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TRIPLE POSITIVE SOLUTIONS OF THREE-POINT BOUNDARY VALUE PROBLEM FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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ABSTRACT. In this paper we consider the existence of triple positive solutions for second-order three-point boundary value problem with impulse effects on the half-line. Main results are besed on fixed point theorem on cone. In particular, the nonlinear term is involved with the first-order derivative.

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1. INTRODUCTION

This paper is concerned with the existence of positive solutions to three-point impulsive boundary value problem (IBVP for short) on the half-line

$$\begin{cases} (\Phi_p(\rho(t)x'(t)))' + q(t)f(t,x(t),x'(t)) = 0, \quad t \neq t_i, t \in J, \\ \Delta x(t_i) = I_i(x(t_i)), \quad -\Delta \Phi_p(\rho(t_i)x'(t_i)) = J_i(x(t_i)),\ i = 1,2,\ldots,m, \\ x'(0) = ax(\xi), \quad \lim\limits_{t\to+\infty} \rho(t)x'(t) = 0, \end{cases} \tag{1.1}$$

here $J = [0,+\infty)$, $\Phi_p x := |x|^{p-2}x$, $p > 1$, $0 = t_0 < t_1 < \cdots < t_m < \infty$, $a > 0$, $0 \le \xi < \infty$, $a\xi < 1$, ρ, I_i, J_i, q, f satisfy the following assumptions

(H1) $\rho \in C[0,+\infty) \cap C^1(0,+\infty)$, $\rho(t) > 0$ is increasing on $[0,+\infty)$, $\int_0^\infty \frac{1}{\rho(t)}dt < \infty$;

(H2) $I_i, J_i \in C(J,J)$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, where $x(t_i^+)$ (respectively $x(t_i^-)$) denote the right limit (respectively left limit) of $x(t)$ at $t = t_i$, $\Delta\Phi_p(\rho(t_i)x'(t_i)) = \Phi_p(\rho(t_i^+)x'(t_i^+)) - \Phi_p(\rho(t_i^-)x'(t_i^-))$, where $x'(t_i^+)$ (respectively $x'(t_i^-)$) denote the right limit (respectively left limit) of $x'(t)$ at $t = t_i$;

(H3) $q \in L^1(J,J)$, $f : J \times J \times J \to J$ is an L^1-Carathédory function, that is,

(i) $t \to f(t,x,y)$ is measurable for any $(x,y) \in J \times J$,

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(ii) $(x,y)\to f(t,x,y)$ is continuous for a.e. $t\in J$,

(iii) for each $r_1,r_2>0$, there exists l_{r_1,r_2} such that $q\cdot l_{r_1,r_2}\in L^1(J)$ and

$|f(t,(1+t)x,y)|\le l_{r_1,r_2}(t)$ for $|x|\le r_1, |y|\le r_2$, a.e. $t\in J$.

In recent years, a great deal of work has been done in the study of the boundary value problems with impulses, by which a number of physical, biological, medical phenomena are described, please refer to [6], [7], [12], [13], [14], [15], [16]. On the other hand, boundary value problems on the half-line occur naturally in the study of radically symmetric solutions of nonlinear elliptic equations, see [5], [11], and various physical phenomena [3], [10], and there are many results, see [1], [2], [8], [9], [17], [19], [20].

As far as we know, there are few papers to study the impulsive boundary value problems on the half-line. In [18], by using Leray-Schauder theorem and fixed point index theory, Yan established the existence of positive solutions of impulsive boundary value problem on the half-line

$$\begin{cases} \frac{1}{p(t)}(p(t)x'(t))'+f(t,x_k)=0, \quad t\neq t_k,\\ \Delta x|_{t=t_k}=I_k(x_{t_k}), \quad k=1,2,\dots,m,\\ \lambda x(0)-\beta\lim\limits_{t\to 0}p(t)x'(t)=a,\\ \gamma x(\infty)+\delta\lim\limits_{t\to\infty}p(t)x'(t)=b,\\ x(t) \text{ is bounded on } [0,+\infty),\end{cases}$$

where $\Phi\in BM_h((-\infty,0],R)$, $x_t(s)=\begin{cases} x(t+s), & t\ge t+s\ge 0,\\ \Phi(t+s), & -\infty<t+s<0,\end{cases}$ and $p\in C([0,+\infty),R)\cap C^1(0,+\infty)$, $p(t)>0$, $\lambda,\beta,\gamma,\delta\ge 0$ with $\beta\gamma+\lambda\delta+\lambda\gamma>0, a,b\ge 0$. But, there are no papers to study multi-point impulsive boundary value problems on the half-line. This paper is to fill this gap. We first transform impulsive boundary value problem into the integral equation. By applying fixed point theorem [4], we get the existence of at least three positive solutions. To apply fixed point theorem [4], it is very important to accomplish three suitable functionals α,β,ψ satisfying the assumptions of fixed point theorem [4] (see Lemma 3.1, Lemma 3.2).

This paper is organized as follows: In Section 2, we present related lemmas. First we state the fixed point theorem in [4] as basic tool. Then we transform the solution of IBVP (1.1) into the fixed point of some operator and verify the completely continuity of the operator. In Section 3, we obtain the main results by defining suitable functionals and applying the fixed point theorem. Besides, an example is presented to illustrate our main result.

2. RELATED LEMMAS

In order to establish the existence of at least three positive solutions for IBVP (1.1), we introduce some notations.

Definition 2.1. The map ψ is said to be a nonnegative continuous concave functional on cone P provided that $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map α is a nonnegative continuous convex functional on P provided that: $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let $r > a > 0, L > 0$ be constants, ψ is a nonnegative continuous concave functional and α, β nonnegative continuous convex functionals on the cone P. Define convex sets

$$P(\alpha, r; \beta, L) = \{y \in P | \alpha(y) < r, \beta(y) < L\},$$
$$\overline{P}(\alpha, r; \beta, L) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L\},$$
$$P(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) < r, \beta(y) < L, \psi(y) > a\},$$
$$\overline{P}(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L, \psi(y) \geq a\}.$$

The following assumptions about the nonnegative continuous convex functionals α, β will be used:
(A1) there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$ for all $x \in P$;
(A2) $P(\alpha, r; \beta, L) \neq \emptyset$ for all $r > 0, L > 0$.

Lemma 2.1 (Bai and Ge [4]). *Let E be a Banach space, $P \subset E$ a cone and $r_2 \geq d > b > r_1 > 0, L_2 \geq L_1 > 0$. Assume that α, β are nonnegative continuous convex functionals satisfying* (A1) *and* (A2), *ψ is a nonnegative continuous concave functional on P such that $\psi(y) \leq \alpha(y)$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2)$, and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$ is a completely continuous operator. Suppose*
(B1) $\{y \in P(\alpha, d; \beta, L_2; \psi, b) | \psi(y) > b\} \neq \emptyset, \psi(Ty) > b$ *for* $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$;
(B2) $\alpha(Ty) < r_1, \beta(Ty) < L_1$ *for all* $y \in \overline{P}(\alpha, r_1; \beta, L_1)$;
(B3) $\psi(Ty) > b$ *for all* $y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ *with* $\alpha(Ty) > d$.
Then T has at least three fixed points y_1, y_2 and y_3 in $\overline{P}(\alpha, r_2; \beta, L_2)$ with

$$y_1 \in P(\alpha, r_1; \beta, L_1), \quad y_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \backslash (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

Let $J' = J \setminus \{t_1, t_2, \dots, t_m\}$,

$$PC(J,R) = \{x : J \to R : x|_{(t_i,t_{i+1})} \in C(t_i, t_{i+1}), x(t_i^-) = x(t_i), \quad \exists\, x(t_i^+),\ i = 1,2,\dots,m\},$$

$$PC^1(J,R) = \{x \in PC(J,R) : x'|_{(t_i,t_{i+1})} \in C(t_i, t_{i+1}), x'(t_i^-) = x'(t_i), \quad \exists\, x'(t_i^+),\ i = 1,2,\dots,m\}.$$

Definition 2.2. A function $x(t) \in PC^1(J,R)$, $(\Phi_p(\rho(t)x'(t)))' \in L^1(J',R)$ is said to be a positive solution of impulsive boundary value problem (1.1), if $x(t) \geq 0$, and x satisfies differential equation

$$(\Phi_p(\rho(t)x'(t)))' + q(t)f(t,x(t),x'(t)) = 0, \quad t \in J'$$

and impulsive condition

$$\Delta x(t_i) = I_i(x(t_i)), \quad -\Delta\Phi_p(\rho(t_i)x'(t_i)) = J_i(x(t_i)),\ i = 1,2,\dots,m,$$

and the three-point boundary conditions $x'(0) = ax(\xi)$, $\lim_{t\to\infty} \rho(t)x'(t) = 0$.

Lemma 2.2. *Assume that $g \in C(J)$ with $\int_0^\infty g(s)ds < \infty$, $a_i, b_i \in C(J,R)$. Then $x \in PC^1(J,R)$, $(\Phi_p(\rho(t)x'(t)))' \in C(J',R)$ is a solution of IBVP*

$$\begin{cases} (\Phi_p(\rho(t)x'(t)))' + g(t) = 0, \quad t \neq t_i, t \in J, \\ \Delta x(t_i) = a_i(t_i), \quad -\Delta\Phi_p(\rho(t_i)x'(t_i)) = b_i(t_i),\ i = 1,2,\dots,m, \\ x'(0) = ax(\xi), \quad \lim_{t\to\infty} \rho(t)x'(t) = 0, \end{cases} \tag{2.1}$$

if and only if $x \in PC(J,R)$ is a solution of the following integral equation

$$\begin{aligned} x(t) = {}& \frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty g(s)ds + \sum_{i=1}^m b_i(t_i)\right] + \sum_{\xi \leq t_i < t} a_i(t_i) \\ & + \int_\xi^t \frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^\infty g(\theta)d\theta + \sum_{t_i \geq s} b_i(t_i)\right] ds, \quad t \in J. \end{aligned} \tag{2.2}$$

Proof. If $x \in PC(J,R)$, $(\Phi_p(\rho(t)x'(t)))' \in C(J)$ is a solution of (2.1), integrating equation in (2.1) from t to ∞, one has

$$-\Phi_p(\rho(t)x'(t)) + \int_t^\infty g(s)ds - \sum_{t_i \geq t} \Delta\Phi_p(\rho(t_i)x'(t_i)) = 0.$$

By the second impulsive condition,

$$-\Phi_p(\rho(t)x'(t)) + \int_t^\infty g(s)ds + \sum_{t_i \geq t} b_i(t_i) = 0,$$

i.e.

$$x'(t) = \frac{1}{\rho(t)}\Phi_p^{-1}\left[\int_t^\infty g(s)ds + \sum_{t_i \geq t} b_i(t_i)\right]. \tag{2.3}$$

Again integrating (2.3) from ξ to t, one has

$$x(t) - \sum_{\xi \le t_i < t} \Delta x(t_i) - x(\xi) = \int_\xi^t \frac{1}{\rho(s)} \Phi_p^{-1} \left[\int_s^\infty g(\theta) d\theta + \sum_{t_i \ge s} b_i(t_i) \right] ds.$$

By the first impulsive condition,

$$x(t) = x(\xi) + \sum_{\xi \le t_i < t} a_i(t_i) + \int_\xi^t \frac{1}{\rho(s)} \Phi_p^{-1} \left[\int_s^\infty g(\theta) d\theta + \sum_{t_i \ge s} b_i(t_i) \right] ds. \tag{2.4}$$

The first boundary condition implies that

$$x(\xi) = \frac{1}{a} x'(0) = \frac{1}{a\rho(0)} \Phi_p^{-1} \left[\int_0^\infty g(s) ds + \sum_{i=1}^m b_i(t_i) \right]. \tag{2.5}$$

Substituting (2.5) into (2.4), x satisfies (2.2).

If $x \in PC(J, R)$ is a solution of integral equation (2.2), then it is easy to see from condition $\int_0^\infty g(s) ds < \infty$ that $x \in PC^1(J, R)$, $(\Phi_p(\rho(t) x'(t)))' \in C(J', R)$ is a solution of problem (2.1). □

Now we define the space $X = \left\{ x \in PC^1(J, R) : \lim_{t \to +\infty} \rho(t) x'(t) = 0, \lim_{t \to +\infty} \frac{|x(t)|}{1+t} < \infty \right\}$ with the norm $\|x\| = \max \left\{ \sup_{t \in [0, +\infty)} \frac{|x(t)|}{1+t}, \sup_{t \in [0, +\infty)} |x'(t)| \right\}$. Evidently, X is a Banach space.

Choose $P \subseteq X$ be a cone defined by

$$P = \{x \in X : x(t) \ge 0, x'(t) \ge 0, t \in J, x'(t) \text{ is nonincreasing on } J'\}.$$

Define the operator $T : P \to X$ by

$$\begin{aligned} (Tx)(t) &= \tfrac{1}{a\rho(0)} \Phi_p^{-1} \left[\int_0^\infty q(s) f(s, x(s), x'(s)) ds + \sum_{i=1}^m J_i(x(t_i)) \right] + \sum_{\xi \le t_i < t} I_i(x(t_i)) \\ &\quad + \int_\xi^t \tfrac{1}{\rho(s)} \Phi_p^{-1} \left[\int_s^\infty q(\theta) f(\theta, x(\theta), x'(\theta)) d\theta + \sum_{t_i \ge s} J_i(x(t_i)) \right] ds, \quad t \in J. \end{aligned}$$

Lemma 2.2 means that $x(t) \in PC^1(J, R)$, $(\Phi_p(\rho(t) x'(t)))' \in L^1(J', R)$ is a solution of IBVP (1.1) if and only if x is a fixed point of the operator T.

Lemma 2.3. *Suppose that* (H1)–(H3) *hold. Then $T : P \to P$ is completely continuous.*

Proof. (1) First we show that the operator $T : P \to P$. By the expression of Tx, it is clear that $(Tx)'(t) \ge 0$ is nonincreasing on J, and

$$\begin{aligned} (Tx)(t) > \frac{1}{a\rho(0)} \Phi_p^{-1} &\left[\int_0^\infty q(s) f(s, x(s), x'(s)) ds + \sum_{i=1}^m J_i(x(t_i)) \right] \\ - \int_0^\xi \frac{1}{\rho(s)} \Phi_p^{-1} &\left[\int_s^\infty q(\theta) f(\theta, x(\theta), x'(\theta)) d\theta + \sum_{t_i \ge s} J_i(x(t_i)) \right] ds \end{aligned}$$

$$> \frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]$$
$$-\frac{\xi}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{i=1}^m J_i(x(t_i))\right]$$
$$=\left(\frac{1}{a}-\xi\right)\frac{1}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{i=1}^m J_i(x(t_i))\right]>0,\ t\in J.$$

(2) We will show that $T:P\to P$ is continuous. For this, let $\{x_n\}\subseteq P, x\in P$ and $x_n\to x$ in X as $n\to\infty$. Then there exists an $M>0$ such that $\|x_n\|\le M$. By expression of Tx, we have

$$\begin{aligned}
&|Tx_n(t)-Tx(t)|\\
&\le \frac{1}{a\rho(0)}\left|\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x_n(s),x_n'(s))ds+\sum_{i=1}^m J_i(x_n(t_i))\right]\right.\\
&\left.-\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]\right|\\
&+\sum_{\xi\le t_i<t}|I_i(x_n(t_i))-I_i(x(t_i))|\\
&+\int_\xi^t\frac{1}{\rho(s)}\left|\Phi_p^{-1}\left[\int_s^\infty q(\theta)f(\theta,x_n(\theta),x_n'(\theta))d\theta+\sum_{t_i\ge s}J_i(x_n(t_i))\right]\right.\\
&\left.-\Phi_p^{-1}\left[\int_s^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s}J_i(x(t_i))\right]\right|ds.
\end{aligned}\tag{2.6}$$

Since f is an L^1-Carathédory function and (H3) holds, we have

$$\int_0^\infty q(s)|f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))|ds\le 2\int_0^\infty q(s)|l_{M,M}(s)|ds<\infty\tag{2.7}$$

and

$$\lim_{n\to\infty}f(t,x_n(t),x_n'(t))=f(t,x(t),x'(t)).\tag{2.8}$$

According to Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_0^\infty q(s)\left|f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))\right|ds=0.\tag{2.9}$$

Since $I_i, J_i\in C(J,J)$, we have

$$\lim_{n\to\infty}I_i(x_n(t_i))-I_i(x(t_i))=0,\quad \lim_{n\to\infty}J_i(x_n(t_i))-J_i(x(t_i))=0.\tag{2.10}$$

Using the continuity of Φ_p^{-1}, (2.6) (2.9) (2.10) mean that

$$\lim_{n\to\infty}\sup_{t\in J}\frac{|Tx_n(t)-Tx(t)|}{1+t}\le\lim_{n\to\infty}\sup_{t\in J}|Tx_n(t)-Tx(t)|=0.$$

Similarly, one has

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{t\in J}|(Tx_n)'(t)-(Tx)'(t)| \\
&\quad=\lim_{n\to\infty}\sup_{t\in J}\frac{1}{\rho(t)}\left|\Phi_p^{-1}\left[\int_t^\infty q(s)f(s,x_n(s),x_n'(s))ds+\sum_{t_i\ge t}J_i(x_n(t_i))\right]\right. \\
&\quad\quad\left.-\Phi_p^{-1}\left[\int_t^\infty q(s)f(s,x(s),x'(s))ds+\sum_{t_i\ge t}J_i(x(t_i))\right]\right| \\
&\quad=0.
\end{aligned}$$

So $T:P\to P$ is continuous.

(3) We will show that $T:P\to P$ is relatively compact.

Given a bounded set $D\subseteq P$. Choose $M>0$ such that $\|x\|\le M$ for all $x\in D$. Then $0\le\frac{x(t)}{1+t}\le M, 0\le x'(t)\le M$ and

$$\begin{aligned}
&\sup_{t\in J}\frac{|Tx(t)|}{1+t}\le\frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]+\sum_{\xi\le t_i\le t}I_i(x(t_i)) \\
&+\sup_{t\in J}\frac{1}{1+t}\int_\xi^t\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s}J_i(x(t_i))\right]ds \\
&\le\frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)l_{M,M}(s)ds+\sum_{i=1}^m\max_{x\in[0,M]}J_i((1+t_i)x)\right]+\sum_{i=1}^m\max_{x\in[0,M]}I_i((1+t_i)x) \\
&+\sup_{t\in J}\frac{|t-\xi|}{1+t}\frac{1}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(\theta)l_{M,M}(\theta)d\theta+\sum_{i=1}^m\max_{x\in[0,M]}J_i((1+t_i)x)\right] \\
&<\infty,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in J}|(Tx)'(t)| &= \sup_{t\in J}\frac{1}{\rho(t)}\Phi_p^{-1}\left[\int_t^\infty q(s)f(s,x(s),x'(s))ds+\sum_{t_i\ge t}J_i(x(t_i))\right] \\
&\le \frac{1}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)l_{M,M}(s)ds+\sum_{i=1}^m\max_{x\in[0,M]}J_i((1+t_i)x)\right] \\
&< \infty.
\end{aligned}$$

So $\{TD(t)\}$ and $\{(TD)'(t)\}$ are uniformly bounded. At the same time, the fact $\{(TD)'(t)\}$ is uniformly bounded implies that $\{TD(t)\}$ is locally equicontinuous on any interval of $[0,\infty)$.

Now we show that $\{(TD)'(t)\}$ is locally equicontinuous on any interval of $[0,\infty)$. For any $\overline{t}>0$, $s_1,s_2\in[0,\overline{t}], s_1<s_2$ and $x\in D$, then

$$\begin{aligned}
&|(Tx)'(s_1)-(Tx)'(s_2)|\\
=\ &\left|\frac{1}{\rho(s_1)}\Phi_p^{-1}\left(\int_{s_1}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_1}J_i(x(t_i))\right)\right.\\
&\left.-\frac{1}{\rho(s_2)}\Phi_p^{-1}\left(\int_{s_2}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_2}J_i(x(t_i))\right)\right|\\
\leq\ &\left|\frac{1}{\rho(s_1)}-\frac{1}{\rho(s_2)}\right|\Phi_p^{-1}\left(\int_{s_1}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_1}J_i(x(t_i))\right)\\
&+\frac{1}{\rho(s_2)}\left|\Phi_p^{-1}\left(\int_{s_1}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_1}J_i(x(t_i))\right)\right.\\
&\left.-\Phi_p^{-1}\left(\int_{s_2}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_2}J_i(x(t_i))\right)\right|.
\end{aligned}$$

Since $\frac{1}{\rho(t)}\in C([0,\infty))$, for any $\varepsilon>0$, there exists $\delta_1>0$ such that

$$\left|\frac{1}{\rho(s_1)}-\frac{1}{\rho(s_2)}\right|<\frac{\varepsilon}{2\Phi_p^{-1}\left(\int_0^{\infty}q(s)l_{M,M}(s)ds+\sum_{i=1}^{m}J_i((1+t_i)x)\right)} \tag{2.11}$$

for $|s_1-s_2|<\delta_1,\ s_1,s_2\in[0,\overline{t}]$.

Since Φ_p^{-1} is continuous, for $\varepsilon>0$, there exists $\delta_2>0$, such that

$$\begin{aligned}
&\left|\Phi_p^{-1}\left(\int_{s_1}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_1}J_i(x(t_i))\right)\right.\\
&\left.-\Phi_p^{-1}\left(\int_{s_2}^{\infty}q(s)f(s,x(s),x'(s))ds+\sum_{t_i\geq s_1}J_i(x(t_i))\right)\right|<\frac{\rho(s_2)\varepsilon}{2}
\end{aligned}$$

for $\left|\int_{s_1}^{s_2}q(s)f(s,x(s),x'(s))ds+\sum\limits_{s_1\leq t_i\leq s_2}J_i(x(t_i))\right|<\delta_2$.

Since f is an L^1-Carathédory function, we have for $\delta_2>0$, there exists $\delta_3>0$ such that

$$\left|\int_{s_1}^{s_2}q(s)f(s,x(s),x'(s))ds+\sum_{s_1\leq t_i\leq s_2}J_i(x(t_i))\right|<\delta_2$$

for $|s_1-s_2|<\delta_3$.

So

$$\begin{aligned}(2.12)\qquad &\Bigg|\Phi_p^{-1}\left(\int_{s_1}^{\infty} q(s)f(s,x(s),x'(s))ds+\sum_{t_i\ge s_1} J_i(x(t_i))\right)\\ &-\Phi_p^{-1}\left(\int_{s_2}^{\infty} q(s)f(s,x(s),x'(s))ds+\sum_{t_i\ge s_2} J_i(x(t_i))\right)\Bigg|<\frac{\rho(s_2)\varepsilon}{2}\end{aligned}$$

for $|s_1-s_2|<\delta_3$.

Let $\delta=\min\{\delta_1,\delta_3\}$, by (2.11) (2.12)

$$|(Tx)'(s_1)-(Tx)'(s_2)|\le\frac{\varepsilon}{2}+\frac{1}{\rho(s_2)}\frac{\rho(s_2)\varepsilon}{2}=\varepsilon$$

for $|s_1-s_2|<\delta$, $s_1,s_2\in[0,\overline{t}]$. Since $\overline{t}$ is arbitrary, $\{(TD)'(t)\}$ is locally equicontinuous on any interval of $[0,\infty)$.

(4) $T:P\to P$ is equiconvergent at ∞.

Now for $x\in D$, one has

$$\begin{aligned}&\lim_{t\to\infty}\frac{|Tx(t)-Tx(\infty)|}{1+t}\\ &=\lim_{t\to\infty}\frac{1}{1+t}\Bigg|\frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^{\infty} q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^{m} J_i(x(t_i))\right]\\ &+\sum_{\xi\le t_i<t} I_i(x(t_i))+\int_{\xi}^{t}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s} J_i(x(t_i))\right]ds\\ &-\frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^{\infty} q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^{m} J_i(x(t_i))\right]-\sum_{\xi\le t_i\le t_m} I_i(x(t_i))\\ &-\int_{\xi}^{\infty}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s} J_i(x(t_i))\right]ds\Bigg|\\ &=\lim_{t\to\infty}\frac{1}{1+t}\Bigg\{\sum_{t\le t_i\le t_m} I_i(x(t_i))\\ &\qquad-\int_t^{\infty}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s} J_i(x(t_i))\right]ds\Bigg\}\\ &=\lim_{t\to\infty}-\frac{1}{1+t}\int_t^{\infty}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{t_i\ge s} J_i(x(t_i))\right]ds.\end{aligned}$$

If $\lim\limits_{t\to\infty}\int_t^{\infty}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum\limits_{t_i\ge s} J_i(x(t_i))\right]ds<\infty$, then $\lim\limits_{t\to\infty}\frac{|Tx(t)-Tx(\infty)|}{1+t}=0$;

If $\lim\limits_{t\to\infty}\int_t^{\infty}\frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^{\infty} q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum\limits_{t_i\ge s} J_i(x(t_i))\right]ds=\infty$, then by L'Hospital rule and $\int_0^\infty \frac{1}{\rho(t)}dt < \infty$

$$\lim_{t\to\infty} -\frac{1}{1+t}\int_t^\infty \frac{1}{\rho(s)}\Phi_p^{-1}\left[\int_s^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta + \sum_{t_i\geq s} J_i(x(t_i))\right] ds$$
$$= \lim_{t\to\infty} \frac{1}{\rho(t)}\Phi_p^{-1}\left(\int_t^\infty q(s)f(s,x(s),x'(s))ds + \sum_{t_i\geq t} J_i(x(t_i))\right) = 0.$$

So, $\lim\limits_{t\to\infty} \frac{|Tx(t)-Tx(\infty)|}{1+t} = 0$. Similarly,

$$\lim_{t\to\infty} |(Tx)'(t) - (Tx)'(\infty)|$$
$$= \lim_{t\to\infty} \frac{1}{\rho(t)}\Phi_p^{-1}\left[\int_t^\infty q(s)f(s,x(s),x'(s))ds + \sum_{t_i\geq t} J_i(x(t_i))\right] = 0.$$

Therefore, $T : P \to P$ is equiconvergent at ∞.

From (1)–(4), $T : P \to P$ is completely continuous. □

3. THE EXISTENCE OF TRIPLE POSITIVE SOLUTIONS

In order to apply Lemma 2.1, we define three functionals as follows

$$\alpha(x) = \sup_{t\in[0,\infty)} \frac{|x(t)|}{1+t}, \quad \beta(x) = \sup_{t\in[0,\infty)} |x'(t)|, \quad \psi(x) = \frac{1}{1+t_1} \inf_{t\in[t_1,t_2]} |x(t)|.$$

Then $\alpha, \beta : P \to [0,\infty)$ are nonnegative continuous convex functionals satisfying (A1), (A2); ψ is a nonnegative continuous concave functional.

Lemma 3.1. *For $x \in P$, $\psi(x) \leq \alpha(x)$.*

Proof. For $x \in P$, $\inf\limits_{t\in[t_1,t_2]} x(t) = x(t_1)$. Then we have

$$\psi(x) = \frac{x(t_1)}{1+t_1} \leq \sup_{t\in[0,\infty)} \frac{|x(t)|}{1+t} = \alpha(x).$$

□

Lemma 3.2. *For $x \in P$, $\psi(x) > \frac{t_1}{(1+t_1)^2}\alpha(x)$.*

Proof. First we claim that $\left\{\frac{|x(t)|}{1+t}\right\}$ has a maximum at the point $\sigma \in [0,\infty)$. In fact, since $\lim\limits_{t\to\infty} \rho(t)x'(t) = 0$ and $\rho(t)$ is increasing on $[0,\infty)$, we have $\lim\limits_{t\to\infty} x'(t) = 0$, which implies that $\lim\limits_{t\to\infty} |x(t)| < +\infty$ and $\left\{\frac{|x(t)|}{1+t}\right\}$ has a maximum at the point $\sigma \in [0,\infty)$, i.e.

$$\sup_{t\in[0,\infty)} \frac{|x(t)|}{1+t} = \frac{x(\sigma)}{1+\sigma}.$$

Following we will show that $\psi(x) > \frac{t_1}{(1+t_1)^2}\alpha(x)$. From $x \in P$, it follows that $\psi(x) = \frac{1}{1+t_1}\inf_{t\in[t_1,t_2]} x(t) = \frac{x(t_1)}{1+t_1}$. Since $x(t)$ is concave on $[t_1, t_2]$, let $\lambda = \frac{t_1}{t_1+\sigma}$, we have

$$\begin{aligned} x(t_1) &= x\left((1-\lambda)\frac{(t_1)^2}{\sigma} + \lambda\sigma\right) \\ &\geq \lambda x(\sigma) = \frac{t_1}{t_1+\sigma}x(\sigma) = \frac{t_1(1+\sigma)}{t_1+\sigma}\cdot\frac{x(\sigma)}{1+\sigma} \\ &> \frac{t_1}{1+t_1}\cdot\frac{x(\sigma)}{1+\sigma} = \frac{t_1}{1+t_1}\alpha(x). \end{aligned}$$

So $\psi(x) = \frac{x(t_1)}{1+t_1} > \frac{t_1}{(1+t_1)^2}\alpha(x)$. □

For convenience, we denote
$P(r) = \sum_{i=1}^{m}\max_{x\in[0,r]} I_i((1+t_i)x)$, $\quad Q(r) = \sum_{i=1}^{m}\max_{x\in[0,r]} J_i((1+t_i)x)$,
$M_i = \frac{1}{\int_0^\infty q(s)ds}\left[\Phi_p\left(\frac{\rho(0)(r_i-P(r_i))}{\frac{1}{a}+\max\{1,\xi\}}\right) - Q(r_i)\right]$, $\quad N_i = \frac{1}{\int_0^\infty q(s)ds}\left[\Phi_p(\rho(0)L_i) - Q(r_i)\right]$,
$i = 1, 2$,
$K = \frac{1}{\int_{t_1}^{t_2} q(s)ds}\Phi_p\left(\frac{b(1+t_i)\rho(0)}{\frac{1}{a}-\xi}\right)$.

Theorem 3.3. *Suppose that* (H1)–(H3) *hold. Assume there exist constants*

$$r_2 \geq \frac{b(1+t_1)^2}{t_1} > b > r_1 > 0, \quad L_2 \geq L_1 > 0$$

such that

$$K < \min\{M_2, N_2\}, \qquad P(r_i) < r_i,$$
$$Q(r_i) < \min\left\{\Phi_p(\rho(0)L_i), \Phi_p\left(\frac{\rho(0)(r_i - P(r_i))}{\frac{1}{a} + \max\{1,\xi\}}\right)\right\}, \quad i = 1, 2.$$

Moreover, assume that:
(C1) $f(s,(1+s)x,y) < \min\{M_1, N_1\}$ *for* $(s,x,y) \in [0,\infty)\times[0,r_1]\times[0,L_1]$*;*
(C2) $f(s,(1+s)x,y) > K$ *for* $(s,x,y) \in [t_1,t_2]\times\left[b, \frac{b(1+t_1)^2}{t_1}\right]\times[0,L_2]$*;*
(C3) $f(s,(1+s)x,y) < \min\{M_2, N_2\}$ *for* $(s,x,y) \in [0,\infty)\times[0,r_2]\times \in [0,L_2]$.
Then problem (1.1) *has at least three positive solutions* x_1, x_2, x_3 *with*

$$0 \leq \frac{x_i(t)}{1+t} \leq r_i, \qquad 0 \leq x_i'(t) \leq L_i, \quad i = 1, 2,$$
$$r_1 \leq \frac{x_3(t)}{1+t} \leq r_2, \qquad 0 \leq x_3'(t) \leq L_2, t \in [0,\infty),$$
$$x_2(t) > (1+t_1)b, \qquad x_3(t) \leq (1+t_1)b, t \in [t_1, t_2].$$

Proof. We will apply Lemma 2.1 to verify the existence of fixed points of the operator T. Lemma 2.3 has shown $T : P \to P$ is completely continuous. Lemma 3.1 has shown $\psi(x) \leq \alpha(x)$ for $x \in P$. Now we will verify that all the conditions of Lemma 2.1 are satisfied. First we show $T:\overline{P}(\alpha,r_2;\beta,L_2)\to\overline{P}(\alpha,r_2;\beta,L_2)$. If $x\in\overline{P}(\alpha,r_2;\beta,L_2)$, then $0\le\frac{x(t)}{1+t}\le r_2, 0\le x'(t)\le L_2$. The assumption (C3) implies

$$\begin{aligned}
\alpha(Tx) &= \sup_{t\in[0,\infty)}\frac{|Tx(t)|}{1+t}\\
&\le \frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]+\sum_{i=1}^m I_i(x(t_i))\\
&\quad+\sup_{t\in[0,\infty)}\frac{|t-\xi|}{1+t}\frac{1}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{i=1}^m J_i(x(t_i))\right]\\
&\le \left(\frac{1}{a}+\sup_{t\in[0,\infty)}\frac{|t-\xi|}{1+t}\right)\frac{1}{\rho(0)}\\
&\quad\times\Phi_p^{-1}\left[\int_0^\infty q(\theta)f(\theta,x(\theta),x'(\theta))d\theta+\sum_{i=1}^m J_i(x(t_i))\right]+\sum_{i=1}^m I_i(x(t_i))\\
&\le \left(\frac{1}{a}+\max\{1,\xi\}\right)\frac{1}{\rho(0)}\\
&\quad\times\Phi_p^{-1}\left[\int_0^\infty q(s)ds\sup_{(s,x,y)\in[0,\infty)\times[0,r_2]\times[0,L_2]}f(s,(1+s)x,y)+Q(r_2)\right]+P(r_2)\\
&< r_2,
\end{aligned}$$

$$\begin{aligned}
\beta(Tx) &= \sup_{t\in[0,\infty)}|(Tx)'(t)|\\
&\le \sup_{t\in[0,\infty)}\frac{1}{\rho(t)}\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]\\
&\le \frac{1}{\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)ds\sup_{(s,x,y)\in[0,\infty)\times[0,r_2]\times[0,L_2]}f(s,(1+s)x,y)+Q(r_2)\right]\\
&\le L_2.
\end{aligned}$$

Hence $T:\overline{P}(\alpha,r_2;\beta,L_2)\to\overline{P}(\alpha,r_2;\beta,L_2)$. In the same way we can show $T:\overline{P}(\alpha,r_1;\beta,L_1)\to\overline{P}(\alpha,r_1;\beta,L_1)$, so the condition (B2) is satisfied.

To check the condition (B1) in Lemma 2.1, we choose $x_0(t)=\frac{b(1+t_1)^2}{t_1}$, $t\in J$. It is easy to see that $x_0(t)=\frac{b(1+t_1)^2}{t_1}\in\overline{P}(\alpha,\frac{b(1+t_1)^2}{t_1};\beta,L_2;\psi,b)$, $\psi(x_0)=\frac{b(1+t_1)^2}{t_1}>b$, and consequently, $\left\{x\in\overline{P}(\alpha,\frac{b(1+t_1)^2}{t_1};\beta,L_2;\psi,b):\psi(x)>b\right\}\ne\emptyset$.

For $x\in\overline{P}(\alpha,\frac{b(1+t_1)^2}{t_1};\beta,L_2;\psi,b)$, then $b\le\frac{x(t)}{1+t}\le\frac{b(1+t_1)^2}{t_1}, t\in[t_1,t_2], 0\le x'(t)\le L_2, t\in J$. Now we will show $\psi(Tx)>b$. By the condition (C2),

$$\begin{aligned}
\psi(Tx) &= \frac{1}{1+t_1}\inf_{t\in[t_1,t_2]}(Tx)(t)=\frac{1}{1+t_1}(Tx)(t_1)\\
&\ge \frac{1}{1+t_1}\left\{\frac{1}{a\rho(0)}\Phi_p^{-1}\left[\int_0^\infty q(s)f(s,x(s),x'(s))ds+\sum_{i=1}^m J_i(x(t_i))\right]\right.
\end{aligned}$$

$$- \int_0^{\xi} \frac{1}{\rho(s)} \Phi_p^{-1} \left[\int_s^{\infty} q(\theta) f(\theta, x(\theta), x'(\theta)) d\theta + \sum_{i=1}^{m} J_i(x(t_i)) \right] ds \Bigg\}$$

$$\geq \frac{1}{1+t_1} \left(\frac{1}{a} - \xi \right) \frac{1}{\rho(0)} \Phi_p^{-1} \left[\int_{t_1}^{t_2} q(s) ds \min_{(s,x,y) \in [t_1,t_2] \times \left[b, \frac{b(1+t_1)^2}{t_1}\right] \times [0,L_2]} f(s, (1+s)x, y) \right]$$

$$> b.$$

Finally, we verify that the condition (B3) in Lemma 2.1 holds. For $x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ with $\alpha(Tx) > \frac{b(1+t_1)^2}{t_1}$, then by the definition ψ and Lemma 3.2 we have

$$\psi(Tx) > \frac{t_1}{(1+t_1)^2} \alpha(Tx) > \frac{t_1}{(1+t_1)^2} \cdot \frac{b(1+t_1)^2}{t_1} = b.$$

Therefore, the operator T has three fixed points $x_i \in \overline{P}(\alpha, r_2; \beta, L_2), i = 1, 2, 3$, with

$$x_1 \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \backslash (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

Also (H3) implies

$$\int_0^{\infty} q(s) f(s, x_i(s), x_i'(s)) ds \leq \int_0^{\infty} q(s) l_{r_1,r_2}(s) ds < \infty.$$

So by Lemma 2.2, problem (1.1) has at least three positive solutions $x_i \in \overline{P}(\alpha, r_2; \beta, L_2)$, $i = 1, 2, 3$ with

$$0 \leq \frac{x_i(t)}{1+t} \leq r_i, \quad 0 \leq x_i'(t) \leq L_i, \ i = 1, 2, \quad r_1 \leq \frac{x_3(t)}{1+t} \leq r_2, \ 0 \leq x_3'(t) \leq L_2, t \in [0, \infty),$$

$$x_2(t) > (1+t_1)b, \quad x_3(t) \leq (1+t_1)b, t \in [t_1, t_2].$$

□

Example 3.4. Consider the following impulsive boundary value problem

$$\begin{cases} (\Phi_3((1+t)^2 x'(t)))' + e^{-t} f(t, x(t), x'(t)) = 0, \quad t \neq 1, t \neq 2, t \in [0, \infty), \\ \Delta x(t_i) = I_i(x(t_i)), \quad -\Delta \Phi_3\left((1+t_i)^2 x'(t_i)\right) = J_i(x(t_i)), \ i = 1, 2 \\ x'(0) = 2x(\frac{1}{4}), \quad \lim_{t \to \infty} (1+t)^2 x'(t) = 0. \end{cases} \tag{3.1}$$

Corresponding to (1.1), $p = 3, \rho(t) = (1+t)^2, q(t) = e^{-t}, t_1 = 1, t_2 = 2, a = 2, \xi = \frac{1}{4}$, $I_1(u) = \frac{u}{12}, I_2(u) = \frac{u}{18}, J_1(u) = \frac{1}{10}\left(\frac{u}{12}\right)^2, J_2(u) = \frac{1}{10}\left(\frac{u}{18}\right)^2$,

$$f(t,u,v) = \begin{cases} \frac{1}{22800}\left(\frac{u}{1+t}\right)(v + 218), & (t,u,v) \in [0,\infty) \times \left[0, \frac{1}{8}\right] \times [0, \infty), \\ \frac{1}{1+t}\left[\left(4 - \frac{1}{22800}\right) u - \frac{1}{2} + \frac{1}{22800 \times 4}\right](v + 218), & (t,u,v) \in [0,\infty) \times \left[\frac{1}{8}, \frac{1}{4}\right] \times [0,\infty), \\ \frac{1}{2}\left(\frac{1}{1+t}\right)(v + 218), & (t,u,v) \in [0,\infty) \times \left[\frac{1}{4}, \infty\right) \times [0,\infty). \end{cases}$$

Let $r_1 = \frac{1}{8}, r_2 = 100, b = \frac{1}{4}, L_1 = 10, L_2 = 20$, we obtain that

$$P(r_1) = \frac{1}{24} < r_1, \quad Q(r_1) = \frac{1}{64 \times 180}, \quad P(r_2) = \frac{100}{3} < r_2, \quad Q(r_2) = \frac{1000}{18},$$

$\min\left\{\Phi_p(\rho(0)L_1), \Phi_p\left(\frac{\rho(0)(r_1 - P(r_1))}{\frac{1}{a} + \max\{1,\xi\}}\right)\right\} = \frac{1}{18^2}$, $\min\left\{\Phi_p(\rho(0)L_2), \Phi_p\left(\frac{\rho(0)(r_2 - P(r_2))}{\frac{1}{a} + \max\{1,\xi\}}\right)\right\} = 400$.

So $Q(r_i) < \min\left\{\Phi_p(\rho(0)L_i), \Phi_p\left(\frac{\rho(0)(r_i - P(r_i))}{\frac{1}{a} + \max\{1,\xi\}}\right)\right\}, i = 1, 2.$

$\min\{M_1, N_1\} = \min\{\frac{1}{18^2} - \frac{1}{64\times180}, 100 - \frac{1}{64\times180}\} > \frac{1}{18\times36}$,

$\min\{M_2, N_2\} = \min\{\frac{160000}{81} - \frac{1000}{18}, 400 - \frac{1000}{18}\} > 340$,

$K = \frac{4e^2}{e-1} < 36$. So $K < \min\{M_2, N_2\}$.

It is easy to verify that all the assumptions in Theorem 3.3 are satisfied. So problem (3.1) has at least three positive solutions.

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