$$\begin{aligned} \dot{r} &= \varepsilon\big(\cos\theta P + \sin\theta Q\big), \\ \dot{\theta} &= r^2 + \varepsilon\frac{1}{r}\big(\cos\theta Q - \sin\theta P\big), \\ \dot{z} &= zr^2 + \varepsilon R, \end{aligned}$$

where $P = P(r\cos\theta, r\sin\theta, z)$, $Q = Q(r\cos\theta, r\sin\theta, z)$ and $R = R(r\cos\theta, r\sin\theta, z)$. This system is equivalent to

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon\frac{1}{r^2}(\cos\theta P + \sin\theta Q) + O(\varepsilon^2),\\ \frac{dz}{d\theta} &= z + \varepsilon\frac{1}{r^3}\big(rR - z(\cos\theta Q - \sin\theta P)\big) + O(\varepsilon^2).\end{aligned} \tag{4.1}$$

System $(4.1)|_{\varepsilon=0}$ restricted in $z = 0$ is a global center around the origin with periodic orbits

$$r = r_0, \quad z = 0, \tag{4.2}$$

satisfying $r(0, r_0) = r_0$ and all of them with period 2π. The fundamental solution matrix $M(\theta)$ of the variational equation of the unperturbed system $(4.1)_{\varepsilon=0}$ with respect to the periodic orbits (4.2) satisfying that $M(0)$ is the identity matrix is

$$M(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\theta} \end{pmatrix}.$$

By the fundamental solution matrix and Theorem 5.1, to prove Theorem 1.3 it is sufficient to estimate the zeros of the function

$$\mathcal{F}(r_0) = \frac{1}{r_0^2}\int_0^{2\pi} \cos\theta P(r_0\cos\theta, r_0\sin\theta, 0) + \sin\theta Q(r_0\cos\theta, r_0\sin\theta, 0)d\theta.$$

Writing

$$P(x, y, 0) = \sum_{l=0}^{n} P_l(x, y), \ Q(x, y, 0) = \sum_{l=0}^{n} Q_l(x, y),$$

where P_l is a homogeneous polynomial of degree l. Then

$$\mathcal{F}(r_0) = \frac{1}{r_0}\sum_{v=0}^{[(n-1)/2]} r_0^{2v} I_{2v+1} \tag{4.3}$$

where

$$I_k = \int_0^{2\pi} \cos\theta P_k(\cos\theta, \sin\theta) + \sin\theta Q_k(\cos\theta, \sin\theta)d\theta,$$

and we have used the fact that the integral $I_k \equiv 0$ if k is even. Hence $\mathcal{F}(r_0)$ has at most $[(n-1)/2]$ positive roots. Consequently, system (1.3) can have at most $[(n-1)/2]$ limit cycles bifurcating from the periodic orbits of the $z = 0$ plane when $\epsilon = 0$.

By choosing

$$P(x, y, z) = \sum_{v=0}^{[(n-1)/2]} a_{2v+1}x^{2v+1}, \qquad Q(x, y, z) \equiv 0,$$

we get that

$$I_{2v+1} = a_{2v+1}\int_0^{2\pi} (\cos\theta)^{2v+2} d\theta.$$

This implies that for convenient choice of a_{2v+1} system (1.3) with the given P and Q can have $m \in \{1, 2, \ldots, [(n-1)/2]\}$ limit cycles. This proves the theorem.

5. APPENDIX

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\mathbf{x}'(t) = F_0(\mathbf{x}, t) + \varepsilon F_1(\mathbf{x}, t) + \varepsilon^2 F_2(\mathbf{x}, t, \varepsilon), \tag{5.1}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \Omega \times \mathbb{R} \to \mathbb{R}^n$ and $F_2 : \Omega \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}'(t) = F_0(\mathbf{x}, t), \tag{5.2}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [11], and of Verhulst [12].

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (5.2) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(\mathbf{x}(t, \mathbf{z}), t)\mathbf{y}. \tag{5.3}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (5.3), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 5.1. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha))\,,\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (5.2) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (5.3) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the right up corner the $k \times (n-k)$ zero matrix, and in the right down corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(\mathbf{x}(t, \mathbf{z}_\alpha), t)dt\right). \tag{5.4}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t, \varepsilon)$ of system (5.1) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 5.1 goes back to Malkin [9] and Roseau [10], for a shorter proof see [3].

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