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SECOND ORDER MOMENTS OF SOLUTIONS OF PARABOLIC INITIAL BOUNDARY VALUE PROBLEMS WITH ε-CORRELATED RANDOM PARAMETERS

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ABSTRACT. Due to the random character of input data of a great variety of technical and economical procedures it seems to be appropriate to model these procedures by stochastic initial boundary value problems (IBVP). This paper deals with IBVP for parabolic partial differential equations where a Neumann boundary condition is assumed to be a random field with a given probability distribution. We assume, that this random field possesses smooth paths and that it is homogeneous and short-range correlated with a small correlation length $\varepsilon > 0$. The main interest lies in the calculation of the moment functions of the solution of the considered problem, which depend on the chosen characteristics of the random influence. Based on the idea of an appropriate FEM discretisation we present several approximation procedures for the computation of the variance and correlation function of the discretised solution. Considering a numerical example the resulting variance functions of the introduced methods are compared with the results of a Monte Carlo simulation.

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1. INTRODUCTION

In this paper a problem of random heat conduction is considered, which is modelled by the following initial boundary value problem (IBVP) for a parabolic PDE

$$
\begin{aligned}
& u_t - \lambda \triangle u = f(t,x), \quad x = (x_1, x_2) \in \mathsf{D} \subset \mathbb{R}^2,\ t \in (0,T] \\
& \text{IC}: u(0,x,\omega) = {}^{\varepsilon}u_0(x,\omega) \quad x \in \bar{\mathsf{D}} \\
& \text{BC}: \frac{\partial u}{\partial N}(t,x,\omega)\Big|_{(\partial \mathsf{D})_2} = {}^{\varepsilon}P(t,x,\omega) \\
& \left(\frac{\partial u}{\partial N}(t,x,\omega) + \alpha(u(t,x) - u_A(t,x))\right)\Bigg|_{(\partial \mathsf{D})_3} = 0\,.
\end{aligned} \tag{1.1}
$$

The function $u(t,x,\omega)$ describes the temperature distribution in the domain D at time $t \in (0,T]$. The two-dimensional domain D possesses a sufficiently smooth boundary $\partial\mathsf{D} = (\partial\mathsf{D})_2 \cup (\partial\mathsf{D})_3$. As random influences the heat flux ${}^{\varepsilon}P$ over the boundary $(\partial\mathsf{D})_2$

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and the initial condition ${}^{\varepsilon}u_0$ are considered and modelled by stochastically independent random fields. Problems of this kind arise for instance in braking or coupling processes of motor vehicles. Due to the occurring diffusion an expedient assumption is to describe the initial condition and the boundary condition as ε-correlated fields. Such random functions have been applied to model numerous problems of physics and engineering. The property of ε-correlation (cf. [5, 6]) means that the correlation function $R(x, y)$, $x, y \in \mathbb{R}^m$ of a random function $f(x, \omega)$, $x \in \mathbb{R}^m$ vanishes if the distance of the arguments exceeds a certain quantity $\varepsilon > 0$, which is called correlation length. In this sense ε-correlated functions are defined as functions without distance effect. In contrast to the white noise model they can possess arbitrarily smoothness. Further considerations according to the concept of ε-correlated functions and its generalizations can be found in [5, 6].

In the following it is appropriate to consider the problem for the random fluctuations

$$
\begin{aligned}
&\overline{u}_t - \lambda \triangle \overline{u} = 0, \quad x = (x_1, x_2) \in \mathsf{D} \subset \mathbb{R}^2,\ t \in (0, T] \\
&\text{IC} : \overline{u}(0, x, \omega) = {}^{\varepsilon}u_0(x, \omega) - \mathbf{E}\{{}^{\varepsilon}u_0(x)\} =: {}^{\varepsilon}\overline{u}_0(x, \omega), \quad x \in \bar{\mathsf{D}} \\
&\text{BC} : \frac{\partial \overline{u}}{\partial N}(t, x, \omega)\Big|_{(\partial \mathsf{D})_2} = {}^{\varepsilon}P(t, x, \omega) - \mathbf{E}\{{}^{\varepsilon}P(t, x)\} =: {}^{\varepsilon}\overline{P}(t, x, \omega) \\
&\left(\frac{\partial \overline{u}}{\partial N}(t, x, \omega) + \alpha\, \overline{u}(t, x, \omega)\right)\Bigg|_{(\partial \mathsf{D})_3} = 0,
\end{aligned}
\tag{1.2}
$$

which holds for the difference

$$\overline{u}(t, x, \omega) := u(t, x, \omega) - w(t, x)$$

of the solution of (1.1) and the solution of the corresponding averaged problem. The latter results from replacing all random quantities by their expectations. The advantage of (1.2) lies in the homogeneous differential equation and the independence of the Robin condition of the surrounding temperature u_A. A classical solution of (1.2) has to satisfy the condition

$$\overline{u} \in C^{(1,2)}((0, T] \times \mathsf{D}) \cap C^{(0,1)}((0, T] \times \overline{\mathsf{D}}) \cap C([0, T] \times \overline{\mathsf{D}}).$$

Moreover some compatibility conditions between the boundary and the initial conditions and between the boundary conditions of different types must be fulfiled (see [4]). It should be mentioned, that due to the chosen properties for ${}^{\varepsilon}\overline{u}_0$ and ${}^{\varepsilon}\overline{P}$ these conditions almost sure will not be satisfied. Nevertheless, for the specific parabolic problem under consideration these irregularities quickly diffuse away (see [1, 2, 6]).

Due to the linearity of the differential equation, of the initial and boundary condition and the independence of the random influences ${}^{\varepsilon}\overline{P}$ and ${}^{\varepsilon}\overline{u}_0$ the IBVP (1.2) can be splitted up into two IBVPs, with random boundary condition ${}^{\varepsilon}\overline{P}$ and homogeneous initial condition ${}^{\varepsilon}\overline{u}_0 \equiv 0$ on the one side and random initial condition ${}^{\varepsilon}\overline{u}_0$ and

homogeneous boundary condition ${}^{\varepsilon}\overline{P} \equiv 0$ on the other side. That means, the solution $\overline{u}$ can be represented as a sum of the solutions of the two problems, each with only a single random influence.

In this paper we are interested in a weak solution of the above IBVP with random boundary condition ${}^{\varepsilon}\overline{P}$ and homogeneous initial condition ${}^{\varepsilon}\overline{u} \equiv 0$ via the Finite Element Method (FEM) and especially in its first and second order moments. Considerations concerning the corresponding problem with random initial condition and homogeneous boundary condition can be found in [4].

The starting point for the FEM is the variational formulation of (1.2). Here the main idea is to transform the infinite dimensional problem into a finite dimensional one. In this way, an approximative solution can be found by using a spatially discretisation. Thereby the domain D is discretised by finite elements $T^{(r)}$. For the seek of simplicity in this paper the domain D is chosen as a rectangle $[-R, R] \times [0, H]$ and the finite elements will be rectangles or triangles (cf. Figure 1).

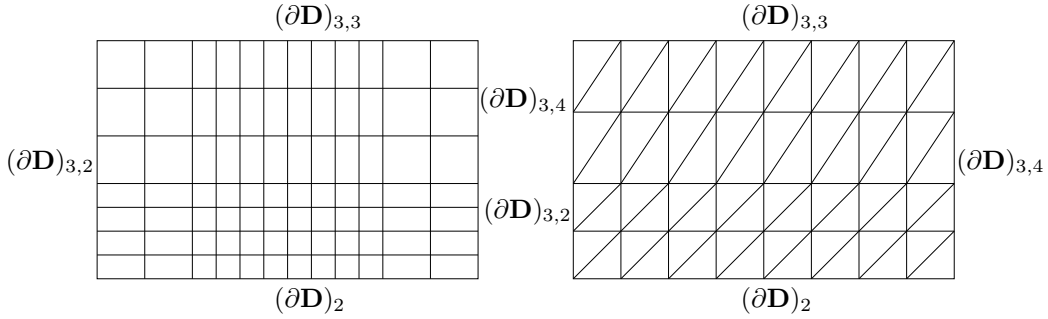

FIGURE 1. Domain D and its discretisation

We are looking for a solution of the form

$$\overline{u}_h(t, x, \omega) = \sum_{i \in \chi_h} u_{h,i}(t, \omega) p_i(x). \tag{1.3}$$

The index set $\chi_h = \{1, \ldots, N_h\}$ describes the chosen discretisation, it contains the numbers of the global nodes P_i which are the corners of the finite elements. Additionally to this global numbering the nodes of the finite elements are numbered from $1, \ldots, \hat{N}_h$ locally, where $\hat{N}_h$ is the number of nodes per element. Then the following relation yields. Starting with the mapping

$$\alpha \longleftrightarrow i = i(r, \alpha), \quad \alpha = 1, \ldots, \hat{N}_h, \ i \in \chi_h \tag{1.4}$$

the matrix $C^{(r)} = \left[C^{(r)}_{\alpha,i}\right]_{\alpha\in\bar{\chi},i\in\chi_h}$ describes the connection between the local and global numbering in every finite element $T^{(r)}$. It is defined as follows,

$$C^{(r)}_{\alpha,i} = \begin{cases} 1 & \text{if } i \text{ is the global number of the node with local number } \alpha \\ & \text{of the element } T^{(r)}\,, \\ 0 & \text{else}\,. \end{cases}$$

The ansatz functions p_i, $i \in \chi_h$, represent a basis of the approximation space

$$V_h = \{v_h(x) : \sum_{i\in\chi_h} v_i p_i(x)\}.$$

A property of the FEM is that these functions possess a finite support. The ansatz functions are defined locally over the finite elements $T^{(r)}$, which contain the node P_i, by element ansatz functions $p^{(r)}_\alpha$. Using the mapping (1.4) again it yields

$$p_i(x) = \begin{cases} p^{(r)}_\alpha(x), & x \in T^{(r)},\ r \in B_i\,, \\ 0 & \text{else}\,. \end{cases}$$

The index set B_i contains the global numbers of all elements with $P_i \in \overline{T}^{(r)}$. A useful relation between the ansatz functions and element ansatz functions is given by

$$\underline{p}(x) = (C^{(r)})^T \underline{p}^{(r)}(x), \quad x \in T^{(r)}. \tag{1.5}$$

The element ansatz functions $p^{(r)}_\alpha,\ \alpha = 1,\dots,\hat{N}_h$ are obtained by the transformation of form functions $\varphi_\alpha,\ \alpha = 1,\dots,\hat{N}_h$ which are defined on a reference element. That means

$$p^{(r)}_\alpha(x) = \varphi_\alpha(\xi_{T^{(r)}}(x)),\ \forall x \in \overline{T}^{(r)}.$$

For a more detailed description of the corresponding FEM techniques it is again referred for instance to [3].

The vector $\underline{u}_h(t,\omega) = \{u_{h,i}(t,\omega)\}_{i\in\chi_h}$ of the time-dependent coefficients in (1.3) is the unique solution of the resulting system of ordinary equations (cf. [4])

$$\begin{aligned} &M_h\dot{\underline{u}}_h(t,\omega) + K_h\underline{u}_h(t,\omega) = \underline{f}_h(t,\omega), \quad t \in (0,T] \\ &\text{IC}: \underline{u}_h(0,\omega) = \mathbf{0}\,. \end{aligned} \tag{1.6}$$

The load vector $\underline{f}_h(t,\omega) = \left[\int_{-R}^{R} p_i(s,0)\,{}^{\varepsilon}\overline{P}(t,s,\omega)ds\right]_{i\in\chi_h}^{T}$ contains the random boundary condition ${}^{\varepsilon}\overline{P}$. Consequently, the solution $\underline{u}_h$ is an integral functional of the form

$$\underline{u}_h(t,\omega) = \int_0^t G_h(t-s)\underline{f}_h(s,\omega)ds = \int_0^t\int_{-R}^{R} G_h(t-s)\Big[p_i(r,0)\Big]_{i\in\chi_h}^{T}\,{}^{\varepsilon}\overline{P}(s,r,\omega)\,drds,$$

with $G_h(t) = \exp(-M_h^{-1}K_h t)M_h^{-1}$. It can bee seen, that the ansatz functions p_i, $i \in \chi_h$, are considered only on the boundary $(\partial\mathsf{D})_2 = [-R,R]$ and so they do not

vanish only for nodes P_i laying on the boundary. For the sake of shortness it is set $p_i(r,0) = \hat{p}_i(r)$, $i \in \chi_h$.

According to (1.3) the approximative solution of the variational formulation is given by

$$\begin{aligned}\overline{u}_h(t,x,\omega) &= \sum_{i\in\chi_h} u_{h,i}(t,\omega)p_i(x)\\ &= \sum_{i\in\chi_h} p_i(x)\int_0^t\int_{-R}^{R}\Big[G_h(t-s)\,\underline{\hat{p}}(r)\Big]_i\,{}^{\varepsilon}\overline{P}(s,r,\omega)\,dr\,ds\,.\end{aligned}$$

The aim is now to calculate the second order moments of the approximative solution $\overline{u}_h$. It yields $\mathbf{E}\{\underline{u}_h(t)\} = \mathbf{0}$, $\forall\, t$ and

$$\begin{aligned}\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} = \int_0^{t_1}\int_{-R}^{R}\int_0^{t_2}\int_{-R}^{R} G_h(t_1-s_1)\underline{\hat{p}}(r_1)\underline{\hat{p}}^T(r_2)G_h^T(t_2-s_2)\\ \mathbf{E}\{{}^{\varepsilon}\overline{P}(s_1,r_1)\,{}^{\varepsilon}\overline{P}(s_2,r_2)\}\,dr_2\,ds_2\,dr_1\,ds_1\,.\end{aligned} \tag{1.7}$$

From the correlation matrix $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ given in (1.7) the correlation function of the approximative solution $\overline{u}_h$ of the IBVP can be calculated easily by

$$\mathbf{E}\{\overline{u}_h(t_1,x)\overline{u}_h(t_2,y)\} = \sum_{i,j\in\chi_h}\mathbf{E}\{\underline{u}_{h,i}(t_1)\underline{u}_{h,j}^T(t_2)\}p_i(x)p_j(y)\,,$$

which follows immediately from (1.3).

2. THE CORRELATION MATRIX $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$

In the following several methods for the calculation of the above derived correlation matrix $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ are introduced and compared. On the one hand, we study asymptotic expansions with respect to the correlation lengths of the random field ${}^{\varepsilon}\overline{P}$. This method was developed in [5, 6]. Derived from this approach we consider a method for the special case of a uniform discretisation. Finally we show an explicit way to calculate the correlation function and compare the results.

The random field ${}^{\varepsilon}\overline{P}$ is modelled as an element of a family of real-valued centered ε-correlated fields $({}^{\varepsilon}\overline{P},\ \varepsilon > 0)$ with the correlation functions

$$R_{{}^{\varepsilon}\overline{P}\,{}^{\varepsilon}\overline{P}}(x,y) = \mathbf{E}\{{}^{\varepsilon}\overline{P}(x)\,{}^{\varepsilon}\overline{P}(y)\},\ x,y\in\mathbb{R}^2\,.$$

It is assumed that $({}^{\varepsilon}\overline{P},\ \varepsilon > 0)$ fulfills the following assumptions.

1. ${}^{\varepsilon}\overline{P}(x,\omega)$ is wide sense homogeneous.
2. The correlation function of ${}^{\varepsilon}\overline{P}$ is generated by a correlation function R of a 1-correlated wide sense homogeneous random field, that means for $x = (x_1,x_2) \in \mathbb{R}^2$ and $y = (y_1,y_2)\in\mathbb{R}^2$ it holds

$$R_{{}^{\varepsilon}\overline{P}\,{}^{\varepsilon}\overline{P}}(x,y) = R_{{}^{\varepsilon}\overline{P}\,{}^{\varepsilon}\overline{P}}(y-x) = R\left(\frac{y_1-x_1}{\varepsilon_1},\frac{y_2-x_2}{\varepsilon_2}\right),$$

with $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$.

3. ${}^{\varepsilon}\overline{P}$ is mean square continuous on $\mathbb{R}^2$, that means the generating correlation function R is continuous on $\mathbb{R}^2$.

Remark 2.1. In the considered case, the components ε_1 and ε_2 of the correlation length ε describe the length of the correlation in temporal and spatial direction, respectively.

Using the given discretisation of the domain D the discretisation of the boundary $(\partial\mathsf{D})_2$ is given by $[-R, R] = \bigcup\limits_{r\in\tilde{\psi}_h} [x_1^{(r)}, x_2^{(r)}]$, where the index set $\tilde{\psi}_h$ contains the numbers of all finite elements laying on the boundary $(\partial\mathsf{D})_2$. Without loss of generality we set $\tilde{\psi}_h = \{1, \dots, \tilde{R}_h\}$ with a certain number $\tilde{R}_h$. Consequently we can formulate the right hand side of equation (1.7) as a sum of integrals over the intervals $I^{(r)} = [x_1^{(r)}, x_2^{(r)}]$, $r \in \tilde{\psi}_h$. Using relation (1.5) it yields

$$
\begin{aligned}
&\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} \\
&\quad = \int_0^{t_1}\int_0^{t_2} G_h(t_1 - s_1) \sum_{r=1}^{\tilde{R}_h}\sum_{s=1}^{\tilde{R}_h} (C^{(r)})^T \Bigg[\int_{I^{(r)}}\int_{I^{(s)}} \underline{\hat{p}}^{(r)}(r_1)(\underline{\hat{p}}^{(s)})^T(r_2) \\
&\qquad R\left(\frac{s_2 - s_1}{\varepsilon_1}, \frac{r_2 - r_1}{\varepsilon_2}\right) dr_2\, dr_1 \Bigg] C^{(s)} G_h^T(t_2 - s_2)\, ds_2\, ds_1 \,.
\end{aligned} \tag{2.1}
$$

By the help of the transformation $(r_1, s_1) \to (u_1, u_2)$, $u_1 = \frac{s_2 - s_1}{\varepsilon_1}$, $u_2 = \frac{r_2 - r_1}{\varepsilon_2}$ the correlation matrix $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ can be written as

$$
\begin{aligned}
\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} = \varepsilon_1\varepsilon_2 \sum_{r,s=1}^{\tilde{R}_h} \int_0^{t_2} \int_{\frac{s_2 - t_1}{\varepsilon_1}}^{\frac{s_2}{\varepsilon_1}} G_h(t_1 - (s_2 - \varepsilon_1 u_1))(C^{(r)})^T \\
\Bigg[\int_{x_1^{(s)} \frac{1}{\varepsilon_2}(r_2 - x_2^{(r)})}^{x_2^{(s)} \frac{1}{\varepsilon_2}(r_2 - x_1^{(r)})} \underline{\hat{p}}^{(r)}(r_2 - \varepsilon_2 u_2)(\underline{\hat{p}}^{(s)})^T(r_2) R(u_1, u_2)\, du_2\, dr_2 \Bigg] \\
C^{(s)} G_h^T(t_2 - s_2)\, du_1\, ds_2 \,.
\end{aligned}
$$

In the further considerations we restrict for the seek of simplicity of the corresponding explicit representations to the case of a uniform discretisation, that means $x_2^{(r)} - x_1^{(r)} = x_2^{(s)} - x_1^{(s)} =: h \;\; \forall r, s = 1, \dots, \tilde{R}_h$.

By changing the order of integration and the additional substitutions $u_1 \to -u_1$, $u_2 \to -u_2$ with paying attention to the homogeneity of the correlation function R we

obtain the following representation of $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ for $0 < t_1 \le t_2$,

$$
\begin{aligned}
&\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} \\
&\quad = \varepsilon_1\varepsilon_2 \sum_{r,s=1}^{\tilde{R}_h} \left[\int_0^{\frac{t_1}{\varepsilon_1}} \int_0^{t_1-\varepsilon_1 u_1} + \int_{\frac{1}{\varepsilon_1}(t_1-t_2)}^{0} \int_{-\varepsilon_1 u_1}^{t_1-\varepsilon_1 u_1} + \int_{-\frac{t_2}{\varepsilon_1}}^{\frac{1}{\varepsilon_1}(t_1-t_2)} \int_{-\varepsilon_1 u_1}^{t_2} \right] \\
&\qquad * \left[\int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} \int_{x_1^{(s)}}^{x_2^{(r)}-\varepsilon_2 u_2} + \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} \int_{x_1^{(r)}-\varepsilon_2 u_2}^{x_2^{(s)}} \right] \\
&\qquad G_h(t_1-(s_2+\varepsilon_1 u_1))(C^{(r)})^T \underline{\hat{p}}^{(r)}(r_2+\varepsilon_2 u_2)(\underline{\hat{p}}^{(s)})^T(r_2) C^{(s)} \\
&\qquad G_h^T(t_2-s_2) R(u_1,u_2)\, dr_2\, du_2\, ds_2\, du_1 \,.
\end{aligned} \tag{2.2}
$$

Remark 2.2. Without loss of generality we consider only the case $0 < t_1 \le t_2$. Since R is the correlation function of a 1-correlated random field, for $t_1 \ge \varepsilon_1$ the domain of integration $\left[0, \frac{t_1}{\varepsilon_1}\right]$ can be restricted to [0,1]. If the correlation function $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ is considered at times t_1 and t_2 with $t_1 + \varepsilon_1 \le t_2$ the integral with domain of integration $\left[-\frac{t_2}{\varepsilon_1}, \frac{1}{\varepsilon_1}(t_1-t_2)\right]$ vanishes completely by the property of R to be 1-correlated. If we consider on the other hand the case $t := t_1 = t_2$, relation (2.2) simplifies because the domain of integration $\left[\frac{1}{\varepsilon_1}(t_1-t_2), 0\right]$ vanishes.

The advantage of equation (2.2) is that the correlation function R depends only on the integration variables u_1 and u_2, but not on r_2 and s_2. To use this property it is meaningful to define the functions $\phi_i^{(r,s)}$ depending on $(z_1, z_2) = (\varepsilon_1 u_1, \varepsilon_2 u_2)$ for $i = 1, \ldots, 6$ as follows,

$$
\begin{aligned}
\phi_1^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[0,t_1-z_1,x_1^{(s)},x_2^{(r)}-z_2]}(z_1,z_2)\,, \\
\phi_2^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[0,t_1-z_1,x_1^{(r)}-z_2,x_2^{(s)}]}(z_1,z_2)\,, \\
\phi_3^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[-z_1,t_1-z_1,x_1^{(s)},x_2^{(r)}-z_2]}(z_1,z_2)\,, \\
\phi_4^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[-z_1,t_1-z_1,x_1^{(r)}-z_2,x_2^{(s)}]}(z_1,z_2), \\
\phi_5^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[-z_1,t_2,x_1^{(s)},x_2^{(r)}-z_2]}(z_1,z_2)\,, \\
\phi_6^{(r,s)}(z_1,z_2) &= \phi^{(r,s)}_{[-z_1,t_2,x_1^{(r)}-z_2,x_2^{(s)}]}(z_1,z_2)\,,
\end{aligned}
$$

where the notation

$$\phi^{(r,s)}_{[a,b,c,d]}(z_1,z_2) :=$$

$$\int_a^b \int_c^d G_h(t_1-(s_2+z_1))(C^{(r)})^T \underline{\hat{p}}^{(r)}(r_2+z_2)(\underline{\hat{p}}^{(s)})^T(r_2)$$
$$C^{(s)} G_h^T(t_2-s_2)\,dr_2\,ds_2$$

has been used. Hence the correlation function $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ can be written as

$$\begin{aligned}
&\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} = \\
&\varepsilon_1\varepsilon_2 \sum_{r,s=1}^{\tilde{R}_h} \Bigg[\int_0^{\frac{t_1}{\varepsilon_1}} \Bigg[\int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} \phi_1^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) + \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} \phi_2^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) \Bigg] \\
&+ \int_{\frac{1}{\varepsilon_1}(t_1-t_2)}^{0} \Bigg[\int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} \phi_3^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) + \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} \phi_4^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) \Bigg] \\
&+ \int_{-\frac{t_2}{\varepsilon_1}}^{\frac{1}{\varepsilon_1}(t_1-t_2)} \Bigg[\int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} \phi_5^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) + \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} \phi_6^{(r,s)}(\varepsilon_1 u_1, \varepsilon_2 u_2) \Bigg] \Bigg] \\
&\qquad R(u_1,u_2)\,du_2\,du_1\,.
\end{aligned} \tag{2.3}$$

3. ASYMPTOTIC EXPANSIONS

The influence of the random boundary condition ${}^{\varepsilon}\overline{P}$ on the vector $\underline{u}_h$ is described by the integral functional

$$\underline{u}_h(t,\omega) = \int_0^t \int_{(\partial \mathsf{D})_2} G_h(t-s)\underline{\hat{p}}(r)\,{}^{\varepsilon}\overline{P}(s,r,\omega)\,dr\,ds.$$

An asymptotic expansion of $\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}$ of order m with respect to the correlation length ε can be derived, if the deterministic function $f(s,r) := G_h(t-s)\underline{\hat{p}}(r)$ for each t fulfills the following assumption DC (cf. [5]).

Assumption: We say that a deterministic function $f(x)$, $x \in \mathbb{R}^2$ satisfies the assumption DC if it yields for the derivatives $D^{(n_1,n_2)}f(x) := \frac{\partial^{n_1+n_2}}{\partial x_1^{n_1}\partial x_2^{n_2}} f(x_1,x_2)$:

1. $D^{(n_1,n_2)}f$ are continuous for $n = n_1+n_2 \le m$.
2. $D^{(n_1,n_2)}f$, $m = n_1+n_2$ are absolute continuous.
3. $D^{(n_1,n_2)}f$ belong to $L_1(\mathsf{D}) \cap L_2(\mathsf{D})$ for $n = n_1+n_2 \le m+1$.

If we use for instance bilinear ansatz functions the above assumption is only fulfiled for $m = 0$ on the domain D. On the other hand, the assumption is clearly fulfiled for

arbitrary m if the domain D is replaced by a single finite element, because the deterministic function $G_h(t-s)\underline{\hat{p}}(r)$ is arbitrarily smooth on the finite elements. Therefore an asymptotic expansion of higher order can be obtained on the base of equation (2.3).

According to [5] the functions $\phi_i^{(r,s)},\ i=1,\dots,6$ are represented by their Taylor expansions with respect to the correlation length at point $\hat{\varepsilon}=(0,0)$

$$\begin{aligned}&\phi_i^{(r,s)}(\varepsilon_1 u_1,\varepsilon_2 u_2)\\ &\quad=\sum_{|\alpha|\le m}\frac{\varepsilon^\alpha}{\alpha!}D^{(\alpha)}\phi_i^{(r,s)}(z_1,z_2)\Big|_{z_1=z_2=0}u^\alpha+\rho_{\phi_i,m+1}(\varepsilon_1 u_1,\varepsilon_2 u_2)\end{aligned}\tag{3.1}$$

with $\alpha=(\alpha_1,\alpha_2)$, $u^\alpha=u_1^{\alpha_1}u_2^{\alpha_2}$, $\alpha!=\alpha_1!\alpha_2!$, $\varepsilon^\alpha=\varepsilon_1^{\alpha_1}\varepsilon_2^{\alpha_2}$, $|\alpha|=\alpha_1+\alpha_2$ and $D^{(\alpha)}\phi_i^{(r,s)}(z_1,z_2)=\frac{\partial^{\alpha_1+\alpha_2}\phi_i^{(r,s)}}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}}(z_1,z_2)$. Due to the domain of the functions $\phi_i^{(r,s)},\ i=1,\dots,6$ the additional assumption

$$\varepsilon_2\le h$$

is required. Since R is the correlation function of a 1-correlated random field, for a fixed index $r\in\tilde{\psi}_h$ some of the summands in the second sum ($s\in\tilde{\psi}_h$) in equation (2.3) vanish. To be precise, under the assumption $u_1\in(-1,1)$ the functions $\phi_1^{(r,s)}$, $\phi_3^{(r,s)}$ and $\phi_5^{(r,s)}$ for fixed r need to be regarded only for $s\in\{r,Nr\}$. Analogously, the functions $\phi_2^{(r,s)}$, $\phi_4^{(r,s)}$ and $\phi_6^{(r,s)}$ need to be regarded for $s\in\{r,Nl\}$, only. Thereby Nr describes the index of the interval right beside $I^{(r)}$ and Nl the index of the interval left beside (cf. Figure 2).

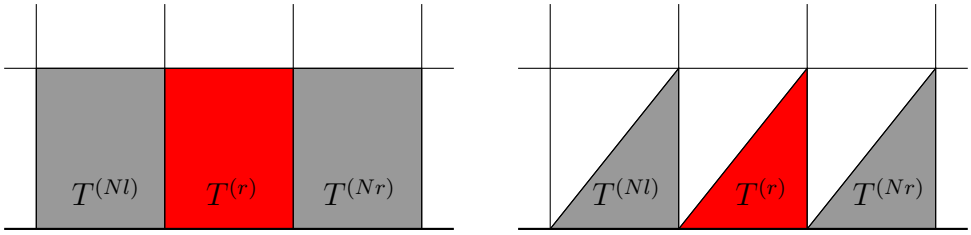

FIGURE 2. Notation of neighbouring elements of $T^{(r)}$ by discretisation by rectangles resp. triangles

A summary of these results leads to the following equation, which is valid for $\varepsilon_2\le h$,

$$\begin{aligned}&\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\}\\ &\quad=\sum_{r=1}^{\tilde{R}_h}\sum_{s=r,Nr}(C^{(r)})^T\sum_{|\alpha|\le m}\frac{\varepsilon^{\alpha+\mathbf{1}}}{\alpha!}\left(\sum_{i=1,3,5}q_{\alpha,i}^{(r,s)}a_{\alpha,i}^{(r,s)}\right)C^{(s)}\\ &\quad+\sum_{r=1}^{\tilde{R}_h}\sum_{s=r,Nl}(C^{(r)})^T\sum_{|\alpha|\le m}\frac{\varepsilon^{\alpha+\mathbf{1}}}{\alpha!}\left(\sum_{i=2,4,6}q_{\alpha,i}^{(r,s)}a_{\alpha,i}^{(r,s)}\right)C^{(s)}\\ &\quad+\tilde{\rho}_{m+1}(\varepsilon_1,\varepsilon_2)\,,\end{aligned}\tag{3.2}$$

with $\mathbf{1} = (1,1)$,

$$q_{\alpha,i}^{(r,s)} := D^{(\alpha)}\phi_i^{(r,s)}(z_1,z_2)\Big|_{z_1=z_2=0},\ i=1,\dots,6\,,$$

as well as

$$a_{\alpha,1}^{(r,s)} = \int_0^{\frac{t_1}{\varepsilon_1}} \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} u^\alpha R(u_1,u_2)du_2du_1\,,$$

$$a_{\alpha,2}^{(r,s)} = \int_0^{\frac{t_1}{\varepsilon_1}} \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} u^\alpha R(u_1,u_2)du_2du_1\,,$$

$$a_{\alpha,3}^{(r,s)} = \int_{\frac{1}{\varepsilon_1}(t_1-t_2)}^{0} \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} u^\alpha R(u_1,u_2)du_2du_1\,,$$

$$a_{\alpha,4}^{(r,s)} = \int_{\frac{1}{\varepsilon_1}(t_1-t_2)}^{0} \int_{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})} u^\alpha R(u_1,u_2)du_2du_1\,,$$

$$a_{\alpha,5}^{(r,s)} = \int_{-\frac{t_2}{\varepsilon_1}}^{\frac{1}{\varepsilon_1}(t_1-t_2)} \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_1^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_1^{(s)})} u^\alpha R(u_1,u_2)du_2du_1\,,$$

$$a_{\alpha,6}^{(r,s)} = \int_{-\frac{t_2}{\varepsilon_1}}^{\frac{1}{\varepsilon_1}(t_1-t_2)} \int_{\frac{1}{\varepsilon_2}(x_1^{(r)}-x_2^{(s)})}^{\frac{1}{\varepsilon_2}(x_2^{(r)}-x_2^{(s)})} u^\alpha R(u_1,u_2)du_2du_1,$$

and remainder $\tilde{\rho}_{m+1}(\varepsilon_1,\varepsilon_2)$.

As a consequence, the expensive evaluation of the quadruple integrals in equation (2.3) can be replaced by sums of corresponding double integrals. The degree m of the expansions has to be chosen according to the desired accuracy. However, the advantage of the usage of ε-correlated random influences lies in the fact, that the number of summands, which have to be considered ($r,s \in \tilde{\psi}_h$), becomes considerable small.

4. COMPARISON WITH THE DIRECT CALCULATION

In the special case of uniform discretisation and bilinear ansatz functions $\underline{p}$ there is an alternative way to calculate the functions $\phi_i^{(r,s)},\ i=1,\dots,6$. In this case, the Taylor expansion can be replaced by the following considerations. For the seek of

simplicity we restrict to the case of a discretisation by rectangles. In the case of triangles, solutions can be obtained in a similar manner.

Let M_h and K_h be the mass and stiffness matrix of the system of ordinary differential equations (1.6), respectively. Considering the generalised eigenvalue problem

$$K_h x = \lambda M_h x\,,$$

there exist matrices Λ_h and V_h with

$$V_h^T K_h V_h = \Lambda_h$$

and

$$V_h^T M_h V_h = I\,,$$

where $\Lambda_h = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_h})$ is a diagonal matrix whose entries are the eigenvalues and V_h consists of the corresponding eigenvectors. Clearly it follows

$$A_h = M_h^{-1} K_h = V_h \Lambda_h V_h^{-1}\,,$$

i.e. the matrix A_h is diagonalisable. Consequently it holds the following relation for the function G_h,

$$G_h(t) = \exp(-M_h^{-1}K_h t)M_h^{-1} = V_h \exp(-\Lambda_h t)V_h^{-1}M_h^{-1} = V_h \exp(-\Lambda_h t)V_h^T.$$

Due to the uniform discretisation the integrals

$$\int_{x_1^{(s)}}^{x_2^{(r)}-z_2} \underline{\hat{p}}^{(r)}(r_2+z_2)(\underline{\hat{p}}^{(s)})^T(r_2)\,dr_2 \quad \text{and} \quad \int_{x_1^{(r)}-z_2}^{x_2^{(s)}} \underline{\hat{p}}^{(r)}(r_2+z_2)(\underline{\hat{p}}^{(s)})^T(r_2)\,dr_2$$

occurring in the definition of the functions $\phi_i^{(r,s)},\ i = 1, \ldots, 6$ in Section 3 now can be transformed to the reference interval $[0,1]$. The ansatz functions $p_i,\ i \in \chi_h$ are assumed to be bilinear, then the form functions $\hat{\varphi}_\alpha,\ \alpha = 1, \ldots, 4$ over $[0,1]$, where $\hat{\varphi}_\alpha(\xi_{I^{(s)}}) := \varphi_\alpha(\xi_{I^{(s)}}, 0)$, can be written as (cf. [3])

$$(\hat{\varphi}_\alpha(\xi_{I^{(s)}}(r_2)))_{\alpha=1,\ldots,4} = \begin{bmatrix} \hat{\varphi}_1(\xi_{I^{(s)}}(r_2)) \\ \hat{\varphi}_2(\xi_{I^{(s)}}(r_2)) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \xi_{I^{(s)}}(r_2) \\ \xi_{I^{(s)}}(r_2) \\ 0 \\ 0 \end{bmatrix}.$$

With the help of the transformations

$$\begin{aligned} &r_2 = h\xi + x_1^{(s)}, \\ &\xi = \xi_{I^{(s)}}(r_2) = \frac{1}{h}(r_2 - x_1^{(s)}), \quad h := x_2^{(s)} - x_1^{(s)} \end{aligned} \tag{4.1}$$

the mapping of the reference interval $[0,1]$ to the interval $I^{(s)} = [x_1^{(s)}, x_2^{(s)}]$ and vice versa are realised, consequently it yields

$$\hat{p}_\alpha^{(s)}(r_2) = \hat{\varphi}_\alpha(\xi_{I^{(s)}}(r_2)) \ \text{ and } \ \hat{p}_\alpha^{(r)}(r_2+z_2) = \hat{\varphi}_\alpha(\xi_{I^{(r)}}(r_2+z_2)),\ \alpha = 1, \ldots, 4\,.$$

For the sake of shortness it is set $\xi_{I^{(s)}}(r_2) = \xi$ and $\xi_{I^{(r)}}(r_2 + z_2) = \bar{\xi}$. With the help of

$$\eta^{(r,s)}(z_2) := \frac{1}{h}(z_2 - k^{(r,s)}), \quad \text{where} \quad k^{(r,s)} := x_1^{(r)} - x_1^{(s)}$$

it holds by using equation (4.1)

$$\bar{\xi} = \frac{1}{h}(r_2 - x_1^{(s)} + z_2 - x_1^{(r)} + x_1^{(s)}) = \xi + \frac{1}{h}(z_2 - k^{(r,s)}) = \xi + \eta^{(r,s)}(z_2).$$

With the notation $\eta := \eta^{(r,s)}(z_2)$ the above integrals now can be straightforwardly calculated as

$$\begin{aligned}
&\int_{x_1^{(s)}}^{x_2^{(r)}-z_2} \underline{\hat{p}}^{(r)}(r_2 + z_2)(\underline{\hat{p}}^{(s)})^T(r_2)dr_2 \\
&\qquad = \frac{h}{6}(1-\eta)\begin{bmatrix} (1-\eta)(2-\eta) & (1-\eta)^2 & 0 & 0 \\ \eta^2 + 4\eta + 1 & (1-\eta)(2+\eta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\qquad =: w_1(z_2, k^{(r,s)}, h)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x_1^{(r)}-z_2}^{x_2^{(s)}} \underline{\hat{p}}^{(r)}(r_2 + z_2)(\underline{\hat{p}}^{(s)})^T(r_2)dr_2 \\
&\qquad = \frac{h}{6}(1+\eta)\begin{bmatrix} -(1+\eta)(2-\eta) & -\eta^2 + 4\eta - 1 & 0 & 0 \\ -(1+\eta)^2 & -(1+\eta)(2-\eta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\qquad =: w_2(z_2, k^{(r,s)}, h),
\end{aligned}$$

where again (4.1) has been used. Consequently the functions $\phi_i^{(r,s)}$, $i = 1, \ldots, 6$ defined in Section 2 can be written as

$$\begin{aligned}
&\phi_1^{(r,s)}(z_1, z_2) \\
&\quad = V_h \int_0^{t_1 - z_1} e^{-\Lambda_h(t_1 - (s_2 + z_1))} V_h^T (C^{(r)})^T w_1(z_2, k^{(r,s)}, h) C^{(s)} V_h e^{-\Lambda_h(t_2 - s_2)} ds_2 V_h^T \\
&\quad = V_h \left[\frac{\overline{W}_{1,ij}^{(r,s)}(z_2)}{\lambda_i + \lambda_j} \left(e^{-\lambda_j(t_2 - t_1 + z_1)} - e^{-\lambda_i(t_1 - z_1) - \lambda_j t_2} \right) \right]_{i,j=1,\ldots N_h} V_h^T, \\
&\phi_2^{(r,s)}(z_1, z_2) \\
&\quad = V_h \int_0^{t_1 - z_1} e^{-\Lambda_h(t_1 - (s_2 + z_1))} V_h^T (C^{(r)})^T w_2(z_2, k^{(r,s)}, h) C^{(s)} V_h e^{-\Lambda_h(t_2 - s_2)} ds_2 V_h^T \\
&\quad = V_h \left[\frac{\overline{W}_{2,ij}^{(r,s)}(z_2)}{\lambda_i + \lambda_j} \left(e^{-\lambda_j(t_2 - t_1 + z_1)} - e^{-\lambda_i(t_1 - z_1) - \lambda_j t_2} \right) \right]_{i,j=1,\ldots N_h} V_h^T,
\end{aligned}$$

$$\phi_3^{(r,s)}(z_1,z_2)$$
$$= V_h \int_{-z_1}^{t_1-z_1} e^{-\Lambda_h(t_1-(s_2+z_1))} V_h^T (C^{(r)})^T w_1(z_2,k^{(r,s)},h) C^{(s)} V_h e^{-\Lambda_h(t_2-s_2)} ds_2 V_h^T$$
$$= V_h \left[\frac{\overline{W}_{1,ij}^{(r,s)}(z_2)}{\lambda_i+\lambda_j} \left(e^{-\lambda_j(t_2-t_1+z_1)} - e^{-\lambda_i t_1 - \lambda_j(t_2+z_1)} \right) \right]_{i,j=1,\dots N_h} V_h^T,$$

$$\phi_4^{(r,s)}(z_1,z_2)$$
$$= V_h \int_{-z_1}^{t_1-z_1} e^{-\Lambda_h(t_1-(s_2+z_1))} V_h^T (C^{(r)})^T w_2(z_2,k^{(r,s)},h) C^{(s)} V_h e^{-\Lambda_h(t_2-s_2)} ds_2 V_h^T$$
$$= V_h \left[\frac{\overline{W}_{2,ij}^{(r,s)}(z_2)}{\lambda_i+\lambda_j} \left(e^{-\lambda_j(t_2-t_1+\varepsilon_1 u_1)} - e^{-\lambda_i t_1 - \lambda_j(t_2+z_1)} \right) \right]_{i,j=1,\dots N_h} V_h^T,$$

$$\phi_5^{(r,s)}(z_1,z_2)$$
$$= V_h \int_{-z_1}^{t_2} e^{-\Lambda_h(t_1-(s_2+z_1))} V_h^T (C^{(r)})^T w_1(z_2,k^{(r,s)},h) C^{(s)} V_h e^{-\Lambda_h(t_2-s_2)} ds_2 V_h^T$$
$$= V_h \left[\frac{\overline{W}_{1,ij}^{(r,s)}(z_2)}{\lambda_i+\lambda_j} \left(e^{-\lambda_i(t_1-t_2-z_1)} - e^{-\lambda_i t_1 - \lambda_j(t_2+z_1)} \right) \right]_{i,j=1,\dots N_h} V_h^T,$$

$$\phi_6^{(r,s)}(z_1,z_2)$$
$$= V_h \int_{-z_1}^{t_2} e^{-\Lambda_h(t_1-(s_2+z_1))} V_h^T (C^{(r)})^T w_2(z_2,k^{(r,s)},h) C^{(s)} V_h e^{-\Lambda_h(t_2-s_2)} ds_2 V_h^T$$
$$= V_h \left[\frac{\overline{W}_{2,ij}^{(r,s)}(z_2)}{\lambda_i+\lambda_j} \left(e^{-\lambda_i(t_1-t_2-z_1)} - e^{-\lambda_i t_1 - \lambda_j(t_2+z_1)} \right) \right]_{i,j=1,\dots N_h} V_h^T.$$

Here, the terms $(\overline{W}_{1,ij}^{(r,s)}(z_2))_{i,j=1,\dots,N_h}$ and $(\overline{W}_{2,ij}^{(r,s)}(z_2))_{i,j=1,\dots,N_h}$ describe the components of the matrices

$$\overline{W}_1^{(r,s)}(z_2) = V_h^T (C^{(r)})^T w_1(z_2,k^{(r,s)},h) C^{(s)} V_h$$

and

$$\overline{W}_2^{(r,s)}(z_2) = V_h^T (C^{(r)})^T w_2(z_2,k^{(r,s)},h) C^{(s)} V_h.$$

Assuming a uniform discretisation of the boundary $(\partial \mathsf{D})_2$ the matrix-valued functions $\phi_i^{(r,s)},\ i=1,\dots,6$ can be calculated in dependence of $z_1=\varepsilon_1 u_1$ and $z_2=\varepsilon_2 u_2$ by the above equations. Again the summands of equation (2.3) become to double integrals.

The basic assumption of the paper consists in the model of an ε-correlated random heat flux ${}^{\varepsilon}\overline{P}$. In this case, the correlation function possesses the property

$$\mathbf{E}\{{}^{\varepsilon}\overline{P}(r_1,s_1), {}^{\varepsilon}\overline{P}(r_2,s_2)\} = 0, \text{ if } |s_2-s_1|>\varepsilon_1 \text{ or } |r_2-r_1|>\varepsilon_2.$$

For every fixed pair (r_1, s_1) the domain of ε-neighbouring points

$$\begin{aligned} D_\varepsilon(r_1, s_1) :=& \{(r_2, s_2) : |r_2 - r_1| < \varepsilon_1 \wedge |s_2 - s_1| < \varepsilon_2\} \\ =& [r_1 - \varepsilon_1, r_1 + \varepsilon_1] \times [s_1 - \varepsilon_2, s_1 + \varepsilon_2] \end{aligned}$$

can be defined. Outside this set the correlation function and therefore the integrand vanish for fixed (r_1, s_1). Consequently the domain of integration can be restricted to the domain $D_\varepsilon(r_1, s_1)$, which reduces the numerical efforts. It yields

$$\begin{aligned} &\mathbf{E}\{\underline{u}_h(t_1)\underline{u}_h^T(t_2)\} \\ &= \int_0^{t_1} \int_{\max\{0, s_1-\varepsilon_2\}}^{\min\{t_2, s_1+\varepsilon_2\}} G_h(t_1 - s_1) \sum_{r=1}^{\tilde{R}_h} \sum_{s=1}^{\tilde{R}_h} (C^{(r)})^T \int_{I^{(r)}} \int_{\min\{\max\{x_1^{(s)}, r_1-\varepsilon_1\}, x_2^{(s)}\}}^{\max\{\min\{x_2^{(s)}, r_1+\varepsilon_1\}, x_2^{(s)}\}} \\ &\underline{p}^{(r)}(r_1)(\underline{p}^{(s)})^T(r_2) R\left(\frac{s_2 - s_1}{\varepsilon_1}, \frac{r_2 - r_1}{\varepsilon_2}\right) dr_2\, dr_1 C^{(s)} G_h^T(t_2 - s_2)\, ds_2\, ds_1 \end{aligned}$$

and this representation of the correlation matrix can be determined by computer algebra systems.

5. NUMERICAL EXAMPLE

In this section we want to verify the analytic results of the previous sections on the basis of the variance functions $\mathbf{E}\{\overline{u}_h(t, x)^2\}$. Note, that it holds $\mathbf{E}\{\overline{u}_h(t, x)\} = 0$.

Additionally to the introduced methods, the variance of the solution of problem (1.2) is evaluated using Monte Carlo methods. Starting from the system of equations (1.6) the random influence ${}^{\varepsilon}\overline{P}$ is modelled based on time-discrete Moving Average fields (see [7]). Then using a realisation of ${}^{\varepsilon}\overline{P}$ the resulting deterministic system of equations is solved. From the corresponding solution $\underline{u}_h^i(t)$ the temperature distribution over the domain D is obtained using $\overline{u}_h^i(t, x) = \sum_{i \in \chi_h} \underline{u}_h^i(t) p_i(x)$. According the law of large numbers the relation

$$E^N(t, x) = \frac{1}{N} \sum_{i=1}^{N} \overline{u}_h^i(t, x)$$

is a suitable estimation for the mean value $\mathbf{E}\{\overline{u}_h(t, x)\}$. An estimate of the correlation function can be obtained by

$$R^N(t_1, t_2, x, y) = \frac{1}{N-1} \sum_{i=1}^{N} \left(\overline{u}_h^i(t_1, x) - E^N(t_1, x_1)\right) \left(\overline{u}_h^i(t_2, y) - E^N(t_2, x_2)\right),$$

and especially for estimation of the variance function it follows

$$V^N(t, x) = R^N(t, t, x, x) = \frac{1}{N-1} \sum_{i=1}^{N} \left(\overline{u}_h^i(t, x) - \frac{1}{N} \sum_{i=1}^{N} \overline{u}_h^i(t, x)\right)^2.$$

The numerical example is based on the following parameters. The considered domain D is the rectangle $[-0.5, 0.5] \times [0, 1]$. The values of the material parameters are chosen as $\lambda = 38.0$ and $\alpha = 10$, respectively. The correlation function of the random influence ${}^{\varepsilon}\overline{P}$ has the form

$$\mathbf{E}\{{}^{\varepsilon}\overline{P}(t_1, x_1){}^{\varepsilon}\overline{P}(t_2, y_1)\}$$

$$= \sigma^2 \begin{cases} \left(1 - \frac{|t_2 - t_1|}{\varepsilon_1}\right)^2 \left(1 - \frac{|y_1 - x_1|}{\varepsilon_2}\right)^2 & |t_2 - t_1| < \varepsilon_1 \text{ and} \\ & |y_1 - x_1| < \varepsilon_2, \\ 0 & \text{else,} \end{cases} \tag{5.1}$$

with $\sigma^2 = 25 \cdot 10^6$. The correlation length is chosen according to $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $\varepsilon_1 = 10^{-5}$ and $\varepsilon_2 = h = \frac{1}{64}$, where ε_1 describes the correlation length in time direction and ε_2 the correlation length in spatial direction.

Figures 3, 4 and 5 show the variance functions plotted over the domain D at time $t = 10^{-2}$, which are calculated by the asymptotic expansion of order 1 (cf. Section 3), the direct calculation (cf. Section 4) and the Monte Carlo simulation. The result of the simulation is obtained by using $N = 10^5$ realisations. It can be seen, that

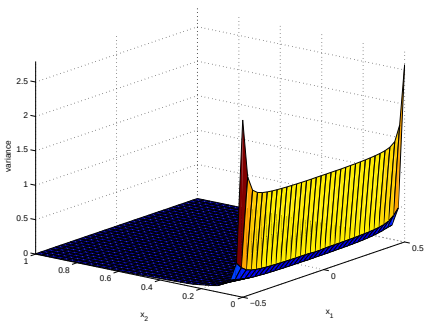

FIGURE 3.
$\mathbf{E}\{\overline{u}_h(t, x)^2\}$ expanded

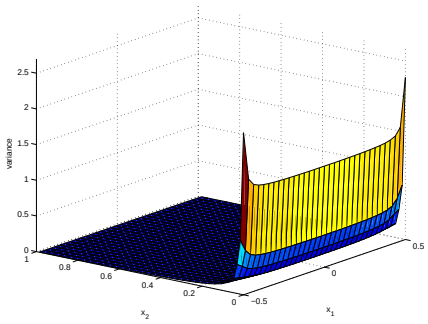

FIGURE 4.
$\mathbf{E}\{\overline{u}_h(t, x)^2\}$ explicit calculated

the variances coincide well, what is emphasised by the two-dimensional cut at point $x_1 = 0$ in x_2-direction shown in Figure 6. This behaviour can be observed also for other parameter constellations, so that we can conclude, that the different ways of calculating the variance function lead to comparable results.

The variance functions have their highest level at the domain around the boundary $(\partial\mathsf{D})_2$. This is due to the fact, that the only random source is the heat flux over this boundary and the temperature of the remaining domain is only weak influenced by ${}^{\varepsilon}\overline{P}$.

Because we are interested in the consequences of the random heat flux we consider a modified domain D, chosen as $\mathsf{D} = [-0.5, 0.5] \times [0, 0.1]$. Due to this comparatively thin domain, ${}^{\varepsilon}\overline{P}$ influences the temperature distribution in the whole domain at time $t = 10^{-2}$ (cf. Figure 7). However, the general behaviour of the variance is the same,

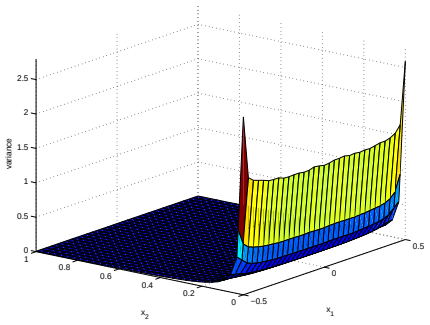

FIGURE 5.
$\mathbf{E}\{\overline{u}_h(t,x)^2\}$ simulated

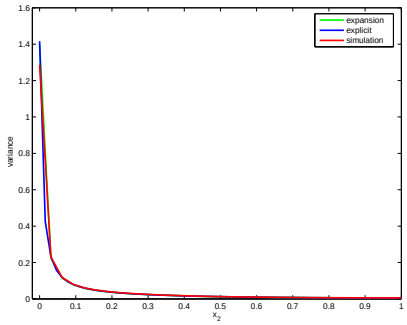

FIGURE 6.
Comparison of the variances at $x_1 = 0$

the highest level is reached in the area around the boundary $(\partial\mathsf{D})_2$, which corresponds to $x_2 = 0$. For higher values of x_2 the variance drops down. To analyse the influence of the deterministic Robin-conditions on the boundary $(\partial\mathsf{D})_3$ we increase the heat transition number α step by step. Higher values of this heat transition number cause a stronger heat flux over the boundary $(\partial\mathsf{D})_3$ and consequently the variance gets smaller in the domain around the boundary and even in the whole domain, cf. Figure 7 for $\alpha = 10$, Figure 8 for $\alpha = 1000$ and Figure 9 for $\alpha = 10000$.

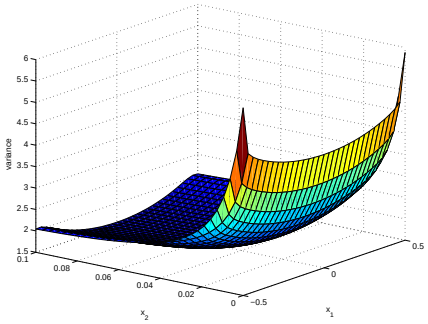

FIGURE 7. $\mathbf{E}\{\overline{u}_h(t,x)^2\}$ for $\alpha = 10$

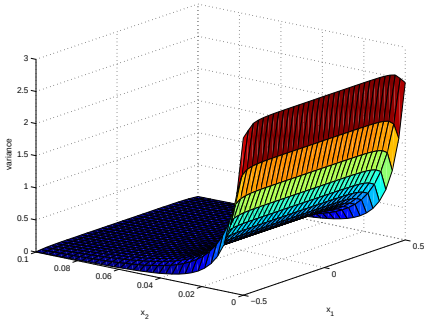

FIGURE 8.
$\mathbf{E}\{\overline{u}_h(t,x)^2\}$ for $\alpha = 1000$

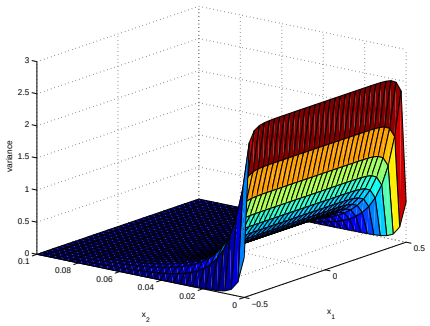

FIGURE 9.
$\mathbf{E}\{\overline{u}_h(t,x)^2\}$ for $\alpha = 10000$

It should be mentioned, that for $\alpha \to \infty$ the Robin-conditions change into Dirichlet-conditions. Then the temperature is fixed on the boundary $(\partial \mathsf{D})_3$ and consequently the variance on $(\partial \mathsf{D})_3$ equals to zero.

Finally, Figure 10 shows the evolution of the variance in time for some points $x = (0\,,\,0.1)\,;\,(0\,,\,0.02)\,;\,(0\,,\,0.04)\,;\,(0\,,\,0.06)\,;\,(0\,,\,0.08)$. The curves illustrate, that the variances grow at the beginning, but stabilise after a certain time. So the system will not explode.

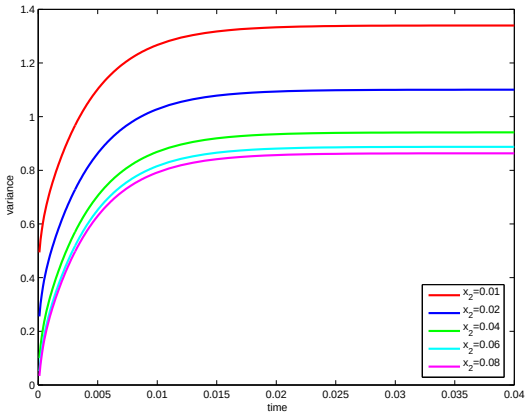

FIGURE 10. $\mathbf{E}\{\overline{u}_h(t,x)^2\}$ for different values of x_2

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