

BLOW-UP PHENOMENA IN SOME POROUS MEDIUM PROBLEMS

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ABSTRACT. We consider a Dirichlet type initial-boundary value problem for the porous medium equation with a power function reaction term. We determine a condition on the initial data which ensures blow-up of the solution in finite time and an upper bound for the blow-up time. We also discuss when blow-up does not occur and a more general initial-boundary value problem where blow-up does occur.

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1. INTRODUCTION

There is a vast number of papers in the literature which deal with nonlinear evolutionary processes and study the behavior of the solution to initial or initial-boundary value problems that model the process. Due to the nonlinearity and/or data, some problems that arise in chemical reactions, gaseous ignition, porous media, ohmic heating, and chemotaxis in biological systems exhibit explosive growth of the solution. Studies of these problems are often concerned with the existence/nonexistence of global solutions or the blow-up of the solution in finite time, where by blow-up, we mean that the solution becomes unbounded in some manner in finite time. Various criteria and conditions on the nonlinearity which imply blow-up does occur have been presented and bounds on the blow-up rate or blow-up time, structure of the blow-up set, and the asymptotic behavior of the solution have been determined. The papers by Levine [6], Galaktionov and Vázquez [3], and Bandle and Brunner [2] have an extensive list of references which deal with these and related investigations and applications. In addition, the text by Straughan [14], which studies the explosive behavior of solutions to problems in mechanics, has a large bibliography (see also [11]). Historically, the study of the phenomena of blow-up in reaction-diffusion equations began with the seminal paper by Kaplan [4].

Many methods have been used in the study of blow-up phenomena (see the list in [5] or [2]) and they often lead to upper bounds on the blow-up time when blow-up does occur. Little attention appears to have been given to the determination of lower bounds for the blow-up time. In fact, lower bounds are more important because of the explosive nature of the process which is being modeled. Recently, Payne and Schaefer

[7] used a first order differential inequality technique on a semilinear parabolic problem under homogeneous Dirichlet condition to determine a lower bound on the blow-up time if blow-up occurs (see also [8] for homogeneous Neumann boundary condition). These results have been extended to more general nonlinear parabolic problems in [9] and [10]. More recently, lower bounds for blow-up time for some porous medium problems have been determined in [13].

In this note we continue the study of porous medium problems. In section 2 we determine a criterion on the initial data which ensures that blow-up does occur and an upper bound for the blow-up time. We cite conditions for which blow-up does not occur in section 3. In section 4 we introduce alternative conditions (to [9]) on the nonlinear function in a more general partial differential equation (that includes the porous medium equation) which leads to blow-up and an upper bound on the blow-up time.

2. CRITERION FOR BLOW-UP

In this section we consider a class of porous medium problems for which a sufficient condition on the initial data is determined which ensures that blow-up of the solution at some finite time does occur. In addition, an upper bound for the blow-up time t^* is determined.

We consider the initial-boundary value problem

$$(2.1) \quad \begin{aligned} u_t &= \Delta(u^m) + ku^p && \text{in } \Omega \times (0, t^*), \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) &= g(x) \geq 0 && \text{in } \Omega, \end{aligned}$$

where Ω is a bounded smooth domain in \mathbb{R}^N . In (2.1), the subscript t denotes partial differentiation with respect to time, Δ is the N -dimensional Laplacian, g is a continuous function in $\overline{\Omega}$ which vanishes on the boundary $\partial\Omega$ (for compatibility), k is a positive constant, and m and p are parameters such that $p > m \geq 1$. It is well known [1] that if $m = 1$, $p > 1$, then the solution blows up in finite time. We assume that a nonnegative classical solution exists for some period of time and aim to show that the solution blows up at some finite time t^* .

We define the auxiliary function

$$(2.2) \quad \varphi(t) = \int_{\Omega} u^{m+1} dx$$

and compute

$$(2.3) \quad \begin{aligned} \varphi'(t) &= (m+1) \int_{\Omega} u^m [\Delta(u^m) + ku^p] dx \\ &= -(m+1) \int_{\Omega} \nabla u^m \cdot \nabla u^m dx + k(m+1) \int_{\Omega} u^{m+p} dx \end{aligned}$$

$$(2.4) \quad > -\frac{(m+1)(m+p)}{2m} \int_{\Omega} |\nabla u^m|^2 dx + k(m+1) \int_{\Omega} u^{m+p} dx,$$

where ∇ is the gradient operator and we have used integration by parts and the fact that $p > m$. We now define the right side of this inequality to be the function

$$(2.5) \quad \psi(t) = -\frac{(m+1)(m+p)}{2m} \int_{\Omega} |\nabla u^m|^2 dx + k(m+1) \int_{\Omega} u^{m+p} dx$$

and compute in a similar manner

$$(2.6) \quad \begin{aligned} \psi'(t) &= -\frac{(m+1)(m+p)}{m} \int_{\Omega} \nabla u^m \cdot (\nabla u^m)_t dx \\ &\quad + k(m+1)(m+p) \int_{\Omega} u^{m+p-1} u_t dx \\ &= \frac{(m+1)(m+p)}{m} \int_{\Omega} (u^m)_t [u_t - k u^p] dx \\ &\quad + k(m+1)(m+p) \int_{\Omega} u^{m+p-1} u_t dx \\ &= (m+1)(m+p) \int_{\Omega} u^{m-1} (u_t)^2 dx. \end{aligned}$$

Since $\psi'(t) \geq 0$, it follows that if $\psi(0) > 0$, we have both

$$(2.7) \quad \psi(t) > 0, \quad \varphi'(t) > 0 \quad \text{for } t > 0.$$

We note that the condition $\psi(0) > 0$ imposes the following constraint on the initial data:

$$(2.8) \quad \int_{\Omega} g(x)^{m+p} dx > \frac{m+p}{2m} \int_{\Omega} |\nabla g^m|^2 dx.$$

Under the assumption $\psi(0) > 0$, the functions $\varphi(t)$, $\psi(t)$, $\varphi'(t)$, and $\psi'(t)$ are all positive valued for $t > 0$. Moreover, we can write

$$(2.9) \quad \begin{aligned} \varphi'(t) &= (m+1) \int_{\Omega} u^{\frac{m+1}{2}} u^{\frac{m-1}{2}} u_t dx \\ &\leq (m+1) \left(\int_{\Omega} u^{m+1} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} u^{m-1} (u_t)^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

by Schwarz's inequality so that by (2.2), (2.6), and (2.9) it follows that

$$(2.10) \quad \varphi(t)\psi'(t) \geq \frac{m+p}{m+1} \varphi'(t)^2 \geq \frac{m+p}{m+1} \psi(t)\varphi'(t), \quad t > 0.$$

We now integrate (2.10) from 0 to t and obtain

$$\frac{\psi(t)}{\psi(0)} \geq \left[\frac{\varphi(t)}{\varphi(0)} \right]^{\frac{m+p}{m+1}}.$$

Letting

$$c = \frac{m+p}{m+1} > 1,$$

we have that

$$\varphi'(t) \geq \psi(t) \geq \psi(0)\varphi(0)^{-c}\varphi(t)^c.$$

A further integration results in

$$(2.11) \quad \frac{1}{\varphi(t)^{c-1}} \leq \frac{1}{\varphi(0)^{c-1}} - (c-1) \frac{\psi(0)}{\varphi(0)^c} t.$$

Since (2.11) cannot hold for all t , we conclude that the solution blows up at some finite time t^* and that

$$(2.12) \quad t^* \leq \frac{1}{c-1} \frac{\varphi(0)}{\psi(0)}.$$

We formalize this result in the following theorem.

Theorem 2.1. *If u is a nonnegative classical solution of (2.1) where $p > m \geq 1$ and the initial data g satisfies (2.8), then the solution blows up in the measure φ in finite time t^* and t^* is bounded above by (2.12).*

We note that Theorem 2.1 holds more generally for the problem (2.1) when the differential equation is replaced by the inequality

$$u_t \geq \Delta(u^m) + f(u),$$

where $f(u) \geq ku^p$. Moreover, the result follows when the Dirichlet boundary condition is replaced by a homogeneous Neumann condition since (2.3) remains valid.

3. NONBLOW-UP

We again consider the problem (2.1) but now ask that $m \geq p > 1$. In this case, we show that the solution remains bounded for all time when a restriction is imposed on the constant k .

We define φ as in (2.2) and make use of the Rayleigh principle

$$\lambda_1 \int_{\Omega} v^2 dx \leq \int_{\Omega} |\nabla v|^2 dx,$$

where λ_1 is the first positive eigenvalue of the fixed membrane problem

$$(3.1) \quad \Delta v + \lambda v = 0, \quad v > 0, \quad \text{in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

From (2.3) we have

$$\begin{aligned} \varphi'(t) &= -(m+1) \int_{\Omega} |\nabla u^m|^2 dx + k(m+1) \int_{\Omega} u^{m+p} dx \\ &\leq -(m+1)\lambda_1 \int_{\Omega} u^{2m} dx + k(m+1) \int_{\Omega} u^{m+p} dx \\ &= (m+1) \left(k \int_{\Omega} u^{m+p} dx - \lambda_1 \int_{\Omega} u^{2m} dx \right) \end{aligned}$$

$$\leq (m + 1)(k - \lambda_1) \int_{\Omega} u^{2m} dx.$$

Thus, when $k \leq \lambda_1$, we have $\varphi'(t) \leq 0$ which implies that u is bounded since otherwise we have a contradiction. We state this result in the following theorem.

Theorem 3.1. *If u is a nonnegative classical solution of (2.1) where $m \geq p > 1$ and $k \leq \lambda_1$, the first positive eigenvalue of (3.1), then u is bounded for all time.*

As in the previous section, we note that the result holds more generally when the differential equation is replaced by the inequality

$$u_t \leq \Delta(u^m) + f(u),$$

where $0 \leq f(u) \leq ku^m$, $m > 1$, and that the result holds under a homogeneous Neumann boundary condition as well. In the latter case, λ_1 is the first positive eigenvalue of the free membrane problem.

4. OTHER BLOW-UP PROBLEMS

In [9] the authors considered the nonlinear problem

$$\begin{aligned} (4.1) \quad u_t &= (\rho(u)u_{,i})_{,i} + f(u) \quad \text{in } \Omega \times (0, t^*), \\ u(x, t) &= 0 \quad \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) &= g(x) \geq 0 \quad \text{in } \Omega, \end{aligned}$$

under a general set and a special set of conditions on f and ρ (see (2.2) and (3.1), respectively, in [9]) and determined lower bounds on blow-up time in each case as well as determined when blow-up does not occur. The arguments there were restricted to $\Omega \subset \mathbb{R}^3$ for technical reasons. The comma i notation in (4.1) denotes spatial differentiation and the repeated index in a term implies summation over the index from 1 to N .

We now assume a different set of conditions on ρ and f in (4.1) for $\Omega \subset \mathbb{R}^N$, namely,

$$\begin{aligned} (4.2) \quad f(0) &= 0, \quad f(s) > 0, \quad f''(s) > 0 \quad \text{for } s > 0, \\ \rho(0) &= 0, \quad \rho(s) > 0, \quad \text{for } s > 0, \\ \int_{\bar{g}}^{\infty} \frac{d\eta}{f(\eta)} &\leq M < \infty, \end{aligned}$$

where \bar{g} denotes the mean value of the initial data over Ω , and show that no global C^1 solution can exist, i.e., that $u \in C^1(\bar{\Omega} \times (0, t^*))$ blows up in finite time t^* and that $t^* \leq M$. We shall use a simple argument that is well suited for a problem like (4.1) with the Dirichlet condition replaced by a homogeneous Neumann condition (see [11]). We note that although the blow-up result for (4.1), (4.2) is extended to a

bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we are restricted by the need to assume that the outward normal derivative $\frac{\partial u}{\partial \nu}$ is bounded on $\partial\Omega$.

We define \bar{u} to be the mean value of u over Ω , i.e.,

$$\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u dx,$$

where $|\Omega|$ denotes the volume of Ω . Then by the divergence theorem and Jensen's inequality,

$$\begin{aligned} \frac{d}{dt} \bar{u} &= \frac{1}{|\Omega|} \int_{\Omega} [(\rho(u)u_{,i})_{,i} + f(u)] dx \\ &= \frac{1}{|\Omega|} \int_{\partial\Omega} \rho(u) \frac{\partial u}{\partial \nu} dx + \overline{f(u)} \\ &\geq f(\bar{u}), \end{aligned}$$

and on integration, we have

$$M \geq \int_{\bar{u}(0)}^{\bar{u}(t)} \frac{d\eta}{f(\eta)} \geq t.$$

However, this inequality cannot hold for all time t and we deduce that u blows up at some finite time $t^* \leq M$.

We summarize this result in the following theorem.

Theorem 4.1. *If $u \in C^2(\Omega \times (0, t^*)) \cap C^1(\bar{\Omega} \times (0, t^*))$ is a nonnegative solution of (4.1), (4.2), then u blows up in finite time t^* in the measure \bar{u} and t^* is bounded above by M .*

We note that since $\Delta(u^m) = (mu^{m-1}u_{,i})_{,i}$, we can let $\rho(u) = mu^{m-1}$, $m > 1$, in (4.1) and that Sato [12] analyzes a problem similar to (4.1), (4.2) by means of supersolutions and subsolutions in order to determine an asymptotic formula for the blow-up time as a parameter in the initial condition goes to infinity.

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