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LONG-TIME BEHAVIOR OF SOLUTIONS OF A NONLINEAR DIFFUSION MODEL WITH TRANSMISSION BOUNDARY CONDITIONS

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Dedicated in fond memory of M. Kolle

ABSTRACT. In order to accurately simulate the transport of growth factor from tumor site into a nearby capillary wall, a recently introduced model of tumor-induced capillary growth incorporates a new form of transmission boundary flux. Growth factor emitted from the tumor may be viewed as a diffusible chemical moving through intersticial space, which is represented as a porous medium. Transmission between the capillary wall and intersticial space gives rise to a type of continuous delay/memory condition at the boundary. Herein, we establish results on global solvability and blow up in finite time for a general nonlinear diffusion model, including such transmission boundary conditions. Although the model appears more closely aligned with models involving nonlinear flux conditions at the boundary, these results bear notable similarities to those with Dirichlet boundary conditions.

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1. PRELIMINARIES

We investigate the global solvability of a nonlinear diffusion model

$$\begin{aligned} &u_t = \nabla \cdot (\nabla \phi(u) + \epsilon \mathbf{f}(u)) + h(u) && \text{on } \Omega_T \\ &(\nabla \phi(u) + \epsilon \mathbf{f}(u)) \cdot \mathbf{n} = g(x,u,v) && \text{on } (\partial\Omega)_T \\ &u = u_0 && \text{on } \overline{\Omega} \times \{0\} \end{aligned} \tag{1.1}$$

with $g_v \geq 0$ on $(\partial\Omega)_T$; $g_v \equiv 0$ on $(\partial\Omega \setminus \Sigma)_T$. Here, T subscripts are used to denote the respective product sets, such as $\Omega_T \equiv \Omega \times (0,T)$. Ω is a bounded domain in $\mathbf{R}^N$ having piecewise smooth boundary $\partial\Omega$, and $\Sigma \subset \partial\Omega$ is a relatively open subset also with piecewise smooth boundary, i.e., locally C^2.

Where $g_v \neq 0$ on Σ, v is determined according to the transmission boundary condition

$$\begin{aligned} &v_t = F(u,v) + G(u)_t && \text{on } \Sigma_T \\ &v = v_0 && \text{on } \Sigma \times \{0\}. \end{aligned} \tag{1.2}$$

Physical motivation for this type of boundary condition arises in efforts to model tumor-induced capillary growth. In such case, $N = 2$ or 3, the capillary wall is located on all or part of Σ, and an early stage (non-vascularized) tumor is positioned on a portion of $\partial\Omega \setminus \Sigma$ at some positive distance from the capillary [6].

Towards understanding the role of the transmission condition in promoting or inhibiting blow up in finite time, we analyze a familiar case of power laws in (1.1). Although these results may be applied to more general nonlinearities, we focus mostly upon the power laws $\phi(u) = u^m$, $h(u) = au^p$, $p, m > 0$, $a \geq 0$. The convective model will be addressed in a one-dimensional case ($N = 1$, $\Omega = (0, 1)$, $\Sigma = \{1\}$) with $\epsilon > 0$, $\mathbf{f}(u) = f(u) = u^n$, $n > 0$.

Regarding the nonlinearities g, F, and G, we introduce general conditions below with particular attention to choices both in connection with those in the capillary growth model and in relation to known results for (1.1) when $g_v \equiv 0$. Specifically, those choices are

$$g(x, u, v) = \begin{cases} l \geq 0, & x \in \partial\Omega \setminus \Sigma \\ -\psi u^q + \beta v, & x \in \Sigma, \end{cases} \tag{1.3}$$

with $\psi, \beta \geq 0$, $q > 0$, $G(u) = Au^q$, and

$$F(u, v) = -\frac{\lambda v}{1 + \nu v} + Bu^q, \tag{1.4}$$

with constants $\lambda, \nu > 0$ and $A, B \geq 0$. If $q = 1$, $\psi = \beta > 0$, and λ is replaced with $\lambda\eta/\eta_0$, then these reduce to the case of the capillary growth model [6]. (Here, η, η_0 refer to endothelial cell densities.) Furthermore, the nonlinear flux model with $\beta = 0$, $\Sigma = \partial\Omega$, $\epsilon = 0$ has been addressed in the literature for the existence of a compact attractor ($\psi > 0$, $q \geq m > p \geq 1$) [5] and for characterization of global solvability ($\psi < 0$, $a > 0$, $p \leq 1$, $q \leq \min\{1, (m+1)/2\}$) [8, 9]. More discussion is provided later on the relationship between these results and those established in the present article.

For the forms of nonlinearities (1.3)–(1.4), one may formally integrate (1.2) and obtain $g \sim (-\psi + A\beta)u^q$ on Σ. Thus, we might anticipate $-\psi + A\beta < 0$ will give rise to a compact attractor and $-\psi + A\beta > 0$ will result in a characterization of blow up according to $q > 1$ or $p > 1$. Guided by these ideas, we first establish that solutions of (1.1)–(1.2) blow up in finite time in the following situations when $-\psi + A\beta > 0$ and u_0 is chosen appropriately "large".

(1) $N \geq 1$, $\epsilon = 0$
- $q, p \geq m > 1$
- $N = 1$, $p > 1$ or $q > \min\{1, (m + 1)/2\}$, and $u_0 \geq \delta > 0$

(2) $N = 1$, $\epsilon > 0$
- $n, q \geq m \geq 1$
- $p \geq \max\{m, n\}$

On the other hand, if $\psi, \beta > 0$, $p \leq m$, then (1.1)–(1.2) possesses an equilibrium solution. Thus, necessary and sufficient conditions $p \leq 1$, $q \leq \min\{1, (m + 1)/2\}$

for the global existence of all positive solutions to the boundary flux problem with $\beta = \epsilon = 0$, $-\psi > 0$ [8, 9] will not continue to allow a similar characterization for (1.1)–(1.2) when $-\psi + A\beta > 0$. In fact, the condition $p \geq m$ for blow up is more reminiscent of results for nonlinear reaction diffusion models with Dirichlet boundary conditions. Currently, we do not have methods developed to conclude global existence of solutions for (1.1)–(1.2) as might be expected in case $-\psi + A\beta < 0$, beyond what may be concluded from the existence of nontrivial equilibria.

Throughout this work, the nonlinear terms in (1.1)–(1.2), $\phi, \mathbf{f}, h, g, F$, and G, are assumed to be continuous, with the diffusion law assumed to satisfy $\phi_u > 0$ for $u > 0$. This is the standard parabolicity restriction sufficiently weakened to allow familiar cases of degenerate diffusion (e.g., porous medium flow). We also assume that $\mathbf{f}_u$, h_u, and g_u exist for $u > 0$, although such assumptions do not overly restrict the power laws which may be included. As we seek nonnegative solutions, it is assumed that $\phi(0) = 0$, $\mathbf{f}(0) = \mathbf{0}$, $h(0) \geq 0$, and $g(\cdot, 0, v) \geq 0$ for $v \geq 0$.

The initial conditions u_0, v_0 are nonnegative, L^∞ functions on $\overline{\Omega}$ and $\overline{\Sigma}$, respectively. As $v - G(u)$ is naturally involved in the weak definition of (1.2), initial conditions are additionally assumed to satisfy $v_0 - G(u_0) \geq 0$ on $\overline{\Sigma}$.

Regarding the nonlinearities in (1.2), we incorporate assumptions from our previously established local existence and comparison theory [1]. In such direction, F and G are required to satisfy the following for $u, v \geq 0$.

(i) G', F_u, and F_v are continuous,
(ii) $F(u, G(u)) \geq 0$,
(iii) $G(0) = 0$, $G(u) \geq 0$, $G'(u) \geq 0$, and
(iv) $F_u + F_v G' \geq 0$.

Rationale for these are provided, predominantly, by the need to build a model possessing nonnegative solutions. Such conditions are sufficient for the existence of a maximal solution, herein referred to as the solution, of (1.1)–(1.2) for which a subsolution comparison is available. However, beyond comparison of solutions, a general supersolution comparison theory is not known. As a result, global existence results are currently limited to development of equilibrium states. A more extensive discussion is available in [1]. In the case of the nonlinearities (1.3)–(1.4), the above assumptions require $q \geq 1$ and $\lambda A \leq B$.

In order to recruit its own blood supply, an early stage tumor emits a growth factor which flows through the intersticial space, or extra-cellular matrix as it is sometimes called, to the capillary wall. The solution of (1.1)–(1.2), u, v, denotes growth factor concentrations in Ω and Σ, respectively. In reality, however, a capillary wall does not have zero thickness as represented by Σ. To reflect this dimensionality difference in the transmission of growth factor taking place at the capillary wall, an important feature of the model is that $u(x,t) \neq v(x,t)$ for $x \in \Sigma$. Instead, the evolution of growth factor concentration in the capillary wall, v, is determined by the concentration (u) and time rate of change in concentration (u_t) of growth factor arriving at the capillary wall. (1.1)–(1.2) comprises a part of the model for growth of new capillary networks ("angiogenesis"), as initiated by a developing solid tumor, describing the evolution of growth factor concentration. See Levine et al. [6] for a discussion of the full system of nonlinear diffusion equations designed to model tumor-induced angiogenesis.

The present work is a portion of an ongoing effort to establish results on global solvability which are applicable to each of the various diffusion models contained within the full angiogenesis system introduced by Levine et. al [6]. Such results are of potential use in allowing future mathematical analyses of the qualitative behavior of solutions for the complete system. Additionally, the transmission condition (1.2) appears to be a new type of delay/memory boundary condition not currently addressed in the literature.

2. SUBSOLUTIONS AND BLOW UP RESULTS

Toward establishing blow up results for (1.1)–(1.2), we consider $v = G(u)$ on Σ. Noting that $F(u,v) = F(u,G(u)) \geq 0$ by the assumptions on F, G, the choice of $v = G(u)$ will therefore formally yield $(v - G(u))_t = 0 \leq F(u,v)$, which, upon precise weak formulation, corresponds with the requirements for a subsolution of (1.2). See [1] for the details. Subsequently, if u is a solution of

$$
\begin{aligned}
&u_t = \nabla \cdot (\nabla\phi(u) + \epsilon\mathbf{f}(u)) + h(u) && \text{on } \Omega_T \\
&(\nabla\phi(u) + \epsilon\mathbf{f}(u)) \cdot \mathbf{n} = g(x,u,G(u)) && \text{on } (\partial\Omega)_T \\
&u = u_0 && \text{on } \overline{\Omega} \times \{0\},
\end{aligned}
\tag{2.1}
$$

then u, $v = G(u)$ is a subsolution of (1.1)–(1.2).

From a slightly different perspective, solutions of (1.1)–(1.2) are nonnegative. Hence, if u is a solution of either

$$
\begin{aligned}
&u_t = \nabla \cdot (\nabla\phi(u) + \epsilon\mathbf{f}(u)) + h(u) && \text{on } \Omega_T \\
&u = 0 && \text{on } (\partial\Omega)_T \\
&u = u_0 && \text{on } \overline{\Omega} \times \{0\},
\end{aligned}
\tag{2.2}
$$

or the mixed problem

$$
\begin{aligned}
&u_t = \nabla \cdot (\nabla\phi(u) + \epsilon\mathbf{f}(u)) + h(u) && \text{on } \Omega_T \\
&(\nabla\phi(u) + \epsilon\mathbf{f}(u)) \cdot \mathbf{n} = g(x,u,G(u)) && \text{on } \Sigma_T \\
&u = 0 && \text{on } (\partial\Omega \setminus \Sigma)_T \\
&u = u_0 && \text{on } \overline{\Omega} \times \{0\},
\end{aligned}
\tag{2.3}
$$

then u, $v = G(u)$ is again a subsolution of (1.1)–(1.2).

To establish blow up for solutions of (1.1)–(1.2), we thus need only determine results on blow up applicable to one of the problems (2.1)–(2.3).

For the general N-dimensional model (2.1) when $\epsilon = 0$, the method of concavity is available. To formulate the terms needed in the method, define

$$\Phi(u) \equiv \int_0^u \phi(v)dv, H(u) \equiv \int_0^u h(v)dv, \text{ and } P(u) \equiv \int_0^u \phi'(v)g(v, G(v))dv.$$

Blow up for solutions of (2.3), and, hence of (1.1)–(1.2) is now a direct application of known concavity results [4, 7].

Theorem 2.1. *Assume Φ^κ is convex for some $0 < \kappa < 1/2$, and both $h(u)/\phi(u)$, $g(u, G(u))/\phi(u)$ are nondecreasing for $u > 0$. If*

$$\frac{1}{2}\int_\Omega |\nabla\phi(u_0)|^2\, dx < \int_\Omega H(u_0)dx + \int_\Sigma P(u_0)dS_x,$$

and $v_0 \geq G(u_0)$, then the solution of (1.1)–(1.2) *blows up in finite time, i.e.,*

$$\limsup_{t\to T^-} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$$

for some $T < \infty$. Since $v \geq G(u)$ [1], *v similarly blows up in finite time.*

As the solution of (2.1) has $\nabla\phi(u)\cdot\mathbf{n} = l \geq 0$ on $\partial\Omega\backslash\Sigma$, we obtain subsolutions of this problem via the boundary condition $\nabla\phi(u)\cdot\mathbf{n} = 0$ on $\partial\Omega\backslash\Sigma$. We may thus utilize blow up results for radial symmetric solutions for $\Omega = \{x : |x| < R\}$ [8, 9], which satisfy $|\nabla u| = 0$ for $|x| = 0$, as subsolutions to establish the following result. Due to the positivity requirement on initial conditions invoked in both of these references, however, the argument currently applies in only the case of $N = 1$.

Theorem 2.2. *Let $N = 1$, $\epsilon = 0$, and $\phi(u) = u^m$ for $m > 0$. Assume there exists $C \geq 0$, $\delta > 0$, $q, p \geq 0$ such that $g(\cdot, u, G(u)) \geq Cu^q$ and $h(u) \geq Cu^p$ for $u \geq \delta$. If $u_0 \geq \delta$ and either $p > 1$ or $q > \min\{1, (m + 1)/2\}$, then the solution of* (1.1)–(1.2) *becomes unbounded in finite time.*

For the remaining cases which will address convective models in one-dimension, $N = 1$, we apply the results from [3] to (2.3), which are established as a combination of concavity and monotonicity methods. The assumptions thus bear similarity to those introduced in Theorem 2.1.

Theorem 2.3. *Assume Φ^κ is convex for some $0 < \kappa < 1/2$, and $g(u, G(u))/\phi(u)$ is nondecreasing for $u > 0$. Additionally, assume $\epsilon > 0$, $f' \geq 0$, and u_0 is nondecreasing on $[0, 1]$. If*

$$\frac{1}{2}\int_0^1 |\phi(u_0)_x|^2\, dx < P(u_0(1)),$$

and $v_0 \geq G(u_0(1))$, then the solution of (1.1)–(1.2) *blows up in finite time.*

Finally, upon consideration of (2.2), again with $N = 1$, we introduce the following assumptions.

$$h \in C^1([0,\infty)),\ h''(u) \geq 0 \text{ for } u > 0,\ \int^\infty 1/h(u)du < \infty. \tag{2.4}$$

$$\phi(u) \leq C(h(u))^\alpha,\ f(u) \leq C(h(u))^\beta,\ \text{for } u > 0, \tag{2.5}$$

with $0 < \alpha \leq 1$; $0 < \beta < 1$.

$$\phi(u) = k_1 h(u),\ f(u) = k_2 h(u),\ \text{for } u \geq 0, \tag{2.6}$$

for constants $k_1, k_2 > 0$. In light of the results obtainable through so-called eigenfunction techniques [2], we now have the following result.

Theorem 2.4. *(i) Assume* (2.4)–(2.5)*, where either* $\alpha < 1$ *or* $\alpha = 1$ *and* a *is sufficiently large. Let* $\lambda = 1/(1-\beta)$*. There exists* $c_0 > 0$ *such that if*

$$\int_0^1 u_0(x)[(\pi/2)\sin(\pi x)]^\lambda dx > c_0,$$

then the solution of (1.1)–(1.2) *becomes unbounded in finite time. (ii) Assume* (2.4) *and* (2.6) *hold. If* a *is sufficiently large, then the solution of* (1.1)–(1.2) *becomes unbounded in finite time for any choice of nontrivial initial data* u_0*.*

We provide a summary of known results for models governed by power laws when $g_v \equiv 0$ (or, equivalently, if $\Sigma = \emptyset$). In such case (1.1)–(1.2) collapses to (1.1) alone, which is a general diffusion model with (localized) nonlinear boundary flux throughout the boundary $\partial\Omega$. Considering the related nonlinear boundary flux model

$$\begin{aligned} &u_t = \Delta u^m + a u^p && \text{on } \Omega_T \\ &\nabla u^m \cdot \mathbf{n} = u^q && \text{on } (\partial\Omega)_T \\ &u = u_0 && \text{on } \overline{\Omega} \times \{0\}, \end{aligned} \tag{2.7}$$

where $m, p, q > 0$, it is known that, for strictly positive initial states u_0, (2.7) is globally solvable if and only if $p \leq 1$, $q \leq \min(1, (m+1)/2)$ [8]. This result is in striking contrast to the same model under Dirichlet boundary conditions,

$$\begin{aligned} &u_t = \nabla \cdot (u^m + \epsilon \mathbf{f}(u)) + a u^p && \text{on } \Omega_T \\ &u = 0 && \text{on } (\partial\Omega)_T \\ &u = u_0 && \text{on } \overline{\Omega} \times \{0\} \end{aligned} \tag{2.8}$$

wherein $p \leq m$ is, roughly, the corresponding necessary and sufficient condition for global solvability when $\epsilon = 0$ [7]. Further, related results for (2.8) suggest that, for $N = 1$ with $f(u) = u^n$ $(n > 0)$, the condition $p \leq \max(m, n)$ comes close to an extension which includes the effects of convection [2].

The analogy in results for (2.7) and (2.8) bear important differences in another direction, which is relevant for the present study. Those results concerning (2.7) developed for positive initial conditions divide into one of two cases, either all solutions are global or all solutions blow up in finite time. On the other hand, results regarding (2.8) are applicable to general choices of nonnegative u_0, and reveal blow up for models which also possess nontrivial equilibria. Thus, sufficiently "large" initial states give rise to solutions that blow up in finite time, while "small" initial states yield global solutions. Our results for the transmission model, in light of the existence of equilibrium states established in the next section, bear more similarities to those for (2.8) than to known results in the case of (2.7).

3. **EQUILIBRIUM SOLUTIONS**

We analyze the global solvability of (1.1)–(1.2) by determining the existence of nontrivial equilibrium states. In particular, we show that the one-dimensional model

$$\begin{array}{ll} u_t = (u^m)_{xx} + au^p & \text{on } (0,1)_T \\ -(u^m)_x(0,t) = l \geq 0 & \text{for } 0 < t < T \\ (u^m)_x(1,t) = -\psi u^q(1,t) + \beta v(t) & \text{for } 0 < t < T \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\} \end{array} \tag{3.1}$$

$$\begin{array}{ll} (v - Au^q)_t = \frac{-\lambda v}{1+\nu v} + Bu^q & \text{for } x = 1, 0 < t < T \\ v = v_0 & \text{for } 0 \leq t \leq T \end{array} \tag{3.2}$$

possesses at least one nontrivial equilibrium state for $q > 0$, $m \geq p$. In fact, we will also see that these equilibria are strictly positive. Thus, the blow up of all positive solutions if $p > 1$ or $q > \min\{1, (m+1)/2\}$, which is known for the model with localized flux (2.7), is no longer true for the related model with transmission conditions (1.1)–(1.2).

If u, v is a nontrivial, classical equilibrium state for (3.1)–(3.2), then

$$v(1)\left[\lambda - \nu Bu^q(1)\right] = Bu^q(1) \tag{3.3}$$

So, we may easily obtain that $\nu Bu^q(1) < \lambda$, $B > 0$, and $u(1), v(1) > 0$. Defining $y(x) \equiv u^m(x)$, (3.1) yields

$$y(1) \equiv k \leq y(x) \leq y(0) \equiv M$$

and $y'(x) < 0$ for all $0 < x < 1$. A simple integration also results in

$$\frac{1}{2}(y'(x))^2 + \frac{am}{m+p}(y(x))^{(p+m)/m} = \tag{3.4}$$

$$\tfrac{1}{2}l^2 + \tfrac{am}{m+p}M^{(p+m)/m} = \tfrac{1}{2}\left[b(k^{q/m})\right]^2 + \tfrac{am}{m+p}k^{(p+m)/m} \tag{3.5}$$

for all $0 \leq x \leq 1$, where

$$b(k) \equiv k\left(-\psi + \frac{\beta B}{\lambda - \nu Bk}\right). \tag{3.6}$$

Since

$$y'(x) = -\left[l^2 + \frac{2am}{m+p}\left(M^{(p+m)/m} - (y(x))^{(p+m)/m}\right)\right]^{1/2},$$

another integration yields

$$\int_{y(x)}^{M}\left[l^2 + \frac{2am}{m+p}\left(M^{(p+m)/m} - \sigma^{(p+m)/m}\right)\right]^{-1/2} d\sigma = x \tag{3.7}$$

Therefore, if $0 < k < M$ satisfies (3.5) and, in addition,

$$\int_{k}^{M}\left[l^2 + \frac{2am}{m+p}\left(M^{(p+m)/m} - \sigma^{(p+m)/m}\right)\right]^{-1/2} d\sigma = 1, \tag{3.8}$$

we may define $y(x)$ according to (3.7), $u(x) \equiv y^{1/m}(x)$, and $v \equiv v(1)$ according to (3.3) to obtain a nontrivial equilibrium solution of (3.1)–(3.2).

Finally, upon performing the substitution $\tau \equiv \sigma^{(p+m)/m} - k^{(p+m)/m}$ and employing

$$c(k) \equiv \frac{m+p}{2am}\left\{k^{2q/m}\left(-\psi + \frac{\beta B}{\lambda - \nu B k^{q/m}}\right)^2 - l^2\right\} = \frac{m+p}{2am}\left\{\left[g(k^{q/m})\right]^2 - l^2\right\},$$

we have the following result.

Theorem 3.1. *Let*

$$I(k) \equiv \int_0^{c(k)} \frac{m}{m+p}\left[\tau + k^{(p+m)/m}\right]^{-p/(p+m)}\left[l^2 + \frac{2am}{m+p}(c(k) - \tau)\right]^{-1/2} d\tau$$

For $0 \le k_0 < k_1 < (\lambda/\nu B)^{m/q}$ such that $c(k_0) = 0$ and $c > 0$ on (k_0, k_1), there exists a classical nontrivial stationary solution, u, v of (3.1)–(3.2) iff $I(\hat{k}) = 1$ for some $\hat{k} \in (k_0, k_1)$. In such case, $u(0) = M^m$ and $u(1) = \hat{k}^m$, where $M^{(p+m)} \equiv c(\hat{k}) + \hat{k}$.

In order to apply this result, we first note that $c(k) = 0$ iff $(b(k^{q/m}))^2 = l^2$. It is easy to see that $b(z_1) = 0$ for

$$z_1 \equiv \max\{0, (\lambda/\nu B) - (\beta/\psi\nu)\},$$

and $b(k) \to \infty$ as $k \to (\lambda/\nu B)^-$. Hence, for some $k_0 \ge (z_1)^{m/q}$, $c(k_0) = 0$ and $c(k) > 0$ for $k_0 < k < (\lambda/\nu B)^{m/q}$. Since

$$\begin{aligned} I(k) &\ge \int_0^{c(k)} \tfrac{m}{m+p}\left[c(k) + k^{\frac{p+m}{m}}\right]^{-\frac{p}{p+m}}\left[l^2 + \tfrac{2am}{m+p}(c(k) - \tau)\right]^{-1/2} d\tau \\ &\ge \tfrac{1}{a}\left[c(k) + k^{\frac{p+m}{m}}\right]^{-\frac{p}{p+m}}\left\{\left[l^2 + \tfrac{2am}{m+p}c(k)\right]^{1/2} - l\right\} \end{aligned} \tag{3.9}$$

Provided $p < m$ or $p = m$ and $a < 1$, there follows $I(k_0) = 0$ and

$$\lim_{k^{q/m} \to (\lambda/\nu B)^-} I(k) > 1$$

Therefore, (3.1)–(3.2) possesses at least one nontrivial stationary solution.

In fact, if $z_1 > 0$, then it is not difficult to show $c(k) = 0$ actually has 3 solutions if

$$\frac{\psi}{\nu}\left[\sqrt{\frac{\lambda}{B}} - \sqrt{\frac{\beta}{\psi}}\right]^2 - l > 0$$

In such case, there exists $0 \le k_0 < k_1 < (z_1)^{m/q}$ such that $c(k_0) = c(k_1) = 0$ and

$$\max_{k_0 \le k \le k_1} c(k) = \frac{m+p}{2am}\left[b(\hat{z})^2 - l^2\right] = \frac{m+p}{2am}\left\{\left(\frac{\psi}{\nu}\right)^2\left[\sqrt{\frac{\lambda}{B}} - \sqrt{\frac{\beta}{\psi}}\right]^4 - l^2\right\},$$

with

$$\hat{z} \equiv \frac{\lambda}{\nu B} - \frac{1}{\nu}\sqrt{\frac{\lambda\beta}{\psi B}}$$

Now, $I(k_0) = I(k_1) = 0$, and, recalling (3.9), we have $I(k) > 1$ iff

$$\left[l^2 + \frac{2am}{m+p}c(k)\right]^{1/2} - l > a\left[c(k) + k^{(p+m)/m}\right]^{p/(p+m)}$$

Provided $p < m$ or $p = m$ and $a < 1$, this will be satisfied for $c(k) \ge C$ with some sufficiently large $C > 0$. Noting that, by the above inequality,

$$c(\hat{z}^{m/q}) = \frac{m+p}{2am}\left\{\left(\frac{\psi}{\nu}\right)^2\left[\sqrt{\frac{\lambda}{B}} - \sqrt{\frac{\beta}{\psi}}\right]^4 - l^2\right\} > C$$

thus gives conditions in which $I(k) = 1$ will have at least two more solutions in addition to the one established in Theorem 3.1

REFERENCES

[1] J. Anderson, Unique solvability of a nonlinear diffusion model with transmission boundary conditions, *Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis*, 13B:159–170, 2006.

[2] J. Anderson and K. Deng, Global existence for nonlinear diffusion equations, *Journal of Mathematical Analysis and Applications*, 196:479–501, 1995.

[3] J. Anderson, Stability and instability for solutions of the convective porous medium equation with a nonlinear forcing at the boundary I, *Journal of Differential Equation*, 104(2):361–385, 1993.

[4] J. Anderson, Local existence and uniqueness of solutions of degenerate parabolic equations. *Communications in Partial Differential Equations*, 16:105–143, 1991.

[5] F. Andreu, J. Mazón, F. Simondon, and J. Toledo, Attractor for a degenerate nonlinear diffusion problem with nonlinear boundary condition, *Journal of Dynamics and Differential Equations*, 10(3):347–377, 1998.

[6] H. Levine, S. Pamuk, B. Sleeman, M. Nilsen-Hamilton, Mathematical modeling of capillary formation and development in tumor angiogenesis: penetration into the stroma, *Bulletin of Mathematical Biology*, 63:801–863, 2001.

[7] H. Levine and P. Sacks, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, *Journal of Differential Equations*, 52 (2):135–161, 1984.

[8] X. Song and S. Zheng, Blow-up and blow-up rate for a reaction-diffusion model with multiple nonlinearities, *Nonlinear Analysis*, 54:279–289, 2003.

[9] N. Wolanski, Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary, *SIAM Journal on Mathematical Analysis*, 24 (2):317–326, 1993.