Dynamic Systems and Applications 18 (2009) 81-102

CRITICAL COEFFICIENTS IN BLOW-UP PROBLEMS

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Dedicated to Juan Luis Vazquez, on his 60th birthday

ABSTRACT. We review some problems in the blow-up theory from the point of view of the coefficients involved. In particular we are interested in problems in which a competition of two opposed terms produces different behaviours; in the critical case of a balance of both terms we study the influence of a coefficient in one of them. Existence of stationary solutions or self-similar solutions plays a fundamental role.

AMS (MOS) Subject Classification. 35B33, 35B35, 35K55, 35K65

1. INTRODUCTION

In this paper we will consider some evolution problems of parabolic type for which the phenomenon of blow-up in finite time has been thoroughly studied in the last forty years. We are in particular interested in describing how this phenomenon can be controlled sometimes, in critical cases, by some coefficients. To center the discussion and to show what kind of results we are looking for, we go to the basis of the phenomenon and consider the following semilinear heat diffusion problem

$$\left\{\begin{array}{ll} u_t = \Delta u + f(u) & \text{for } x \in \Omega \subset \mathbb{R}^N,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{array}\right. \tag{1.1}$$

By *blow-up* in a finite time T we mean that the solution $u = u(x,t)$ is regular and bounded for $0 < t < T$, but becomes unbounded at $t = T$. A necessary condition to have blow-up is that the reaction is superlinear in the following sense

$$\int_b^\infty \frac{ds}{f(s)} < \infty, \qquad \text{for some } b > 0; \tag{1.2}$$

otherwise a constant in time supersolution can be constructed easily. In fact, the above condition is also sufficient if convexity is also assumed.

Theorem 1.1. *If f is convex and* (1.2) *holds, then there exist solutions to problem* (1.1) *that blow up in finite time.*

The proof, due to Kaplan [43], uses the following argument: Let (μ, Φ) be the first eigenvalue and eigenfunction of $-\Delta$ in Ω, with $\Phi > 0$, $\int_\Omega \Phi = 1$. Putting $J(t) = \int_\Omega u\Phi$, we have

$$J'(t) = \int_\Omega \Delta u \, \Phi + \int_\Omega f(u) \, \Phi \geq -\mu J + f(J) \equiv H(J).$$

If $J(0) = \int_\Omega u_0\Phi$ is large we have $J' \geq cf(J)$, for some $c > 0$, which by the integral condition (1.2) implies that J (and hence u) blows up.

We now fix our attention in the exponential case $f(u) = \lambda e^u$, $\lambda > 0$. We have here $H(J) = -\mu J + \lambda e^J \geq (\lambda - \mu/e)e^J$ for every $J \geq 0$ whenever $\lambda > \mu/e$. Thus in this case there exists blow-up not only if u_0 is large, but for every u_0 if λ is large.

On the other hand, if λ is small there exist stationary supersolutions (w satisfying $\Delta w + 1 = 0$ in Ω is a supersolution to the equation $\Delta v + \lambda e^v = 0$ if $\lambda < e^{-\|w\|_\infty}$), and hence stationary solutions by a well-known iterative method.

By comparison, the set of values $\lambda \in (0, \infty)$ in problem (1.1), with $f(u) = \lambda e^u$, is divided into two disjoint intervals for which we have global existence for small initial data (λ small) or blow-up for every initial value (λ large). The borderline value $\lambda = \lambda^*$ is the so called *critical coefficient* for the global existence of solutions.

In the problem just discussed, we observe a competition between the tendency to blow-up produced by the exponential reaction term and the diffusion of the zero boundary value. The larger is the domain (the further is the boundary for a given point), the weaker is the effect at that point of cooling of the boundary. This competition is measured in terms of the coefficient λ of the reaction.

Throughout the following pages we present and discuss a series of examples appeared in the literature, together with some open problems, in which a competition between two different effects in a superlinear parabolic problem can be controlled, in critical cases, by a coefficient. The appearance of a critical coefficient will mark the borderline between blow-up or global existence. We will show examples of a competition between reaction and diffusion as above, Section 2, reaction vs. convection, Section 3, or even a reaction in the form of a boundary flux against diffusion, convection or absorption, Section 4. We have unified the point of view of all the problems, changing sometimes the notation from the works cited, thus putting the controlling coefficient in front of the reaction. In that way the larger is the coefficient the higher is the tendency of the blow-up to occur. We finally introduce in Section 5 some examples of other phenomena in which again a critical coefficient appears, like grow-up, instantaneous blow-up, continuation or superfast blow-up.

Most of the problems are considered in bounded domains, where the role played by the coefficient is usually relevant, but for some examples with convection, where the presence of the coefficient also affects the final behaviour of the solutions even if the domain is the whole space. In unbounded domains there is another typical critical parameter, the so called *Fujita exponent*, cf. [32]. We do not pay attention here to this exponent, see the excellent surveys [21, 48].

A crucial property, as we have seen in the above example, is the existence or nonexistence of stationary solutions for the precise problem studied. Also the existence of self-similar solutions turns to be of some importance.

2. **REACTION vs. DIFFUSION**

A GENERAL RESULT

We start this series of examples with a general result on the classical reaction-diffusion model.

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and convex C^1 function, and u_0 is bounded nontrivial. We consider classical solutions defined in a maximal time interval $[0, T)$, $0 < T \leq \infty$. We also consider classical or weak solutions of the stationary associated problem (nontrivial), in a standard way. In other problems below, degenerate ones, we must take into account also weak solutions of the evolution problem. In [13] it is proved the following very general result. See also [9, 45].

Theorem 2.1. *There exist global solution to problem* (2.1) *if and only if there exist weak stationary solutions. In the existence case there also exist classical stationary solutions with* $f(u)$ *replaced by* $\varepsilon\, f(u)$ *for any* $0 < \varepsilon < 1$.

The proof uses a modification of Kaplan's argument to get uniform estimates for a global solution of problem (2.1), in order to pass to the limit and obtain a weak stationary solution. Replacing $f(u)$ by $\varepsilon f(u)$ is made by a tricky comparison.

As an immediate consequence we have a characterization of the blow-up question for the problem with a reaction coefficient

$$\begin{cases} u_t = \Delta u + \lambda g(u) & \text{in } \Omega,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{2.2}$$

Corollary 2.2. *Assume that* g *is convex and satisfies the assumption* (1.2). *Then there exists some* $\lambda^* > 0$ *such that if* $\lambda > \lambda^*$ *then every solution to the problem* (2.2) *blows up in finite time, while if $\lambda \le \lambda^*$ then there exist global solutions and also blow-up solutions.*

In the case $g(u) = e^u$, we already know $\lambda^* \le \mu/e$. In the sequel μ denotes, as in the introduction, the first eigenvalue of $-\Delta$ in Ω. If Ω is the unit ball in $\mathbb{R}^N$, with $N \ge 10$ then it is also known the exact value $\lambda^* = 2(N-2)$, cf. [42]. Moreover the corresponding stationary solution $v_*(x) = -2\log|x|$ is only a weak solution. Observe that in this case $\lambda = \lambda^*$, the stationary problem admits no classical solutions but there exists a global classical solution of the evolutionary problem. On the other hand, in [53] it is proved that the stationary solution corresponding to λ^* is indeed classical if $N \le 9$.

Following the ideas of the above work [13], in [16] the following convex-concave problem is considered, for which Theorem 2.1 does not apply directly. See also [11, 14].

$$\begin{cases} u_t = \Delta u + \lambda u^q + u^p & \text{in } \Omega,\ 0<t<T, \\ u = 0 & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2.3}$$

with $0 < q < 1 < p$. In terms of the corresponding stationary problem, in the mentioned paper it is proved

Theorem 2.3. *If $\lambda > \lambda^*(p,q,\Omega)$ then every solution to problem* (2.3) *blows up in finite time, while if $\lambda \le \lambda^*$ then there exist small global solutions and also blow-up solutions.*

<u>Degenerate diffusion</u>

Now we present some problems related to degenerate diffusion, like the porous medium equation, or the σ-laplacian diffusion equation. Here and in the sequel, a power u^m means $|u|^{m-1}u$, though in most of the cases the maximum principle implies $u \ge 0$ if the initial datum satisfies $u_0 \ge 0$. Also, as we have commented upon in the introduction, the notion of weak solutions has to be taken into account in these degenerate problems. Consider first the porous medium type problem.

$$\begin{cases} u_t = \Delta u^m + \lambda u^p & \text{in } \Omega,\ 0<t<T, \\ u = 0 & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2.4}$$

with $m > 1$. The critical case is $p = m$; if $p < m$ the solutions are all global, while if $p > m$ there exist global solutions and also blow-up solutions, see for instance the book [64]. When $p = m$ the blow-up question depends on λ, and the critical coefficient turns to be $\lambda^*(\Omega) = \mu$.

Theorem 2.4. *Let $p = m > 1$. If $\lambda > \lambda^*(\Omega) = \mu$ then every solution to problem* (2.4) *blows up in finite time, while if $\lambda \le \lambda^*$ then there exist global solutions and also blow-up solutions.*

See [33]. The fact that the critical coefficient is $\lambda^* = \mu$ in the case $p = m$ is not surprising, since the stationary problem for $v = u^m$ becomes

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The global existence result follows by comparison with stationary supersolutions. The blow-up result uses the *energy method* of Levine, also called *concavity method*, see [47]. Though this method is more general, it is particularly easy to apply in this case $p = m$. Define, for a solution u to problem (2.4), the functions

$$J(t) = \int_\Omega u^{m+1}, \qquad E(t) = \int_\Omega |\nabla u^m|^2 - \lambda \int_\Omega u^{2m}.$$

The function $E(t) = E_u(t)$ is called the energy. Assuming $E(t_0) < 0$ for some t_0, it is a calculus matter to prove that the function $J^{-\gamma}(t)$ is positive, decreasing and concave for $\gamma = (m-1)/(m+1) > 0$, and thus $J(t)$ must blow up in finite time. To get $E(t) < 0$ we observe that the first eigenvalue $\mu = \mu(\Omega)$ satisfies

$$\mu = \inf\{ \int_\Omega |\nabla\varphi|^2 \,:\, \int_\Omega |\varphi|^2 = 1 \},$$

and the infimum is achieved by the corresponding first eigenfunction. Take $\widetilde{\Omega} \subset \Omega$ such that the first eigenvalue $\widetilde{\mu} = \widetilde{\mu}(\widetilde{\Omega})$ satisfies $\mu < \widetilde{\mu} < \lambda$. If $\widetilde{\Phi}$ is the corresponding eigenfunction in $\widetilde{\Omega}$ (for instance with $\|\widetilde{\Phi}\|_\infty = 1$), extended by zero outside $\widetilde{\Omega}$, we have that the energy of $g = \widetilde{\Phi}^{1/m}$ is

$$E(t) = E_g(t) = (\mu(\Omega) - \lambda) \int_\Omega \widetilde{\Phi}^2 < 0.$$

We have thus proved that the corresponding solution to the evolution problem with initial value g blows up in finite time. Finally, for any initial function we can get the initial comparison $u_0 \geq \varepsilon g$ by taking $\varepsilon > 0$ small enough.

This method works verbatim when replacing the diffusion by the σ-laplacian operator:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{\sigma-2}\nabla u) + \lambda u^p & \text{in } \Omega,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2.5}$$

with $\sigma > 2$. The critical case is $p = \sigma - 1$, and in this case the critical coefficient becomes $\lambda^* = \mu_\sigma(\Omega)$, the first eigenvalue of the generalized eigenvalue problem

$$\begin{cases} \operatorname{div}(|\nabla v|^{\sigma-2}\nabla v) + \lambda v^{\sigma-1} = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

In fact μ_σ minimizes an analogous Rayleigh quotient, i.e.,

$$\mu_\sigma = \inf\{ \int_\Omega |\nabla\varphi|^\sigma \,:\, \int_\Omega |\varphi|^\sigma = 1 \},$$

and the corresponding energy is

$$E_u(t) = \int_\Omega |\nabla u|^\sigma - \lambda \int_\Omega |u|^\sigma.$$

The same kind of proof as for the porous medium operator leads to the following result, see [51].

Theorem 2.5. *Let $p = \sigma - 1 > 1$. If $\lambda > \lambda^*(\Omega) = \mu_\sigma$ then every solution to problem* (2.5) *blows up in finite time, while if $\lambda \le \lambda^*$ then there exist global solutions and also blow-up solutions.*

Nondivergence diffusion

Another type of reaction-diffusion problems of interest take the non-divergence form

$$\text{(2.6)} \qquad \begin{cases} u_t = u^s \Delta u + \lambda u^p & \text{in } \Omega,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } \overline{\Omega}. \end{cases}$$

If $0 < s < 1$, the change of variables $z = u^{1-s}$ transforms this problem into problem (2.4) with $m = 1/(1-s) > 1$ and a reaction cz^q with $c = (1-s)$, $q = (p-s)/(1-s)$. We thus have in this case that the critical exponent is $p = s+1$ and the critical coefficient is μ. But we now observe that problem (2.6) also makes sense when $s \ge 1$ or even when $s < 0$. The latter case corresponds to fast diffusion $0 < m < 1$, where analogous results can be proved. But $s > 1$ implies $m < 0$, singular diffusion, and $c < 0$, absorption instead of reaction. Finally, the case $s = 1$ is genuinely non-divergence. On the other hand, when $s > 1$ the boundary condition transforms into $v = \infty$ on $\partial\Omega$. This means that we are facing a very different problem.

The first work on this type of problems goes back to [30] where the case $s = 2$ was studied. Later on a great number of authors contributed to the theory of blow-up for nondivergence problems like (2.6). Curiously the result is almost exactly the same as for problem (2.4): the critical case is $p = s+1$, the critical coefficient is again μ, but below the critical coefficient all solutions are global, not only the small ones.

Theorem 2.6. *Let $s \ge 1$ and $p = s+1$. If $\lambda > \lambda^*(\Omega) = \mu$ then every solution to problem* (2.6) *blows up in finite time, while if $\lambda \le \lambda^*$ then the solutions are global.*

See [30, 71, 76, 78]. More precisely, when $\lambda < \lambda^*$, all the solutions tend to zero (like a negative power of t), and if $\lambda = \lambda^*$, the solution is globally bounded and stays away from zero. In the proof of blow-up the first eigenfunction plays its role, as before. The non blow-up case is based in a careful study of certain elliptic problem with a singular nonlinearity. We remark also by passing that a striking property, that distinguishes this problem $s > 1$ from the porous medium equation (2.4), $0 < s < 1$, is the constancy of the support, see [20, 80].

Nonlocal equations

We end this section by showing a similar result for a non-local example.

$$
\begin{cases}
u_t = \Delta u + \lambda \dfrac{f(u)}{(\int_\Omega f(u))^p} & \text{in } \Omega,\ 0 < t < T, \\
u = 0 & \text{on } \partial\Omega,\ 0 < t < T, \\
u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}.
\end{cases}
\tag{2.7}
$$

In [8] the authors consider the case $f(u) = e^u$ and dimensions $N = 1, 2$. They prove that the critical exponent is $p = 1$, but the coefficient is important whenever $0 < p < 1$. When $p \geq 1$ all the solutions are global. For related results see also [44, 57, 65].

Theorem 2.7. *Let $N \leq 2$ and $0 < p < 1$. If $\lambda > \lambda^*(p, \Omega)$ then every solution to problem* (2.7) *with $f(u) = e^u$ blows up in finite time, while if $\lambda \leq \lambda^*$ then there exist global solutions and also blow-up solutions.*

The proof follows from the characterization of the stationary solutions, as in Theorem 2.1, see [8].

3. REACTION vs. FIRST ORDER TERMS

Hamilton-Jacobi type terms

We begin this section by introducing a term involving the gradient, in Hamilton-Jacobi type, in the previously studied reaction-diffusion models. This new term will act as an absorption, and we want to study the competition between this absorption and the reaction. A model problem under consideration is

$$
\begin{cases}
u_t = \Delta u + \lambda u^p - |\nabla u|^q & \text{in } \Omega \subseteq \mathbb{R}^N,\ 0 < t < T, \\
u = 0 & \text{on } \partial\Omega \text{ (if not } \mathbb{R}^N),\ 0 < t < T, \\
u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}.
\end{cases}
\tag{3.1}
$$

This problem was posed by [18]. We fix $p > 1$, necessary to get blow-up. The general question of when there exist blow-up solutions is not completely solved for general domains. The critical exponent seems to be $p^* = q$, [61], though the exponent $p^{**} = q/(2-q)$ also appears when $\Omega = \mathbb{R}^N$, [25, 67]. See also [2, 68, 69].

Proposition 3.1. *Consider problem* (3.1). *If $p > q \geq 1$ then large solutions blow up. If $p \leq q$ and Ω is bounded, then all solutions are global. The same holds when $\Omega = \mathbb{R}^N$ if the initial datum has compact support.*

As to the subject of this paper, in [69] the critical case $p = q$ is considered, and the following result on no blow-up is obtained. It requires some condition on the domain such as being bounded in one direction. More precisely, it imposes the following Poincaré inequality

$$\|v\|_{L^q(\Omega)} \le C(q,\Omega)\|\nabla v\|_{L^q(\Omega)}, \qquad \forall\, v \in W_0^{1,q}(\Omega). \tag{3.2}$$

Observe that (3.2) does not hold if $\Omega = \mathbb{R}^N$.

Theorem 3.2. *Let Ω be a domain for which the Poincaré inequality* (3.2) *holds. Assume also $p = q$ and $\lambda > 0$ is small. Then all the solutions to problem* (3.1) *are global.*

The existence of blow-up solutions when $p = q$ and λ is large (or $\Omega = \mathbb{R}^N$) is at present unknown.

Besides the condition u_0 being large to have blow-up if $p > q$, other conditions of more specific type are used, often with the restriction $p \ge p^{**}$. Regarding the role of the coefficient λ in the equation in the case $p = p^{**}$, in [67] the following result is included.

Theorem 3.3. *Let $\Omega = \mathbb{R}^N$. Assume $p = p^{**}$, with $1 \le q < 2$, $1 < p < (N+2)/((N-2)_+)$, and $\lambda > 0$ large. If the initial datum is such that $u_t \ge 0$, then the solution to problem* (3.1) *blows up.*

The number $p^{**} = q/(2-q)$ comes from dimensional analysis. In fact if $p \ne p^{**}$ the coefficient λ can be scaled out by a simple change of variables. Even more, if $p = p^{**}$, and u solves the equation in (3.1) in $\mathbb{R}^N$, then so does $u_\alpha(x,t) = \alpha^{2/(p-1)}u(\alpha^2 x, \alpha t)$. This is related with self-similarity. It also appears as a limit case in energy estimates, see [18, 66]. Finally, it appears in the study of the corresponding stationary problem in $\mathbb{R}^N$, [27]. On the other hand, the number $p_S = (N+2)/((N-2)_+)$ is called the *critical Sobolev exponent*, and it is also related with the stationary problem.

Generalizations of problem (3.1) usually consider substituting the diffusion by the porous medium operator Δu^m, and the gradient by ∇u^n, thus putting

$$\begin{cases} u_t = \Delta u^m + \lambda u^p - |\nabla u^n|^q & \text{in } \Omega \subseteq \mathbb{R}^N,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega \text{ (if not } \mathbb{R}^N),\ 0 < t < T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } \overline{\Omega}. \end{cases} \tag{3.3}$$

We consider $m \ge 1$, $p > 1$, $n \ge 1$ and $1 \le q < 2$. See [4, 5, 68, 76, 77]. The critical exponent must be $p^* = nq$. There exist only very partial results concerning blow-up, and the complete picture is far to be understood. In [5] the corresponding result to Proposition 3.1 has been proved. The critical case $p = p^*$ has been considered also in that paper, but with the restrictions $q = 1$ and $m = n$. They show that the coefficient λ plays its role in that case.

Theorem 3.4. *Fix the exponents $q = 1$, $m = n > 1$, and $p = n$ in problem (3.3) and let Ω be bounded. Then if $\lambda < 1/4$ the solution is global, while there exists some $\lambda^* > 0$ such that if $\lambda > \lambda^*$ then large solutions blow up.*

The case $\Omega = \mathbb{R}^N$ is open.

Convection

We introduce now a problem very similar to the one considered above:

$$(3.4)\qquad \begin{cases} u_t = \Delta u^m + \lambda u^p + a \cdot \nabla u^q & \text{in } \Omega \subseteq \mathbb{R}^N,\ 0 < t < T, \\ u = 0 & \text{on } \partial\Omega \text{ (if not } \mathbb{R}^N),\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases}$$

Here the gradient term is of generalized Burgers' type, with $a \in \mathbb{R}^N$. If the initial value is not monotone, for instance with compact support, then this new term neither acts as an absorption nor as a reaction. Even the semilinear case $m = 1$ is not completely solved, though it is clear that the critical coefficient must be always $p^* = q$. We refer for instance to the works [1, 31, 50, 69, 70]. In particular in [1] the case $m = 1$ and $\Omega = \mathbb{R}^N$ is considered: there exist blow-up solutions whenever $p \geq q$ (provided as usual $p > 1$), and even all solutions blow up if $q \leq p \leq p_c$ for some $p_c = p_c(q, N)$. Blow-up when $p < q$ remains an open problem as far as we know, but if this blow-up occurs it necessarily has to take place at the point of infinity. The case $m > 1$ has been treated in [70], where the restriction $p \geq \max\{m, 2q - m\}$ is needed to get blow-up solutions. This means in particular $p \geq q$.

We observe that for problem (3.4), the similarity number is $p^{**} = 2q - m$. We want to describe the situation, not included in the above results, $p = 2q - m < m$, in which the coefficient λ reveals its importance. Recall that when $m = 1$ there is no room to choose $1 < p = 2q - 1 < 1$. In one dimension (put $a = 1$), and looking at the form of the possible self-similar solutions, as well as to the solutions in travelling wave form, [58], we anticipate the result in that case: the critical coefficient must be $\lambda^* = 2\sqrt{m/q}$.

4. BOUNDARY FLUX AS A REACTION

This section is devoted to study problems where blow-up is caused by a nonlinear boundary flux, rather than by a reaction in the interior of the domain. The best reference for this kind of problems is the survey [26]. See also, [35, 49, 62, 73, 81].

The model blow-up problem in this setting is

$$(4.1)\qquad \begin{cases} u_t = \Delta u & \text{in } \Omega,\ 0 < t < T, \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative. Our interest here lies in the study of this problem when some damping term in the form of an absorption in the equation or a negative flux in part of the boundary is added. More general diffusion equations are also considered. We focus our attention in the role played by a coefficient in critical cases of a balance between the opposite terms in the problem.

In this kind of problems, the following argument, absolutely nonrigorous, is helpful in order to understand the influence of each term. Differentiating the boundary condition, if it would hold extended from this boundary to the interior in some nontrivial set, we have that near the boundary the diffusion behaves like $f(u)f'(u)$. In fact, the analogous to condition (1.2) in order to get blow-up becomes here

$$\int_b^\infty \frac{ds}{f(s)f'(s)} < \infty, \qquad \text{for some } b > 0,$$

which has been proved in [73].

Flux vs. absorption

We begin by studying the following problem with absorption

$$\left\{\begin{array}{ll} u_t = \Delta u - u^q & \text{in } \Omega,\ 0<t<T, \\ \dfrac{\partial u}{\partial \nu} = \lambda u^p & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{array}\right. \tag{4.2}$$

Following the above heuristic argument, our equation becomes, near the boundary

$$u_t \sim \lambda^2 p u^{2p-1} - u^q.$$

The critical case is then $p^* = (q+1)/2$. It has been shown in [17], see also [52], that this is indeed the case, at least in the one dimensional or radial cases. For $p \neq p^*$ the effect of λ is irrelevant in first approximation: if $p > p^*$ there exist blow up solutions, while if $p < p^*$ then all solutions are global. As for the critical case $p = p^*$, the critical coefficient is $\lambda^* = 1/\sqrt{p}$, see [17]:

Theorem 4.1. *Let $p = (q+1)/2$ in problem (4.2). If $\lambda > \lambda^* = 1/\sqrt{p}$ then there exist blow up solutions, while if $\lambda < \lambda^*$ then all solutions are global.*

The case $\lambda = \lambda^*$ is only known in dimension one, and it belongs to the global existence case (though the solutions are unbounded, blow-up in infinite time). The study of stationary solutions is fundamental in the previous results. In fact, another way to see why the critical numbers p^* or λ^* appear, is to look at the corresponding stationary problem in one dimension (symmetric):

$$\left\{\begin{array}{ll} \varphi'' = \varphi^q & \text{for } 0<x<L, \\ \varphi'(0) = 0, & \\ \varphi'(L) = \lambda\varphi^p(L). & \end{array}\right.$$

A simple quadrature gives the following relation between the values of φ at both ends of the interval:

$$\varphi^{q+1}(0) = \varphi^{q+1}(L) - \frac{\lambda^2(q+1)}{2}\varphi^{2p}(L).$$

We recognize immediately the critical exponent relation $q+1=2p$, and if this holds, the critical coefficient relation is $\lambda^2(q+1)/2 = 1$.

The situation is more interesting in the quasilinear analogue

$$\begin{cases} u_t = \Delta u^m - u^q & \text{in } \Omega,\ 0<t<T, \\ \dfrac{\partial u^m}{\partial \nu} = \lambda u^p & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = u_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases}$$

See [6]. The critical exponent, in the same sense as above, turns to be in this case $p^* = \min\{\,(m+q)/2, \max\{q,1\}\,\}$. Recall that $p=p^*$ with $q\leq 1$ means $p=1$, and no solution can blow up in this case. When $p=p^*$, $q>1$, the authors in [6] show that the critical coefficient is explicit, $\lambda^* = \sqrt{m/p}$, if $q \geq m$, while it depends on the domain if $1<q<m$. The proofs in that work depend entirely on comparison arguments. We remark that the borderline case $\lambda = \lambda^*$ is not dealt with in the mentioned reference.

Closely related to the previous problems is the following one, whose absorption and flux terms include logarithmic nonlinearities:

$$\begin{cases} u_t = \Delta u - u\log^q u & \text{in } \Omega,\ 0<t<T, \\ \dfrac{\partial u}{\partial \nu} = \lambda u\log^p u & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = u_0(x) \geq 1 & \text{on } \overline{\Omega}. \end{cases} \tag{4.3}$$

This problem has been analyzed in [63] when $2p=q$, and in [23] for general p and q in the particular case of the half-line $\Omega = \mathbb{R}^+$. The introduction of logarithmic terms in blow-up problems, as an example of slightly superlinear reaction, is not merely academic, and shows behaviours typically limited to nonlinear diffusion, like regional or global blow-up sets, see for instance [36, 46].

The difference between this problem and the previous situation is easily seen when performing the change of variables $v = \log u$. Thus problem (4.3) translates into problem (4.2) with a Hamilton-Jacobi added term.

$$\begin{cases} v_t = \Delta v + |\nabla v|^2 - v^q & \text{in } \Omega,\ 0<t<T, \\ \dfrac{\partial v}{\partial \nu} = \lambda v^p & \text{on } \partial\Omega,\ 0<t<T, \\ v(x,0) = v_0(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases}$$

The result depends here on a competition between three terms. The argument shown at the beginning of this section gives the equation

$$v_t \sim \lambda^2 p v^{2p-1} + \lambda^2 v^{2p} - v^q.$$

First of all, $2p > 1$ is necessary to have blow-up. Also, we discover two critical cases, $2p-1 = q$ and $2p = q$. When $2p-1 = q$ the critical coefficient must be $\lambda^2 p = 1$, while if $2p = q$ it must be $\lambda^2 = 1$, though the coefficient λ is important whenever $2p-1 \le q \le 2p$. In fact, the characterization of stationary solutions in this interval depends on λ. In [23] the following result is obtained for the problem posed in the half-line.

Theorem 4.2. *Let $\Omega = \mathbb{R}^+$ in problem (4.3). If $p < \max\{q/2, 1/2\}$ or $p = 1/2$, then every solution is global. If $p > (q+1)/2$, then there exist both global and blow-up solutions. In the intermediate case $q/2 \le p \le (q+1)/2$ (with $p > 1/2$), the existence of global or blow-up solutions depends on λ. More precisely, the picture is:*

- if $p = q/2$ there exist blow-up solutions if and only if $\lambda > \lambda^* = 1$ or $\lambda = \lambda^*$ and $p > 1$;

- if $q/2 < p < (q+1)/2$, all solutions are global if $p > 2$ or $\lambda \le \lambda^*$; all solutions blow up if $1/2 < p \le 1$ and $\lambda > \lambda^*$;

- if $p = (q+1)/2$ all solutions are global if $p > 2$ or $\lambda < \lambda^* = 1/\sqrt{p}$; all solutions blow up if $1/2 < p < 1$ and $\lambda \ge \lambda^*$.

Therefore λ^* is explicit when $p = q/2$ or $p = (q+1)/2$. On the other hand, it remains as an open problem if every solution blows up in the range

$$\frac{q}{2} < p \le \frac{q+1}{2}, \quad 1 \le p \le 2, \quad \lambda > \lambda^*.$$

<u>Flux vs. convection</u>

In the following problem we replace the absorption term introduced in the previous examples by a convection, but acting again as an absorption. Thus we consider the problem in one dimension restricted to spatially nonincreasing solutions. We study in that way the balance between the boundary flux and the convection. The problem

$$\text{(4.4)} \qquad \begin{cases} u_t = (u^m)_{xx} + (u^q)_x & \text{in } (0,L),\ 0 < t < T, \\ -(u^m)_x(0,t) = \lambda u^p(0,t) & \text{for } 0 < t < T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } [0,L], \end{cases}$$

with $m \ge 1$, p, $q > 0$ has been studied, with different conditions on $x = L$ if L is finite, among others by [3, 59, 74]. We mention here the case of zero derivative $(u^m)_x = 0$ on $x = L$ and the case of the half-line, $L = \infty$. We remark that the real flux is $F = (u^m)_x + u^q$, thus the condition at the left border reads

$$-F(0,t) = \lambda u^p(0,t) - u^q(0,t).$$

The critical case is, obviously, $p = q$, with critical coefficient $\lambda^* = 1$.

Theorem 4.3. *Consider problem* (4.4) *with the condition* $(u^m)_x(L,t) = 0$. *If* $p < q$ *and* $q > m$, *then the solution is global. If* $p > q$ *or* $q \le m$, *then the solution blows up. If* $p = q > m$ *and* $\lambda = q$ *then the solution blows up.*

See [74]. The proofs are based entirely on comparison with subsolutions and supersolutions. The case $\lambda \ne q$ when $p = q > m$ is not treated in that paper, though it can be shown that the critical coefficient is $\lambda^* = 1$.

As to the half line, the result for $p \ne q$ is easy: If $p \le (m+1)/2$ or $p < q$, then the solution is global. If $p > \max\{(m+1)/2,\, q\}$, then there exist global solutions as well as blow-up solutions, see [59]. Assume now $p = q$, with $p > (m+1)/2$. In the mentioned paper the following result is included:

Theorem 4.4. *Consider problem* (4.4) *with* $L = \infty$ *and put* $p = q > (m+1)/2$. *If* $\lambda < \lambda^* = 1$, *then the solution is global. Let* $\lambda > 1$. *If* $p \le m+1$, *then all solutions blow up, while if* $p > m+1$ *then there exist global solutions as well as blow-up solutions. Let* $\lambda = 1$. *If* $p \le m+1$, *then the solution is global.*

It remains as an open problem if for the case $p = q$, $\lambda = 1$ and $p > m+1$ there exist blow-up solutions or not, though it can be shown that there cannot exist blow-up subsolutions in self-similar form. On the other hand, in the analogous conservative problem posed in a bounded interval, i.e., with the condition $(u^m)_x(L,t) = u^p(L,t)$, the driving quantity is the mass: if the mass is large enough, then the solution blows up, see for instance [19] in the case $m = 1$. In problem (4.4) there exist global solutions with arbitrarily large mass. Also, if some solution blows up, then it can be proved that there exist blowing-up solutions with arbitrarily small mass.

Positive and negative fluxes

We now consider the joint effect of two opposite fluxes in the quasilinear heat equation in an interval. More precisely, we study, for m, p, q, λ, $L > 0$, the following problem:

$$
\begin{cases}
u_t = (u^m)_{xx} & \text{in } (0,L),\ 0 < t < T, \\
-(u^m)_x(0,t) = \lambda u^p(0,t) & \text{for } \ 0 < t < T, \\
(u^m)_x(L,t) = -u^q(L,t) & \text{for } \ 0 < t < T, \\
u(x,0) = u_0(x) \ge 0 & \text{on } [0,L].
\end{cases}
\tag{4.5}
$$

If we put $M(t) = \int_0^L u(x,t)\,dx$, we easily have

$$
M'(t) = \lambda u^p(0,t) - u^q(L,t).
\tag{4.6}
$$

Therefore, at first sight the critical exponent should be $p = q$, at least if the solution behaves in the same way at all the points in $[0,L]$. This has been proved in [24] by showing that when $p \ne q$ the picture does not depend on λ: all solutions blow up if

$m < p < q$, all solutions are global if $p \le \max\{1, \min\{q, m\}\}$ and there exist both global and blow-up solutions if $p > \max\{1, q\}$.

When $p = q$, the authors in [24] prove that the critical coefficient is $\lambda^* = 1$ if $p \ne m$ and $\lambda^* = 1/(L+1)$ if $p = m$, though the critical value not always belongs to the blow-up case. Observe that again the critical value depends on the domain in some cases and in some other cases does not.

Theorem 4.5. *Consider problem* (4.5) *and let* $p = q > 1$. *All solutions are global if* $p \le (m+1)/2,\ \lambda \le 1$, *or* $(m+1)/2 < p < m,\ \lambda < 1$, *or* $p = m,\ \lambda \le 1/(L+1)$. *All solutions blow up if* $p \le (m+1)/2,\ \lambda > 1$, *or* $p = m,\ \lambda > 1/(L+1)$, *or* $p > m,\ \lambda > 1$. *In the complement there exist blow-up solutions as well as global solutions.*

This result can be stated in other words as follows: There exist blow-up solutions if $\lambda > 1$ for $1 < p \le (m+1)/2$ or $p > m$; if $\lambda \ge 1$ for $(m+1)/2 < p < m$; if $\lambda > 1/(L+1)$ for $p = m$. Let us see how the numbers $\lambda^* = 1$ and $\lambda^* = 1/(L+1)$ appear. Assume for simplicity that the solution is spatially nondecreasing (it holds by only assuming this property to the initial datum). Then in (4.6) we have, when $p = q$,

$$M'(t) \ge (\lambda - 1)u^p(0,t) \ge cM^p(t),$$

provided $\lambda > 1$. Therefore $p > 1$ implies blow-up. In the case $p = q = m$ this estimate can be sharpened if we assume in addition the convexity of u^m (again this holds if $(u_0^m)'' \ge 0$). Thus we have

$$\frac{u^m(L,t) - u^m(0,t)}{L} \le (u^m)_x(L,t) = -u^m(L,t).$$

This implies $(L+1)u^m(L,t) \le u^m(0,t)$, which subtituted in (4.6) gives

$$M'(t) \ge (\lambda - 1/(L+1))u^p(0,t) \ge cM^p(t),$$

provided $\lambda > 1/(L+1)$. Another point of view is to look at the stationary solutions, which are particularly simple in the case $p = q$. See [24].

Nondivergence diffusion with boundary flux

We end this section with an example that combines a nonlinear boundary flux like the above with a diffusion in nondivergence form, as in problem (2.6) of the precedent section. We show here that the critical coefficient depends again on the length of the interval considered. In particular we consider the problem

$$\tag{4.7} \begin{cases} u_t = u^s u_{xx} & \text{in } (0,L),\ 0 < t < T, \\ -u_x(0,t) = \lambda u^p(0,t) & \text{for } \ 0 < t < T, \\ u(L,t) = 0 & \text{for } \ 0 < t < T, \\ u(x,0) = u_0(x) \ge 0 & \text{on } [0,L], \end{cases}$$

with $s \geq 1$, $p > 0$. It is easy to see that the critical power in the flux imposed is $p = 1$ for every s. In fact if $p < 1$ the solution is global, while if $p > 1$ the solution blows up if only $u_0(0)$ is large. As to the exponent s in the equation, critical values are $s = 1$ and $s = 2$, see [22, 23, 30]. We recall that in [22] the problem considered is not directly (4.7), but the quenching problem associated to the function $v = u^{1-s}$, $s > 1$, i.e., the question of the blow-up of v_t as v decreases to zero. We fix here $s = p = 1$ in order to describe easily the results involving the coefficient λ. We find that the critical coefficient depends again on the domain [23]:

Theorem 4.6. *Consider problem* (4.7) *with* $s = p = 1$. *If* $\lambda > \lambda^*(L) = 1/L$ *then every solution blows up in finite time, while if* $\lambda \leq \lambda^*$ *then the solution is globally bounded.*

The basis of that theorem is the stationary character of the support, see [20, 23], contrary to what occurs for $0 < s < 1$. This can be interpreted in the following way: a local (in space) solution defined for $0 < x < R$ with $u(R,t) = 0$ can be extended by zero getting a solution defined in $(0, \infty)$. In particular this allows for the existence of stationary solutions

$$U(x) = A(1 - \lambda x)_+ \,, \qquad A > 0.$$

Also there exist blow-up solutions in self-similar form

$$V(x,t) = \frac{(x^2 - 2\lambda B x + 2B)\chi_{[0,R]}(x)}{2(T-t)},$$

$$R = \lambda B - \sqrt{\lambda^2 B^2 - 2B}, \quad B \geq \frac{2}{\lambda^2}.$$

Observing that $R \to 1/\lambda$ as $B \to \infty$, the theorem follows.

Another consequence of the constancy of the support is that we can pose the problem in $\mathbb{R}^+$ with compactly supported initial data. In this way the value L above comes from the support of u_0.

5. **OTHER PHENOMENA**

There exist a lot of other directions in which focus the investigation of the blow-up phenomenon related to the influence of some coefficient in the problem. We mention here just a few.

Size of the initial data

When in some problem the property of blow-up holds only for large solutions, this implies that for each initial datum φ_0 there exists a critical coefficient $\lambda^* = \lambda^*(\varphi_0)$ such that the initial value $\lambda\varphi_0$ produces blow-up for $\lambda > \lambda^*$ while it produces a global solution whenever $\lambda < \lambda^*$. For instance, we consider the problem

$$(5.1) \qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \Omega,\ 0<t<T, \\ u = 0 & \text{on } \partial\Omega,\ 0<t<T, \\ u(x,0) = \lambda\varphi_0(x) \ge 0 & \text{on } \overline{\Omega}. \end{cases}$$

The study of the solution corresponding to $\lambda = \lambda^*$ has attracted a great interest since the work [56] for problem (5.1). In some cases a global unbounded solution is obtained. This phenomenon is called *grow-up*, or blow-up in infinite time. See also [34, 37].

In the same way, a possible continuation of the solution, beyond the blow-up time, could also depend on the size of the initial data. This is the subject of [54] for problem (5.1) in $\mathbb{R}^N$. *Continuation* is a very active area in blow-up problems, with interest by itself, see for instance the survey [38].

Instantaneous blow-up

Throughout this paper we always understood blow-up in L^∞ sense. Thus the solution is bounded for all times $0<t<T$ and becomes unbounded at $t=T$. On the other hand, we have already seen in Section 2 for the exponential reaction problem that in some cases there exists a singular stationary solution. This means that we can consider a different framework and treat solutions in the so-called energy space H^1. At this respect it is interesting to consider functions close to the singular stationary one and study the evolutionary problem. More generally, we take the log function as initial datum, even in the cases when it is not a stationary solution. We thus consider the problem

$$(5.2) \qquad \begin{cases} u_t = \Delta u + \lambda e^u & \text{in } \mathbb{R}^N,\ 0<t<T, \\ u(x,0) = u_0(x) \approx U(x) = -2\log|x| & \text{in } \mathbb{R}^N. \end{cases}$$

The condition on the initial value is understood as $\lim_{x\to 0}(u_0(x) - U(x)) = 0$, u_0 bounded for $x \ne 0$. By a limiting procedure, it has been proved that the corresponding solution can be identically infinite for every $t>0$. This is called *instantaneous blow-up*. This phenomenon depends again on λ.

Theorem 5.1. *If $\lambda > \lambda^*$ then the solution to problem* (5.2) *blows up instantly, while if $\lambda \le \lambda^*$ then the solution exists at least locally in time. The critical coefficient is explicit $\lambda^* = 2(N-2)$ if $N \ge 10$. Also, $\lambda^* > 2(N-2)$ if $3 \le N \le 10$, $\lambda^* > 1$ if $N=2$, and $0<\lambda^*<1$ if $N=1$.*

See [12, 42, 60, 72]. We remark that $U(x)$ is a solution to the equation in problem (5.2) only if $\lambda = 2(N-2)$, $N \ge 3$, and it is the unique solution with that initial value only if $N \ge 10$, [42]. In this case, $N \ge 10$ and $\lambda = 2(N-2)$, the singular solution $U(x)$ is an attractor from below [60]. Also, in that paper it is proved that if $u(x,0) \ge U(x)$, $u_0 \not\equiv U$, then instantaneous blow-up occurs. This implies $\lambda^* = 2(N-2)$. The case $3 \le N \le 9$ follows from the construction of a solution in the self-similar form $w(x,t) = f(r) - \log t$, $r = |x|/\sqrt{t}$, see [72]. The behaviour $f(r) \sim -2\log r + c$ for $r \to \infty$ is crucial. It is translated into an initial condition $w(x,0) = -2\log|x| + c$. Observe that, by rescaling, for each $\lambda > 0$ the singular function $U(x)$ corresponds to $f(r) = -2\log r + \log\lambda - \log(2(N-2))$. In fact, such self-similar solutions exist only for $c \le c^*$, and some $c^* > 0$. The critical coefficient is then $\lambda^* = 2(N-2)e^{c^*}$. In the lower dimensions $N = 1$ or $N = 2$, the same analysis applies, but here with $\lambda^* = e^{c^*}$.

A similar investigation has been performed for the linearized problem around the singular solution. Here a typical inverse square potential appears. See [7, 15].

<u>Superfast blow-up</u>

An important aspect in the description of the blow-up solutions for a given problem is the speed at which these solutions go to infinity. This is called the *blow-up rate*. When this rate, in a reaction-diffusion problem, is the same as for the pure reactive (ODE) equation, we say that the rate is *natural*, or of *Type I*. In general, dimensional considerations have to be taken into account. For instance, in the power reaction example (1.1) with $f(u) = u^p$, the natural rate is $\alpha = 1/(p-1)$, in the sense that

$$\|u(\cdot,t)\|_\infty \le c(T-t)^{-1/(p-1)}, \tag{5.3}$$

see [29, 39, 75]. In the boundary flux case (4.1), again with $f(u) = u^p$, the natural rate is $\alpha = 1/(2(p-1))$. See [41].

The first example of a solution not satisfying (5.3) for (1.1) is due to [40] for $N > 10$ and what is called supercritical exponents, $p > (N+2)/(N-2)$. They denoted this example as *superfast blow-up*. It is now also known as blow-up of *Type II*, see also [28, 55].

Prior to this result for problem (1.1), an example of superfast blow-up was presented in [30] for the nondivergence problem (2.6), with $s = 2$ and $p = 3$. For the case $s > 2$, $p = s + 1$ we refer to [10, 78, 79].

We present here a simple example, related with (2.6), see also (4.7), in which the question of whether blow-up is natural or superfast depends on a critical coefficient. We consider diffusion in nondivergence form and a nonlinear boundary flux at $x = 0$. But contrary to problem (4.7), we impose here a zero flux condition at $x = L$.

$$\begin{cases} u_t = u^s u_{xx} & \text{in } (0,L),\ 0 < t < T, \\ -u_x(0,t) = \lambda u(0,t) & \text{for } \ 0 < t < T, \\ u_x(L,t) = 0 & \text{for } \ 0 < t < T, \\ u(x,0) = u_0(x) & \text{on } [0,L]. \end{cases} \tag{5.4}$$

We also assume an initial condition $u_0 \ge \delta > 0$, decreasing and convex. In this way the solution always blows up, at least at $x = 0$, and is increasing in t, decreasing and convex in x. Dimensional analysis gives that the natural rate is $\alpha = 1/s$, see also the argument at the beginning of Section 4. The critical case here is $s = 2$, [22].

Theorem 5.2. *Consider problem* (5.4). *If $s < 2$ then blow-up is always natural, i.e.,*

$$\|u(\cdot,t)\|_\infty = u(0,t) \le c(T-t)^{-1/s}.$$

Let $s \ge 2$. If $\lambda \le \lambda^*(L)$ then blow-up is natural, while if $\lambda > \lambda^*$ then blow-up is superfast, i.e.,

$$\lim_{t\to T}(T-t)^{1/s}u(0,t) = \infty. \tag{5.5}$$

Moreover, $\lambda^* = 1/L$ if $s \ge 4$, $\lambda^* > 1/L$ if $2 \le s < 4$.

In fact (5.5) holds in the whole blow-up set, which if $\lambda > \lambda^*$ is exactly the interval $[0, 1/\lambda]$. In this set the solution blows up with the same (unknown) rate. The following asymptotic behaviour shows this fact

$$\lim_{t\to T}\frac{u(x,t)}{u(0,t)} = (1-\lambda x)_+ \qquad \text{for every } \ 0 \le x \le L.$$

See [22]. The existence or nonexistence of self-similar solutions plays a fundamental role in the proof of the above results.

ACKNOWLEDGMENTS

This research work is partially supported by grant MTM2005-08760-C02-01 (Spain).

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