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ON THE OSCILLATION OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. This paper is focused on the following nonlinear neutral differential equation with positive and negative coefficients

$$[x(t) - R(t)f(x(t-r))]' + P(t)g(x(t-\tau)) - Q(t)g(x(t-\sigma)) = 0,$$

where $R(t), P(t), Q(t) \in C([t_0,\infty),\mathbb{R}^+)$, $r > 0$, $\tau \geq 0$, $\sigma \geq 0$. For this equation, oscillation criteria are established.

Key Words. Differential equation, neutral, nonlinear, oscillation

AMS (MOS) Subject Classification. 34K40, 34K99, 34C10

1. INTRODUCTION

In this paper, we investigated the oscillatory behavior of all solutions of the following neutral differential equation with positive and negative coefficients

$$(1)\qquad [x(t) - R(t)f(x(t-r))]' + P(t)g(x(t-\tau)) - Q(t)g(x(t-\sigma)) = 0,\ \ t \geq t_0$$

where $P, Q, R \in C([t_0,\infty),\mathbb{R}^+)$, $r > 0$, $\tau \geq \sigma \geq 0$.

We will also assume that f, g are real continuous functions defined on $\mathbb{R}$ such that $xf(x) > 0$, $xg(x) > 0$ for $x \neq 0$ and

$$M_1 \leq \frac{f(x)}{x} \leq M_2, \qquad N_1 \leq \frac{g(x)}{x} \leq N_2$$

for $x \in \mathbb{R}$, where M_1, M_2, N_1 and N_2 are some fixed positive integers.

When f and g are the identity functions, equation (1) reduces to linear equation

$$(2)\qquad [x(t) - R(t)x(t-r)]' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0,$$

which oscillatory and nonoscillatory behaviors of solutions of equation (2) has been studied by many authors [1, 3–17]. As far as we know, nothing has been done for the oscillatory behavior of equation (1). However, in [2] Cheng et al. obtained some results deal with existence and nonexistence of positive solutions of equation (1).

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In this paper, our aim is to give some new sufficient conditions for the oscillation of equation (1). Our results improve the known results in the literature.

Let $m = \max\{r, \tau\}$, by a solution of equation (1), we mean a function $x(t) \in C([t_1 - m, \infty), \mathbb{R})$ which is defined for some $t \geq t_1$, such that $x(t) - R(t)f(x(t-r))$ is continuously differentiable on $[t_1, \infty)$ and satisfies equation (1) for $t \geq t_1$. We recall that a nontrivial solution of equation (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

Throughout this paper, we let

$$\alpha = \begin{cases} r, & \text{when } Q(t) \equiv 0 \\ \max\{r, \tau\}, & \text{otherwise} \end{cases}$$

$$\beta = \begin{cases} r, & \text{when } Q(t) \equiv 0 \\ \min\{r, \sigma\}, & \text{otherwise} \end{cases}$$

and the following condition holds:

$$H(t) = P(t) - Q(t - \tau + \sigma) \geq 0, \text{ and not identically zero.}$$

2. OSCILLATION OF EQUATION (1)

In this section, firstly we shall give some basic lemmas which we will use in proofs of Theorems.

LEMMA 1. Assume that

$$M_2 R(t) + N_2 \int_{t-\tau+\sigma}^{t} Q(s)ds \leq 1 \tag{3}$$

for large t. Let $x(t)$ be an eventually positive solution of the differential inequality

$$[x(t) - R(t)f(x(t-r))]' + P(t)g(x(t-\tau)) - Q(t)g(x(t-\sigma)) \leq 0. \tag{4}$$

and the $z(t)$ defined by

$$z(t) = x(t) - R(t)f(x(t-r)) - \int_{t-\tau+\sigma}^{t} Q(s)g(x(s-\sigma))\,ds. \tag{5}$$

Then we have eventually

$$z'(t) \leq 0 \quad \text{and} \quad z(t) > 0.$$

PROOF. Let $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Then, from (4) and (5), we have

$$z'(t) \leq -\left(P(t) - Q(t-\tau+\sigma)\right) g(x(t-\tau)) = -H(t)g(x(t-\tau)) \leq 0, \ \ t \geq t_1. \tag{6}$$

So, $z(t)$ is nonincreasing for $t \geq t_1$. If $z(t) > 0$ does not hold, then eventually $z(t) < 0$, which implies that there exists a constant $\mu > 0$ and $t_2 \geq t_1$ such that $z(t) \leq -\mu$ for $t \geq t_2$. By (5) we have

$$x(t) \leq -\mu + R(t) f\left(x(t-r)\right) + \int_{t-\tau+\sigma}^{t} Q(s) g\left(x(s-\sigma)\right) ds\,, \quad t \geq t_2. \tag{7}$$

We consider the two possible cases.

CASE 1. $x(t)$ is unbounded, that is $\limsup_{t\to\infty} x(t) = \infty$. Thus, there exists a sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $s_i \geq t_2 + \alpha$ $i = 1, 2, 3, \ldots$, $s_i \to \infty$, $x(s_i) \to \infty$ as $i \to \infty$ and

$$x(s_i) = \max\{x(t) : t_2 \leq t \leq s_i\}, \quad i = 1, 2, 3, \ldots$$

From (3), (7) and above equality, we get

$$\begin{aligned} x(s_i) &\leq -\mu + R(s_i) f\left(x(s_i - r)\right) + \int_{s_i-\tau+\sigma}^{s_i} Q(s) g\left(x(s-\sigma)\right) ds \\ &\leq -\mu + \left(M_2 R(s_i) + N_2 \int_{s_i-\tau+\sigma}^{s_i} Q(s) ds \right) x(s_i) \\ &\leq -\mu + x(s_i), \end{aligned}$$

which is a contradiction.

CASE 2. $x(t)$ is bounded, that is $\limsup_{t\to\infty} x(t) = L \in [0, \infty)$. Let $\{\bar{s}_i\}_{i=1}^{\infty}$ be a sequence of points such that $\bar{s}_i \to \infty$ and $x(\bar{s}_i) \to L$ as $i \to \infty$. Let ξ_i be such that

$$x(\xi_i) = \max\{x(s) :\ \bar{s}_i - \alpha \leq s \leq \bar{s}_i - \beta\},\ \bar{s}_i - \alpha \leq \xi_i \leq \bar{s}_i - \beta,\ i = 1, 2, \ldots$$

Then $\xi_i \to \infty$ as $i \to \infty$ and $\limsup_{i\to\infty} x(\xi_i) \leq L$. Thus, by (3) and (7), we have

$$\begin{aligned} x(\bar{s}_i) &\leq -\mu + R(\bar{s}_i) f\left(x(\bar{s}_i - r)\right) + \int_{\bar{s}_i-\tau+\sigma}^{\bar{s}_i} Q(s) g\left(x(s-\sigma)\right) ds \\ &\leq -\mu + x(\xi_i). \end{aligned}$$

Taking the superior limit as $i \to \infty$ on both sides, we obtain

$$L \leq -\mu + \limsup_{i\to\infty} x(\xi_i) \leq -\mu + L,$$

which is also a contradiction. Thus, the proof is complete.

LEMMA 2. Assume that

$$M_1 R(t) + N_1 \int_{t-\tau+\sigma}^{t} Q(s) ds \geq 1 \tag{8}$$

for large t. Let $x(t)$ be an eventually positive solution of (4) and $z(t)$ be defined by (5). If the following second order differential inequality

$$y''(t) + \frac{N_1}{\alpha} H(t) y(t) \leq 0 \tag{9}$$

does not have eventually positive solution, then eventually

$$z'(t) \leq 0 \text{ and } z(t) < 0.$$

PROOF. We give only the proof in the case $Q(t) \equiv 0$ does not hold. For the case $Q(t) \equiv 0$, the proof is quite similar and is omitted. Then $\alpha = \max\{r, \tau\}$. By (4) and (5), (6) holds for large t, then $z'(t) \leq 0$ for large t.

Suppose to the contrary that $z(t)$ is eventually positive, then there is a constant $T > \alpha$ such that $x(t) > 0, z(t) > 0$ and $z'(t) \leq 0$ for any $t \geq T - \alpha$.

Set $K = \min\{x(t) : T - \alpha \leq t \leq T\} > 0$. From (5) and (8), we have

$$\begin{aligned} x(T) &= z(T) + R(T) f\left(x(T-r)\right) + \int\limits_{T-\tau+\sigma}^{T} Q(s) g\left(x(s-\sigma)\right) ds \\ &\geq K \left(M_1 R(T) + N_1 \int\limits_{T-\tau+\sigma}^{T} Q(s) ds \right) \geq K. \end{aligned}$$

In general, we can obtain

$$x(t) \geq K \left(M_1 R(T) + N_1 \int\limits_{T-\tau+\sigma}^{T} Q(s) ds \right) \geq K, \qquad T \leq t \leq T + \alpha.$$

By the induction we have for $T + n\alpha \leq t \leq T + (n+1)\alpha$

$$x(t) \geq K, \qquad T + n\alpha \leq t \leq T + (n+1)\alpha$$

where $n = [(t-T)/\alpha]$ and $[\cdot]$ denotes the greatest integer function. Hence

$$x(t) \geq K \text{ for any } t \geq T - \alpha. \tag{10}$$

Let $\lim_{t\to\infty} z(t) = a$. Then there exist two possible cases.

CASE 1. $a = 0$. Let $T_1 > T$ such that $z(t) < K/2$ for $t \geq T_1$. Then, from (10), we have

$$x(t) \geq \frac{1}{\alpha} \int\limits_{T_1}^{t+\alpha} z(s) ds, \qquad T_1 \leq t \leq T_1 + \alpha.$$

CASE 2. $a > 0$. Since $z'(t) \leq 0$ for $t \geq T$, we have $z(t) \geq a$ for $t \geq T$. From (5), (8) and (10) we have

$$x(t) \geq a + R(t) f\left(x(t-r)\right) + \int\limits_{t-\tau+\sigma}^{t} Q(s) g\left(x(s-\sigma)\right) ds \geq a + K, \qquad t \geq T_1.$$

In general, we have

$$x(t) \geq na + K, \qquad t \geq T_1 + n\alpha$$

and so $\lim_{t\to\infty} x(t) = \infty$, which implies that there is a $T_2 > T_1$ such that

$$x(t) \geq \frac{1}{\alpha} \int_{T_2}^{t+\alpha} z(s)ds, \qquad T_2 \leq t \leq T_2 + \alpha.$$

Combining the cases 1 and 2, we see that there exists a $T^* > T_2$ such that

$$x(t) \geq \frac{1}{\alpha} \int_{T^*}^{t+\alpha} z(s)ds, \qquad T^* \leq t \leq T^* + \alpha. \tag{11}$$

For $T^* + \alpha \leq t \leq T^* + \alpha + \beta$, by (5), (8) and (11) we have

$$\begin{aligned} x(t) &= z(t) + R(t)f\left(x(t-r)\right) + \int_{t-\tau+\sigma}^{t} Q(s)g\left(x(s-\sigma)\right)ds \\ &\geq z(t) + \left(M_1 R(t) + N_1 \int_{t-\tau+\sigma}^{t} Q(s)ds\right) \frac{1}{\alpha} \int_{T^*}^{t} z(s)ds \\ &\geq \frac{1}{\alpha} \int_{t}^{t+\alpha} z(s)ds + \frac{1}{\alpha} \int_{T^*}^{t} z(s)ds \\ &= \frac{1}{\alpha} \int_{T^*}^{t+\alpha} z(s)ds. \end{aligned}$$

By induction, for $T^* + \alpha + (n-1)\beta \leq t \leq T^* + \alpha + n\beta$, we can show that

$$x(t) \geq \frac{1}{\alpha} \int_{T^*}^{t+\alpha} z(s)ds,$$

hence

$$x(t) \geq \frac{1}{\alpha} \int_{T^*}^{t+\alpha} z(s)ds, \ \ t \geq T^*. \tag{12}$$

Let $y(t) = \int_{T^*}^{t} z(s)ds$, then $y(t) > 0$ for large t, and $y'(t) = z(t)$, $y''(t) = z'(t)$ for $t \geq T^*$. Thus, by (6), (12), we have

$$z'(t) \leq -H(t)g\left(x(t-\tau)\right) \leq \frac{-H(t)N_1}{\alpha} \int_{T^*}^{t-\beta+\alpha} z(s)ds \leq \frac{-H(t)N_1}{\alpha} \int_{T^*}^{t} z(s)ds,$$

and so

$$y''(t) + \frac{N_1}{\alpha} H(t)y(t) \leq 0, \qquad t \geq T^* + \alpha.$$

It is clear that, $y(t)$ is an eventually positive solution of (9). Thus, this is a contradiction. Therefore the proof is complete.

Now, we have the following result, which can be found in [1].

LEMMA 3. Let $p(t) \in C([t_0,\infty),\mathbb{R}^+)$. If

$$\liminf_{t\to\infty} t\int_t^\infty p(s)ds > \frac{1}{4}, \tag{13}$$

then every solution of second order differential equation

$$y''(t)+p(t)y(t)=0, \qquad t\geq t_0, \tag{14}$$

oscillates.

THEOREM 1. Assume that (3) and

$$N_1\liminf_{t\to\infty} t\int_t^\infty H(s)ds > \frac{\alpha}{4} \tag{15}$$

hold. Suppose further that $M_1N_1N_2^{-1}\geq 1$ and

$$R(t-\tau)H(t)\geq H(t-r) \tag{16}$$

for large t. Then every solution of (1) oscillates.

PROOF. Suppose on the contrary that $x(t)$ is an an eventually positive solution of (1). By Lemma 1, we have $z(t)>0$. In view of (6), we see that

$$-H(t)N_2x(t-\tau)\leq z'(t)=-H(t)g\left(x(t-\tau)\right)\leq -N_1H(t)x(t-\tau)$$

for large t. By means of (16), we obtain

$$\begin{aligned}
z'(t) &\leq -N_1H(t)x(t-\tau)\\
&= -N_1H(t)\left(\begin{array}{c} z(t-\tau)+R(t-\tau)f\left(x(t-\tau-r)\right)\\ +\int\limits_{t-\tau+\sigma}^{t} Q(s-\tau)N_2x(s-\tau-\sigma)ds\end{array}\right)\\
&\leq -N_1H(t)z(t-\tau)-N_1M_1H(t)R(t-\tau)x(t-\tau-r)\\
&\leq -N_1H(t)z(t-\tau)-N_1M_1H(t-r)x(t-\tau-r)\\
&\leq -N_1H(t)z(t-\tau)+N_1M_1N_2^{-1}z'(t-r)\\
&\leq -N_1H(t)z(t-\tau)+z'(t-r).
\end{aligned}$$

That is, $z(t)$ is an eventually positive solution of the inequality

$$[u(t)-u(t-r)]'+N_1H(t)u(t-\tau)\leq 0,$$

which is a contradiction by Lemmas 1 and 2. The proof is complete.

Theorem 1 extends results in [2, 7].

THEOREM 2. Assume that (8) and (15) hold, and that $\{Q(t)/H(t-\tau+\sigma)\}$ is nondecreasing. Suppose further that there are nonnegative constants λ_1, λ_2 such that

$$R(t-\tau)H(t) \le \lambda_1 H(t-r), \tag{17}$$

$$H(t)Q(t-\tau) \le \lambda_2 H(t-\sigma), \tag{18}$$

and

$$\lambda_1 M_2 N_1^{-1} N_2 + \lambda_2 N_2^2 N_1^{-1}(\tau-\sigma) = 1. \tag{19}$$

Then every solution of (1) oscillates.

PROOF. If the above conclusion does not hold, equation (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (5). From Lemma 2, we have $z(t) < 0$. By (17) and (18) and by using the nondecreasing of $\{Q(t)/H(t+\tau-\sigma)\}$, we get the following

$$\begin{aligned}
z'(t) &= -H(t)g\left(x(t-\tau)\right) \ge -H(t)N_2 x(t-\tau) \\
&= -H(t)N_2 \begin{pmatrix} z(t-\tau) + R(t-\tau)f\left(x(t-\tau-r)\right) \\ + \int\limits_{t-\tau+\sigma}^{t} Q(s-\tau)N_2 x(s-\tau-\sigma)ds \end{pmatrix} \\
&\ge -H(t)N_2 z(t-\tau) - \lambda_1 H(t-r)N_2 f\left(x(t-\tau-r)\right) \\
&\quad -H(t)N_2^2 \int\limits_{t-\tau+\sigma}^{t} \frac{N_1 Q(s-\tau)H(s-\sigma)}{N_1 H(s-\sigma)} x(s-\tau-\sigma)ds \\
&\ge -H(t)N_2 z(t-\tau) + \lambda_1 M_2 N_1^{-1} N_2 z'(t-r) \\
&\quad -H(t)N_2^2 N_1^{-1} \frac{Q(t-\tau)}{H(t-\sigma)} \int\limits_{t-\tau+\sigma}^{t} \left[-z'(s-\sigma)\right] ds \\
&\ge -H(t)N_2 z(t-\tau) + \lambda_1 M_2 N_1^{-1} N_2 z'(t-r) + \lambda_2 N_2^2 N_1^{-1} \int\limits_{t-\tau+\sigma}^{t} z'(s-\sigma)ds \\
&= -H(t)N_2 z(t-\tau) + \lambda_1 M_2 N_1^{-1} N_2 z'(t-r) + \lambda_2 N_2^2 N_1^{-1}\left(z(t-\sigma) - z(t-\tau)\right) \\
&= -\left(H(t)N_2 + \lambda_2 N_2^2 N_1^{-1}\right) z(t-\tau) + \lambda_1 M_2 N_1^{-1} N_2 z'(t-r) + \lambda_2 N_2^2 N_1^{-1} z(t-\sigma).
\end{aligned}$$

That is

$$\left[z(t) - \lambda_1 M_2 N_1^{-1} N_2 z(t-r)\right]' + \left(H(t)N_2 + \lambda_2 N_2^2 N_1^{-1}\right) z(t-\tau) - \lambda_2 N_2^2 N_1^{-1} z(t-\sigma) \ge 0,$$

which implies $-z(t)$ is an eventually positive solution of the inequality

$$\left[u(t) - \lambda_1 M_2 N_1^{-1} N_2 u(t-r)\right]' + \left(H(t)N_2 + \lambda_2 N_2^2 N_1^{-1}\right) u(t-\tau) - \lambda_2 N_2^2 N_1^{-1} u(t-\sigma) \le 0,$$

which will yield a contradiction by Lemmas 1 and 2. The proof is complete.

Theorem 2 extends results in [2, 7, 8, 12, 13].

EXAMPLE. Consider the equation

$$\left[x(t) - \frac{1}{10} f\left(x(t-1)\right)\right]' + \left(e^{10t} + \frac{1}{2}\right) g\left(x(t-2)\right) - \frac{1}{2} g\left(x(t-1)\right) = 0, \tag{20}$$

where

$$2 \leq \frac{f(x)}{x} \leq 5, \qquad \frac{1}{2} \leq \frac{g(x)}{x} \leq 1.$$

As it is seen here, $r = 1$, $\tau = 2$, $\sigma = 1$, $M_1 = 2$, $M_2 = 5$, $N_1 = \frac{1}{2}$, $N_2 = 1$ and $R(t) = \frac{1}{10}$, $P(t) = e^{10t} + \frac{1}{2}$, $Q(t) = \frac{1}{2}$. It is easy to calculate that (3) holds with

$$5\frac{1}{10} + 1\frac{1}{2}\int_{t-1}^{t} ds = 1$$

and (15) holds with

$$\liminf_{t\to\infty} t \int_{t}^{\infty} e^{10s} ds > 1.$$

Also, $M_1 N_1 N_2^{-1} \geq 1$ and $\frac{1}{10} e^{10t} \geq e^{10(t-1)}$ holds. So, all conditons of Theorem 1 are satisfied thus every solution of (20) oscillates.

REFERENCES

[1] T. A. Chanturia, Integral criteria for the oscillation of higher order differential equations, Differencialnye Uravnenija **16**, (1980), 470–482.

[2] S. S. Cheng, X. P. Guan and J. Yang, Positive solutions of a nonlinear neutral equation with positive and negative coefficients, Acta Math. Hungar. **86** (3), (2000), 169–192.

[3] Q. Chuanxi and G. Ladas, Oscillation in differential equations with positive and negative coefficients, Canad. Math. Bull. **33**, (1990), 442–450.

[4] K. Farrell, E. A. Grove and G. Ladas, Neutral delay differential equations with positive and negative coefficients, Appl. Anal. **27**, (1988), 181–197.

[5] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation theory for functional differential equations, Marcel Dekker, (1995).

[6] I. Gyori and G. Ladas, Oscillation theory of delay differential equations with applications, Clarendon Press, Oxford, (1991).

[7] Z. G. Luo, J. H. Shen and X. Z. Liu, Oscillation criteria for a class of forced neutral equations, Dynamics of Continuous, Discrete and Impulsive Systems **7**, (2000), 489–501.

[8] Z. G. Luo and J. H. Shen, Oscillation and nonoscillation of neutral differential equations with positive and negative coefficints, Czech. Math. J. **54**, (2004), 79–93.

[9] Ö. Öcalan, Oscillation of neutral differential equations with positive and negative coefficients, J. Math. Anal. Appl. 331 (2007) 644–654.

[10] S. G. Ruan, Oscillations for first order neutral differential equations with variable coefficients, Bull. Austral. Math. Soc. **43**, (1991), 147–152.

[11] S.H. Saker, Oscillation of higher order neutral delay differential equations with variable coefficients, Dynamic Systems and Applications 11 (2002) 107–125.

[12] J. H. Shen and L. Debnath, Oscillations of solutions of neutral differential equations with positive and negative coefficients, Applied Mathematics Letters **14**, (2001), 775–781.

[13] J. H. Shen and Z. C. Wang, Oscillation and nonoscillation for a class of nonlinear neutral differential equation, Diff. Eq. Dyna. Sys., 2, (1994), 347–360.

[14] X. H. Tang and J. S. Yu, On the positive solutions to a kind of neutral equations with positive and negative coefficients, Appl. Math. Wuhan **12**(2), (1999), 97–102.

[15] J. S. Yu, Neutral differential equations with positive and negative coefficients, Acta Math. Sinica **34**, (1991), 517–523.

[16] J. S. Yu and Z. C. Wang, Some further results on oscillation of neutral differential equation, Bull. Austral. Math. Soc. **46**, (1992)149–157.

[17] J. S. Yu and J. Yan, Oscillation in first order neutral differential equations with an "integrally small" coefficients, J. Math. Anal. Appl. **187**, (1994), 361–370.