Dynamic Systems and Applications 18 (2009) 121-128

QUENCHING CRITERIA FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

C. Y. CHAN

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, LA 70504-1010, USA

ABSTRACT. A criterion for the quenching of the solution for a degenerate semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source situated at b is given. The locations of b for global existence of the solution and for the quenching of the solution are given.

AMS (MOS) Subject Classification. 35K60, 35K57, 35K65, 35B35

1. INTRODUCTION

Let q, a, T and b be any numbers such that $q \geq 0$, $a > 0$, $T > 0$, and $0 < b < 1$. Also, let D denote the interval $(0, 1)$, and $\bar{D}$ be its closure. We consider the following degenerate semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source situated at b,

$$(1.1) \qquad \begin{cases} x^q u_t - u_{xx} = a\delta(x-b) f(u(x,t)) \text{ in } D \times (0, T], \\ u(x, 0) = 0 \text{ on } \bar{D}, \\ u(0,t) = u(1,t) = 0 \text{ for } 0 < t \leq T, \end{cases}$$

where $\delta(x)$ is the Dirac delta function, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$. The case $q = 0$ was studied by Deng and Roberts [7] by analyzing its corresponding nonlinear Volterra equation at the site b of the concentrated source. Instead of studying a solution $u(b,t)$ of the nonlinear Volterra equation, we would like to investigate a solution $u(x,t)$ of the degenerate problem (1.1).

The right-hand side of the partial differential equation in (1.1) has the term $\delta(x-b)$. This implies that u_x has a jump discontinuity at $x = b$. Thus, a solution of the problem (1.1) is a continuous function satisfying (1.1). In the proof of Theorem 3 of Chan and Jiang [4], it is shown that $u_{xx} \geq 0$ for $x \in (0, b)$ and $x \in (b, 1)$. It follows from the differential equation in (1.1) that $u_t(b,t) = \infty$ for each $t > 0$. Hence, we say that a solution u of the problem (1.1) is said to quench if there exists some t_q such that

$$\max\{u(x,t) : x \in \bar{D}\} \to c^- \text{ as } t \to t_q$$

(cf. Chan and Liu [5]). If t_q is finite, then u is said to quench in a finite time. On the other hand, if $t_q = \infty$, then u is said to quench in infinite time.

Let $G(x,t;\xi,\tau)$ denote Green's function corresponding to the problem (1.1), and t_q denote the supremum of all t_1 such that on $[0,t_1]$, the integral equation,

$$u(x,t) = a\int_0^t G(x,t;b,\tau)\, f(u(b,\tau))\, d\tau, \tag{1.2}$$

corresponding to the problem (1.1) has a unique nonnegative continuous solution. For ease of reference, we summarize the main results of Theorems 1, 2 and 3 of Chan and Jiang [4] as Theorem 1.1 below.

Theorem 1.1. *There exists some t_q $(\le \infty)$ such that for $0 \le t < t_q$, the integral equation* (1.2) *has a unique nonnegative continuous solution $u(x,t)$, which is a strictly increasing function of t in D. Before a quenching occurs, u is the solution of the problem* (1.1)*, and attains its maximum at (b,t) for each $t > 0$. If t_q is finite, then u quenches at t_q. Furthermore, if u quenches, then b is the single quenching point.*

In Section 2, we give a criterion for the quenching. It turns out that the forcing term $f(u)$ need not be superlinear in u for a quenching to occur. This is in sharp contrast with the blow-up phenomenon, which requires the forcing term to be superlinear (cf. Chan and Tian [6]). In Section 3, we find the exact position b^* for the problem (1.1) such that u never quenches for $b \in (0,b^*] \cup [1-b^*,1)$, and u always quenches in a finite time for $b \in (b^*, 1-b^*)$. For illustration, an example is given.

2. A QUENCHING CRITERION

Let

$$\mu(t) = \int_D x^q \phi(x)\, u(x,t)\, dx,$$

where ϕ denotes the normalized fundamental eigenfunction of the problem,

$$\phi'' + \lambda x^q \phi = 0, \ \ \phi(0) = \phi(1) = 0,$$

with λ denoting its corresponding eigenvalue, which is positive (cf. Chan and Chan [2]). Below is a quenching criterion.

Theorem 2.1. *If there exist constants $c_1\,(>0)$ and $c_2\,(\ge 0)$ such that*

$$\sqrt{1+q}\,\phi(b)\, f(u(b,t)) \ge c_1 + c_2 u(b,t), \tag{2.1}$$

$$\frac{\lambda}{a} > c_2, \ \frac{ac_1}{\lambda - ac_2} > c, \tag{2.2}$$

then u quenches in a finite time. Furthermore, an upper bound for the quenching time is given by

$$\frac{1}{\lambda - ac_2} \ln\left[1 - \frac{(\lambda - ac_2)\,c}{ac_1}\right]^{-1}.$$

Proof. Multiplying the partial differential equation in (1.1) by ϕ, and integrating with respect to x over D, we obtain

$$\mu'(t) + \lambda\mu(t) = a\phi(b) f(u(b,t)). \tag{2.3}$$

Since $u(x,t) \le u(b,t)$, we have

$$\mu(t) \le \left(\int_D x^q \phi(x)\, dx\right) u(b,t).$$

It follows from the Schwarz inequality and $\int_D x^q \phi^2(x)\, dx = 1$ that

$$\mu(t) \le \left(\int_D x^q \phi^2(x)\, dx\right)^{1/2} \left(\int_D x^q dx\right)^{1/2} u(b,t)$$

$$= \frac{1}{\sqrt{1+q}} u(b,t).$$

By (2.1),

$$a\phi(b) f(u(b,t)) \ge \frac{a}{\sqrt{1+q}} (c_1 + c_2 u(b,t))$$
$$\ge a\left(\frac{1}{\sqrt{1+q}} c_1 + c_2\mu(t)\right).$$

From (2.3),

$$\mu'(t) + (\lambda - ac_2)\mu(t) \ge \frac{a}{\sqrt{1+q}} c_1.$$

Since $\mu(0) = 0$, we obtain

$$\mu(t) \ge \frac{ac_1}{\sqrt{1+q}(\lambda - ac_2)} \left[1 - e^{-(\lambda - ac_2)t}\right].$$

Hence,

$$u(b,t) \ge \sqrt{1+q}\,\mu(t) \ge \frac{ac_1}{\lambda - ac_2} \left[1 - e^{-(\lambda - ac_2)t}\right].$$

From (2.2), there exists some finite t_q such that u quenches at (b, t_q). An upper bound for the quenching time follows by setting the right-hand side of the above inequalities equal to c to evaluate t. □

3. CRITICAL POSITION b^*

Let $\lim_{t\to\infty} u(x,t)$ be denoted by $U(x)$. For ease of reference, let us summarize the main results of Section 3 of Chan and Jiang [4] in the following theorem.

Theorem 3.1. *There exists a critical length a^* such that u exists on $\bar{D}$ for all $t > 0$ if $a \le a^*$, and u quenches in a finite time if $a > a^*$. The critical length a^* is determined as the supremum of all positive values a for which a solution U of the nonlinear two-point boundary value problem,*

$$-U''(x) = a\delta(x-b) f(U(x)) \text{ in } D,\ U(0) = U(1) = 0, \tag{3.1}$$

exists. Furthermore, $u(x,t) < U(x)$ *in* $D \times (0,\infty)$,

$$U(x) = ag(x;b)\, f(U(b)), \tag{3.2}$$

where

$$g(x;\xi) = \begin{cases} \xi(1-x), \ 0 \le \xi \le x, \\ x(1-\xi), \ x < \xi \le 1, \end{cases}$$

is Green's function corresponding to the problem (3.1),

$$a^* = \frac{1}{b(1-b)} \max_{0 \le s \le c} \left(\frac{s}{f(s)} \right) \ \textit{for a given } b \in D. \tag{3.3}$$

As a consequence of the above theorem, the solution u does not quench in infinite time. We note from (3.3) that a^* depends on b. For a given $a\,(> a^*)$, there exists a position b such that the problem (1.1) quenches in a finite time. Chan and Boonklurb [1] studied the critical position of the concentrated source for a blow-up problem. Here, we give an analogous argument for the quenching problem (1.1). To find a position b for the same given $a\,(> a^*)$ such that the solution u exists for all $t > 0$, let us first consider the problem (1.1) with $q = 0$, namely,

$$\begin{cases} v_t - v_{xx} = a\delta(x-b) f(v(x,t)) \ \text{in } D \times (0,T], \\ v(x,0) = 0 \text{ on } \bar{D},\ v(0,t) = v(1,t) = 0 \text{ for } 0 < t \le T. \end{cases} \tag{3.4}$$

From Theorem 1.1, the quenching set is the single point $x = b$, and

$$v(b,t) = a \int_0^t G_0(b,t;b,\tau) f(v(b,\tau)) d\tau, \tag{3.5}$$

where

$$G_0(x,t;\xi,\tau) = 2 \sum_{n=1}^{\infty} (\sin n\pi x)(\sin n\pi\xi) e^{-n^2\pi^2(t-\tau)} \text{ for } t > \tau$$

is Green's function corresponding to the problem (3.4). From Olmstead and Roberts [9],

$$\int_0^t G_0(b,t;b,\tau)d\tau = b(1-b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\pi b}{n^2} e^{-n^2\pi^2 t}.$$

Since $\sum_{n=1}^{\infty} (\sin^2 n\pi b) e^{-n^2\pi^2 t}/n^2$ and $2\sum_{n=1}^{\infty} (\sin^2 n\pi b) e^{-n^2\pi^2 t}$ converge uniformly in $(0,t)$, we have

$$\frac{\partial}{\partial t} \left(\int_0^t G_0(b,t;b,\tau)d\tau \right) = 2 \sum_{n=1}^{\infty} (\sin^2 n\pi b) e^{-n^2\pi^2 t} > 0,$$

$$\begin{aligned} \lim_{t\to\infty} \int_0^t G_0(b,t;b,\tau)d\tau \\ &= b(1-b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\pi b}{n^2} \lim_{t\to\infty} e^{-n^2\pi^2 t} \\ &= b(1-b). \end{aligned} \tag{3.6}$$

From Theorem 1.1, $v(x,t)$ attains its maximum M at (b,θ) for $0 \le t \le \theta$. Thus given any positive number $M\,(<c)$, it follows from (3.5) and (3.6) that for $0 \le t \le \theta$,

$$v(b,t) \le af(M)\int_0^t G_0(b,t;b,\tau)d\tau \le af(M)\,b(1-b).$$

In order that $af(M)\,b(1-b) \le M$ so that v exists for all $t>0$, we choose b in such a way that

$$0 < b \le \frac{1}{2}\left(1-\sqrt{1-\frac{4M}{af(M)}}\right) \text{ or } \frac{1}{2}\left(1+\sqrt{1-\frac{4M}{af(M)}}\right) \le b < 1. \tag{3.7}$$

Since v is a nondecreasing function of t, we have for $0 \le x \le 1$ and $q>0$,

$$x^q v_t - v_{xx} \le v_t - v_{xx},$$

which implies that the solution of the problem (1.1) is a lower solution of the problem (3.4). Thus under the above condition (3.7) on b, the solution of (1.1) exists for all $t>0$.

Let us consider the function

$$\psi(U(b)) = \frac{U(b)}{f(U(b))}.$$

Since $\psi(U(b)) > 0$ for $0 < U(b) < c$, and $\psi(0) = 0 = \lim_{U(b)\to c^-}\psi(U(b))$, a direct computation shows that $\psi(U(b))$ attains its maximum when $\psi(U(b)) = 1/f'(U(b))$, where $U(b) \in (0,c)$ by Rolle's Theorem. Thus, $\max(U(b)/f(U(b)))$ occurs when

$$\frac{U(b)}{f(U(b))} = \frac{1}{f'(U(b))}, \text{ where } 0 < U(b) < c. \tag{3.8}$$

This also implies that $U(x)$ exists when $a = a^*$.

From (3.2), $U(b) = ab(1-b)f(U(b))$. We would like to know how $U(b)$ behaves as b varies when $a > a^*$. A direct calculation gives

$$U'(b) = \frac{a(1-2b)f(U(b))}{1-ab(1-b)f'(U(b))}. \tag{3.9}$$

Since $a > a^*$, and $1/4 \ge b(1-b)$, we have

$$1-\frac{4}{a}\max_{0\le U\le c}\frac{U(b)}{f(U(b))} > 1-\frac{4}{a^*}\max_{0\le U\le c}\frac{U(b)}{f(U(b))} \ge 0.$$

Thus for

$$b \in \left(0, \frac{1}{2}\left(1-\sqrt{1-\frac{4}{a}\max_{0\le U\le c}\frac{U(b)}{f(U(b))}}\right)\right),$$

the numerator is positive. Also,

$$b < \frac{1}{2}\left(1-\sqrt{1-\frac{4}{a}\max_{0\le U\le c}\frac{U(b)}{f(U(b))}}\right)$$

gives

$$b-\frac{1}{2}<-\frac{1}{2}\sqrt{1-\frac{4}{a}\max_{0\leq U\leq c}\frac{U(b)}{f\left(U\left(b\right)\right)}}<0.$$

We have

$$\left(b-\frac{1}{2}\right)^2>\frac{1}{4}\left(1-\frac{4}{a}\max_{0\leq U\leq c}\frac{U(b)}{f\left(U\left(b\right)\right)}\right),$$

which by (3.8) gives

$$1-ab\left(1-b\right)f'\left(U\left(b\right)\right)>0,$$

and hence, $U'\left(b\right)>0$. Thus for a given $a>a^*$, the function $U\left(b\right)$ is a strictly increasing function of b for

$$b\in\left(0,\frac{1}{2}\left(1-\sqrt{1-\frac{4}{a}\max_{0\leq U\leq c}\frac{U(b)}{f\left(U\left(b\right)\right)}}\right)\right).$$

Similarly for a given $a>a^*$, the function $U\left(b\right)$ is a strictly decreasing function of b for

$$b\in\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4}{a}\max_{0\leq U\leq c}\frac{U(b)}{f\left(U\left(b\right)\right)}}\right),1\right).$$

Hence on the interval $(0,1/2)$, the position b for global existence of u is closer to 0 than the position b for the quenching of u in a finite time. On the other hand, on the interval $(1/2,1)$, the position b for global existence of u is closer to 1 than the position b for the quenching of u in a finite time. Thus, there exists $b^*\in(0,1/2)$ such that the steady state $U\left(x\right)$ exists for $b\in(0,b^*)\cup(1-b^*,1)$, and does not exist for $b\in(b^*,1-b^*)$. We note that

$$b^*=\frac{1}{2}\left(1-\sqrt{1-\frac{4}{a}\max_{0\leq U\leq c}\frac{U(b)}{f\left(U\left(b\right)\right)}}\right),$$

and is attained for $0<U\left(b\right)<c$ by (3.8). Since $u\left(x,t\right)\leq U\left(x\right)=\lim_{t\to\infty}u\left(x,t\right)$ in $D\times(0,\infty)$ when U exists, we have for $b\in(0,b^*]\cup[1-b^*,1)$, u exists for $0\leq t<\infty$, and for $b\in(b^*,1-b^*)$, u quenches in a finite time.

The above discussion gives the following result.

Theorem 3.2. *For $a>a^*$, the solution of the problem* (1.1) *exists globally for $b\in(0,b^*]\cup[1-b^*,1)$, and quenches in a finite time for $b\in(b^*,1-b^*)$.*

For illustration, let $f(u)=(1-u)^{-p}$. A direct computation shows that

$$a^*=\frac{p^p}{b(1-b)(1+p)^{1+p}},$$

$$b^*=\frac{1}{2}\left(1-\sqrt{1-\frac{4p^p}{a(1+p)^{1+p}}}\right).$$

When $p=1$ and $b=1/2$, we have $a^*=1$, and $b^*=\left(1-\sqrt{1-a^{-1}}\right)/2$ for $a>1$. We note that the concept of the quenching was introduced by Kawarada [8] through the following problem, which arises in the study of a polarization phenomenon in ionic conductors:

$$u_t - u_{xx} = \frac{1}{1-u} \text{ in } (0,a) \times (0,T],$$
$$u(x,0) = 0 \text{ on } 0 \le x \le a,$$
$$u(0,t) = u(a,t) = 0 \text{ for } 0 < t \le T.$$

Its $a^* = 1.5303$ (to five significant figures) (cf. Chan and Chen [3]). Thus, the presence of the concentrated source shortens the critical length.

REFERENCES

[1] C. Y. Chan and R. Boonklurb, A blow-up criterion for a degenerate parabolic problem due to a concentrated nonlinear source, Quart. Appl. Math., 65:781–787, 2007.

[2] C. Y. Chan and W. Y. Chan, Existence of classical solutions for degenerate semilinear parabolic problems, Appl. Math. Comput., 101:125–149, 1999.

[3] C. Y. Chan and C. S. Chen, A numerical method for semilinear singular parabolic quenching problems, Quart. Appl. Math., 47:45–57, 1989.

[4] C. Y. Chan and X. O. Jiang, Quenching for a degenerate parabolic problem due to a concentrated nonlinear source, Quart. Appl. Math., 62:553–568, 2004.

[5] C. Y. Chan and H. T. Liu, Does quenching for degenerate parabolic equations occur at the boundaries?, Dynam. Contin. Discrete Impuls. Systems (Series A), 8:121–128, 2001.

[6] C. Y. Chan and H. Y. Tian, Single-point blow-up for a degenerate parabolic problem due to a concentrated nonlinear source, Quart. Appl. Math., 61:363–385, 2003.

[7] K. Deng and C. A. Roberts, Quenching for a diffusive equation with a concentrated singularity, Differential Integral Equations, 10:369–379, 1997.

[8] H. Kawarada, On solutions of initial boundary problem for $u_t = u_{xx} + 1/(1-u)$, Res. Inst. Math. Sci., 10:729–736, 1975.

[9] W. E. Olmstead and C. A. Roberts, Explosion in a diffusive strip due to a concentrated nonlinear source, Methods Appl. Anal., 1:435–445, 1994.