Dynamic Systems and Applications 18 (2009) 55-62

QUENCHING FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE ON A SEMI-INFINITE INTERVAL

C. Y. CHAN AND T. TREEYAPRASERT

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, Louisiana 70504-1010, USA
Department of Mathematics and Statistics, Thammasat University
Rangsit Center, Pathumthani 12121, Thailand

ABSTRACT. Let α, b, and T be positive numbers, $D = (0, \infty)$, $\bar{D} = [0, \infty)$, and $\Omega = D \times (0, T]$. This article studies the first initial-boundary value problem with a concentrated nonlinear source situated at b,

$$u_t - u_{xx} = \alpha\delta(x-b) f(u(x,t)) \text{ in } \Omega,$$
$$u(x,0) = 0 \text{ on } \bar{D},$$
$$u(0,t) = 0 \text{ and } u(x,t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function and f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$.

The problem has a unique continuous solution u before $\sup\{u(x,t) : 0 \leq x < \infty\}$ reaches c^-, and u is a strictly increasing function of t in Ω. It is shown that if

$$\sup\{u(x,t) : 0 \leq x < \infty\}$$

reaches c^-, then u attains the value c in a finite time only at the point b. A criterion for u to exist globally and a criterion for u to quench in a finite time are given. It is also shown that there exists a critical position b^* for the nonlinear source to be placed such that for $b \leq b^*$, u exists for $0 \leq t < \infty$, and for $b > b^*$, u quenches in a finite time. This also implies that u does not quench in infinite time. The formula for computing b^* is also derived.

AMS (MOS) Subject Classification. 35K60, 35K57

1. INTRODUCTION

Let α, b, and T be positive numbers, $D = (0, \infty)$, $\bar{D} = [0, \infty)$, $\Omega = D \times (0, T]$, and $Lu = u_t - u_{xx}$. We consider the following semilinear parabolic first initial-boundary value problem,

$$(1.1) \qquad \begin{cases} Lu = \alpha\delta(x-b) f(u(x,t)) \text{ in } \Omega, \\ u(x,0) = 0 \text{ on } \bar{D}, \\ u(0,t) = 0 \text{ and } u(x,t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T, \end{cases}$$

where $\delta(x)$ is the Dirac delta function, and f is a given function such that $\lim_{u\to c^-} f(u) = \infty$ for some positive constant c. We assume that $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \le u < c$. A solution u of the problem (1.1) is a continuous function satisfying (1.1). A solution u of the problem (1.1) is said to quench if there exists some t_q such that $\sup\{u(x,t) : 0 \le x < \infty\} \to c^-$ as $t \to t_q$. If t_q is finite, then u is said to quench in a finite time. On the other hand, if $t_q = \infty$, then u is said to quench in infinite time. The position b^* is called the critical position of the nonlinear source if a unique global solution u exists for $b < b^*$, and if the solution u quenches in a finite time for $b > b^*$.

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.1) is given by

$$G(x,t;\xi,\tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} \text{ for } t > \tau$$

(cf. Duffy [2, p. 183]). To derive the integral equation from the problem (1.1), we consider the adjoint operator L^*, which is given by $L^*u = -u_t - u_{xx}$. Using Green's second identity, we obtain

$$u(x,t) = \alpha \int_0^t G(x,t;b,\tau) f(u(b,\tau))\, d\tau. \tag{1.2}$$

Blow-up is a phenomenon related to quenching. Olmstead and Roberts [4] studied the blow-up phenomenon for the following semilinear problem with a concentrated source at b on a bounded domain,

$$\begin{gathered} Lw = \delta(x-b) g(w(x,t)) \text{ in } (0,d) \times (0,T], \\ w(x,0) = w_0(x) \text{ on } [0,d], \\ w(0,t) = 0 = w(d,t) \text{ for } 0 < t \le T, \end{gathered}$$

where d is a positive number, and $w_0(x)$ and $g(w)$ are given functions. They showed that blow-up can always be prevented by placing the nonlinear source sufficiently close to the edge $x = 0$ or $x = d$. Our main purpose here is to find the exact b^* for the problem (1.1) such that u never quenches for $b \le b^*$ and u always quenches in a finite time for $b > b^*$. The fact that u does not quench for $b = b^*$ implies that u does not quench in infinite time. We also note that the proof does not depend on existence of a solution for the steady-state problem corresponding to the problem (1.1). The formula for computing b^* is derived. For illustration, an example is given.

By modifying the techniques used in proving Theorems 1 and 2 of Chan and Jiang [1] for a bounded domain to a semi-infinite interval, we obtain the following result.

Theorem 1.1. *There exists some* $t_q\,(\le \infty)$ *such that the integral equation* (1.2) *has a unique nonnegative continuous solution* u *for* $0 \le t < t_q$, *and* u *is a strictly increasing* *function of t. If t_q is finite, then u quenches in $[0, t_q)$. Furthermore, u is the solution of the problem* (1.1).

2. SINGLE-POINT QUENCHING, AND CRITICAL POSITION FOR THE NONLINEAR SOURCE

We modify the technique used in proving Theorem 3 of Chan and Jiang [1] for a bounded domain to obtain the following result for an unbounded domain.

Theorem 2.1. *The solution $u(x,t)$ attains its absolute maximum value at (b,θ) for $0 \le t \le \theta < t_q$. If in addition, u quenches, then b is the single quenching point.*

Proof. Since $u(b,t)$ is known, let it be denoted by $\eta(t)$. We can rewrite (1.1) as follows:

$$\begin{cases} Lu = 0 \text{ in } (0,b)\times(0,t_q),\ u(x,0)=0 \text{ for } 0\le x\le b, \\ u(0,t)=0 \text{ and } u(b,t)=\eta(t) \text{ for } 0<t<t_q, \end{cases} \tag{2.1}$$

$$\begin{cases} Lu = 0 \text{ in } (b,\infty)\times(0,t_q),\ u(x,0)=0 \text{ for } x\ge b, \\ u(b,t)=\eta(t) \text{ and } u(x,t)\to 0 \text{ as } x\to\infty \text{ for } 0<t<t_q. \end{cases} \tag{2.2}$$

It follows from the strong maximum principle (cf. Friedman [3, p. 34]) and Theorem 1.1 that the solution $u(x,t)$ of the problem (2.1) attains its maximum value at (b,θ) for $0 < t \le \theta < t_q$. Since $u(x,t) \to 0$ as $x \to \infty$, it follows from the Phragmèn-Lindelöf Principle and the Remark (ii) (cf. Protter and Weinberger [5, pp. 183–185]) that u must attain its maximum at b for the problem (2.2). We claim that the solution $u(x,t)$ of the problem (2.2) attains its absolute maximum value at (b,θ) for $0 < t \le \theta < t_q$. To show this, let us assume that $u(x,t)$ attains its absolute maximum value at (r,t) for some positive number $r > b$ and $t \in (0,\theta]$. Let l be a positive number such that $l > r$. By Theorem 1.1, $u(l,t)$ is known. Let us denote it by $\gamma(t)$. We then consider the following problem:

$$\begin{gathered} Lu = 0 \text{ in } (b,l)\times(0,t_q),\ u(x,0)=0 \text{ for } b\le x\le l, \\ u(b,t)=\eta(t) \text{ and } u(l,t)=\gamma(t) \text{ for } 0<t<t_q. \end{gathered}$$

Since $u(x,t)$ attains its absolute maximum value at (r,t), we have by the strong maximum principle that $u(x,t) \equiv u(r,t)$ for all $(x,t) \in (b,l) \times (0,t]$, for which we have a contradiction. Since r is an arbitrary point in (b,∞), we conclude that $u(x,t)$ of the problem (2.2) attains its absolute maximum value only at (b,θ) for $0 < t \le \theta < t_q$. Therefore, the solution $u(x,t)$ attains its absolute maximum value at (b,θ) for $0 < t \le \theta < t_q$ for each of the problems (2.1) and (2.2). Hence, if u quenches, then it quenches at $x = b$.

To show that b is the only quenching point, let us consider the problem (2.1). By the parabolic version of Hopf's lemma (cf. Friedman [3, p. 49]), $u_x(0,t) > 0$ for any arbitrarily fixed $t \in (0, t_q)$. For any $x \in (0, b)$, $u_{xx} = u_t$, which is nonnegative by Theorem 1.1. Hence, u is concave up. Similarly, for the problem (2.2), we have that for any arbitrarily fixed $t \in (0, t_q)$, $u_x(b, t) < 0$. For any $x \in (b, \infty)$, $u_{xx} = u_t \geq 0$, and hence u is concave up. Thus, if u quenches, then b is the single quenching point. □

By using Mathematica Version 6.0, we have

$$\int_0^t G(b, t; b, \tau) d\tau = b + \left(1 - e^{-b^2/t}\right)\sqrt{\frac{t}{\pi}} - b\text{Erf}\left(\frac{b}{\sqrt{t}}\right),$$

$$\frac{d}{dt}\int_0^t G(b, t; b, \tau) d\tau = \frac{1 - e^{-b^2/t}}{2\sqrt{\pi t}} > 0. \tag{2.3}$$

By the L'Hôpital rule,

$$\lim_{t\to\infty}\left(1 - e^{-b^2/t}\right)\sqrt{\frac{t}{\pi}} = \lim_{t\to\infty}\frac{2b^2}{\sqrt{\pi t}}e^{-b^2/t} = 0.$$

Since $\lim_{t\to\infty}\text{Erf}\left(b/\sqrt{t}\right) = 0$, we have

$$\lim_{t\to\infty}\int_0^t G(b, t; b, \tau) d\tau = b. \tag{2.4}$$

Given any positive constant $M\ (< c)$, we would like to choose b such that $u(b, t) \leq M$ for all $t > 0$. From (1.2), (2.3) and (2.4), we have $u(b, t) \leq \alpha f(M) b$. Thus, if

$$\alpha f(M) b \leq M, \tag{2.5}$$

then u exists globally. The above discussion gives us the following sufficient condition for global existence of u.

Theorem 2.2. *Given any positive number $M\ (< c)$, if (2.5) holds, then u exists for all $t > 0$.*

We now give a sufficient condition for u to quench in a finite time.

Theorem 2.3. *If $b > c/f(0)$, then u quenches in a finite time.*

Proof. From Theorem 1.1, there exists some t_1 such that

$$u(b, t) = \int_0^t G(b, t; b, \tau) f(u(b, \tau))\, d\tau < c \text{ for } t \in (0, t_1].$$

Since f is an increasing function and $u(b, t) > 0$ for $t \in (0, t_1]$, we have for $t \in (0, t_1]$,

$$f(0)\int_0^t G(b, t; b, \tau) d\tau < \int_0^t G(b, t; b, \tau) f(u(b, \tau))\, d\tau < c, \tag{2.6}$$

which gives

$$\int_0^t G(b, t; b, \tau) d\tau < \frac{c}{f(0)} \text{ for } t \in (0, t_1].$$

If there exists some $t_2 \in (t_1, \infty)$ such that

$$\int_0^{t_2} G(b, t_2; b, \tau) d\tau \geq \frac{c}{f(0)}, \tag{2.7}$$

then (2.6) is contradicted for $t = t_2$, and hence the continuous solution $u(b, t)$, which is a strictly increasing function of t, cannot be continued to the interval $(0, t_2)$. Thus, there exists some $\hat{t} \in (t_1, t_2)$ such that $u(b, t) \to c^-$ as $t \to \hat{t}$. Therefore, $u(b, t)$ quenches at $\hat{t}$.

Since $b > c/f(0)$, it follows from (2.3) and (2.4) that (2.7) can always be satisfied. The theorem is then proved. □

Theorem 2.4. *If* $\lim_{t\to\infty} u(b, t) < c$, *then*

$$U(b) = \alpha b f(U(b)), \tag{2.8}$$

where $U(b)$ *denotes* $\lim_{t\to\infty} u(b, t)$. *Furthermore,* $u(b, t) < U(b)$ *for* $t \in (0, \infty)$.

Proof. From (1.2),

$$U(b) = \lim_{t\to\infty} u(b, t) = \lim_{t\to\infty} \alpha \int_0^t G(b, t; b, \tau) f(u(b, \tau)) d\tau.$$

We want to show that

$$\lim_{t\to\infty} \alpha \int_0^t G(b, t; b, \tau) f(u(b, \tau)) d\tau = \alpha b f(U(b)).$$

By using Mathematica version 6.0,

$$\int_{\frac{t}{2}}^t G(b, t; b, \tau) d\tau = b + \sqrt{\frac{t}{2\pi}} \left(1 - e^{-\frac{2b^2}{t}}\right) - b\mathrm{Erf}\left(b\sqrt{\frac{2}{t}}\right),$$

$$\frac{d}{dt}\left[b + \sqrt{\frac{t}{2\pi}} \left(1 - e^{-\frac{2b^2}{t}}\right) - b\mathrm{Erf}\left(b\sqrt{\frac{2}{t}}\right)\right] = \frac{1 - e^{-\frac{2b^2}{t}}}{2\sqrt{2\pi t}} > 0.$$

By the L'Hôpital rule,

$$\lim_{t\to\infty} \sqrt{\frac{t}{2\pi}} \left(1 - e^{-\frac{2b^2}{t}}\right) = \lim_{t\to\infty} \frac{2\sqrt{2} b^2 e^{-b^2/t}}{\sqrt{\pi t}} = 0.$$

Since $\lim_{t\to\infty} \mathrm{Erf}\left(b\sqrt{2/t}\right) = 0$, we have

$$\lim_{t\to\infty} \int_{\frac{t}{2}}^t G(b, t; b, \tau) d\tau = b.$$

It follows from the continuity of f that

$$\lim_{t\to\infty} f(u(b, t)) = f\left(\lim_{t\to\infty} u(b, t)\right) = f(U(b)).$$

Thus given any positive number ε, there exists some positive number $\tilde{t}$ such that for $t > \tilde{t}$,

$$0 < b - \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau < \frac{\varepsilon}{2f\left(U\left(b\right)\right)}, \tag{2.9}$$

$$0 < f\left(U\left(b\right)\right) - f\left(u\left(b,t\right)\right) < \frac{\varepsilon}{2b}. \tag{2.10}$$

Thus,

$$\begin{aligned}
& bf\left(U\left(b\right)\right) - \int_0^t G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau \\
&= bf\left(U\left(b\right)\right) - f\left(U\left(b\right)\right) \int_0^t G(b,t;b,\tau)d\tau \\
&+ \int_0^t G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau \\
&= bf\left(U\left(b\right)\right) - f\left(U\left(b\right)\right) \int_0^{\frac{t}{2}} G(b,t;b,\tau)d\tau - f\left(U\left(b\right)\right) \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau \\
&+ \int_0^{\frac{t}{2}} G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau \\
&+ \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau \\
&= bf\left(U\left(b\right)\right) - f\left(U\left(b\right)\right) \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau - \int_0^{\frac{t}{2}} G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau \\
&+ \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau \\
&\leq f\left(U\left(b\right)\right)\left(b - \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau\right) \\
&+ \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau.
\end{aligned}$$

It follows from (2.9), f being an increasing function of τ, and (2.10) that for $t > 2\tilde{t}$,

$$\begin{aligned}
& bf\left(U\left(b\right)\right) - \int_0^t G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau \\
&< f\left(U\left(b\right)\right) \frac{\varepsilon}{2f\left(U\left(b\right)\right)} + \left(f\left(U\left(b\right)\right) - f\left(u\left(b,\frac{t}{2}\right)\right)\right) \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2b} \int_{\frac{t}{2}}^{t} G(b,t;b,\tau)d\tau \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2b} b = \varepsilon.
\end{aligned}$$

On the other hand,

$$bf\left(U\left(b\right)\right)-\int_{0}^{t}G(b,t;b,\tau)f\left(u\left(b,\tau\right)\right)d\tau$$
$$=f\left(U\left(b\right)\right)\left(b-\int_{0}^{t}G(b,t;b,\tau)d\tau\right)+\int_{0}^{t}G(b,t;b,\tau)\left(f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right)d\tau.$$

By (2.3) and (2.4), $0<b-\int_{0}^{t}G(b,t;b,\tau)d\tau$. By (2.10), $f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)>0$. It follows that the right-hand side is positive. Hence for $t>2\tilde{t}$,

$$0<bf\left(U\left(b\right)\right)-\int_{0}^{t}G(b,t;b,\tau)f\left(u\left(b,\tau\right)\right)d\tau<\varepsilon.$$

Since ε is arbitrary, we have (2.8). It follows from $u\left(b,t\right)$ being a strictly increasing function of t that $u\left(b,t\right)<U\left(b\right)$ for $t\geq 0$. □

Let $\phi\left(s\right)=s/f\left(s\right)$ for $0\leq s<c$. Since $\phi'\left(s\right)=\left(f\left(s\right)-sf'\left(s\right)\right)/f^{2}\left(s\right)$, the critical value s of $\phi\left(s\right)$ is given by $s=f\left(s\right)/f'\left(s\right)$. Evaluating $d^{2}\phi\left(s\right)/ds^{2}$ at this critical value, we have

$$\frac{d^{2}}{ds^{2}}\phi\left(\frac{f\left(s\right)}{f'\left(s\right)}\right)=-\frac{f\left(s\right)f''\left(s\right)}{f'\left(s\right)f^{2}\left(s\right)}<0.$$

Therefore, $\phi\left(s\right)$ attains its relative (namely in this case, absolute) maximum at this critical value. From (2.8), $b=U\left(b\right)/\left(\alpha f\left(U\left(b\right)\right)\right)$, where $0\leq U\left(b\right)<c$. Let

$$b^{*}=\frac{1}{\alpha}\sup_{0\leq U(b)<c}\frac{U\left(b\right)}{f\left(U\left(b\right)\right)}. \tag{2.11}$$

For $b>b^{*}$, it follows from Theorem 2.4 that $U\left(b\right)$ does not exist. Since

$$\sup_{0\leq U(b)<c}\frac{U\left(b\right)}{f\left(U\left(b\right)\right)}$$

is attained at $U\left(b\right)=f\left(U\left(b\right)\right)/f'\left(U\left(b\right)\right)$, we have

$$b^{*}=\frac{f\left(U\left(b\right)\right)}{f'\left(U\left(b\right)\right)}\frac{1}{\alpha f\left(U\left(b\right)\right)}=\frac{1}{\alpha f'\left(U\left(b\right)\right)}. \tag{2.12}$$

Theorem 2.5. *If $b<b^{*}$, then $U\left(b\right)$ increases as b increases.*

Proof. Differentiating (2.8) with respect to b yields

$$U'\left(b\right)=\alpha\left(f\left(U\left(b\right)\right)+bf'\left(U\left(b\right)\right)U'\left(b\right)\right),$$

which, by (2.12) and $b<b^{*}$, gives

$$U'\left(b\right)=\frac{\alpha f\left(U\left(b\right)\right)}{1-\alpha bf'\left(U\left(b\right)\right)}>0.$$

Hence, $U'\left(b\right)>0$. The theorem is proved. □

To obtain the following result, we modify the technique used in proving Theorem 7 of Chan and Jiang [1] for the critical length for a bounded domain.

Theorem 2.6. *For $b \leq b^*$, u exists for all $t > 0$. For $b > b^*$, u quenches in a finite time.*

Proof. For $b < b^*$, it follows from Theorem 2.5 that $U(b)$ exists, and hence u exists for $0 \leq t < \infty$. Since $\phi(s) > 0$ for $s \in (0, c)$, and $\phi(0) = 0$, and $\lim_{s \to c^-} \phi(s) = 0$, it follows that $\phi(s)$ attains its maximum with $s \in (0, c)$. This implies $U(b)$ exists when $b = b^*$. Hence for $b \leq b^*$, u exists globally. For $b > b^*$, $U(b)$ does not exist. By Theorem 1.1, u quenches in a finite time for $b > b^*$. □

The next result follows from Theorem 2.6.

Corollary 2.7. *The solution u of the problem* (1.1) *does not quench in infinite time.*

Example. Let us consider the problem (1.1) with $f(u) = (1-u)^{-p}$, where p is a positive number. Since

$$\frac{d}{ds}\left(\frac{s}{(1-s)^{-p}}\right) = (1-s)^{p-1}(1-s-ps),$$

the critical value is given by $s = 1/(p+1)$. From (2.11),

$$b^* = \frac{p^p}{\alpha(p+1)^{1+p}}.$$

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