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STRATONOVICH STOCHASTIC INCLUSION

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ABSTRACT. The purpose of the paper is to investigate the existence of strong solutions for the Stratonovich type stochastic inclusion with maximal monotone and upper separated set-valued functions.

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1. PRELIMINARIES

The purpose of the paper is to prove the existence of strong solutions for the following Stratonovich type stochastic inclusion with maximal monotone A and upper separated set-valued functions F and G:

$$x(t) \in x(0) + \int_0^t F(x(s)) \circ dz(s) + \int_0^t G(x(s))da(s) - \int_0^t A(x(s))d[m,m](s),$$

where z is a continuous semimartingale, $[m,m]$ is a quadratic variation process of a local continuous martingale part of a semimartingale z and a is a continuous finite variation process. Let us notice that classical existence results dealing with the Stratonovich type stochastic equation require C^2-regularity of coefficients (see e.g., [6]). J. San Martin in [7] considered the single dimensional Stratonovich equation with coefficients taken from the class of uniformly antiderivative functions (UAD). M. Michta and J. Motyl in [3] introduced the concept of upper separated set-valued functions and proved the existence of solutions to the following inclusion:

$$x(t) \in x(0) + \int_0^t F(x(s)) \circ dz(s) + \int_0^t G(x(s))da(s)$$

with set-valued functions F, G taking their values in $Comp\mathbb{R}$, the space of nonempty, compact and convex subsets of reals. On the other hand, R. Pettersson in [4] proved the existence of solutions to the inclusion of the type:

$$x(t) \in x(0) + \int_0^t b(s,x(s))ds + \int_0^t \sigma(s,x(s))dB(s) - \int_0^t A(x(s))ds$$

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with b, σ being Lipschitz and single valued functions and A-maximal monotone set-valued operator. The inclusion considered in the paper generalizes results obtained in [7] and [3] in the direction inspired by [4].

2. UPPER SEPARATED SET-VALUED FUNCTIONS

The paper is devoted the investigation of the existence of strong solutions of the Stratonovich stochastic inclusion. To this end, it is essential to define first a set-valued Stratonovich integral. We describe the class of upper separated set-valued functions for which a set-valued Stratonovich integral is well defined

Definition 2.1. Let F be a set-valued function from $\mathbb{R}$ into nonempty subsets of $\mathbb{R}$. We define upper and lower bounds of F by the following formulas

$$U_F : X \to \mathbb{R}, \quad U_F(x) = \sup\{b_1 : \quad b_1 \in F(x)\}$$

$$L_F : X \to \mathbb{R}, \quad L_F(x) = \inf\{b_2 : \quad b_2 \in F(x)\}.$$

We say that F is upper separated if for every x and $\epsilon > 0$ there exists a hyperplane $H_{x,\epsilon}$ strongly separating a point $(x, L_F(x) - \epsilon)$ from the set $Epi(U_F) = \{(x,b) \in X \times \mathbb{R} : \quad U_F(x) \leq b\}$.

Let $Comp\mathbb{R}$ denote a space of all nonempty compact and convex subsets of $\mathbb{R}$.

Proposition 2.2. *The upper separated set-valued function $F : \mathbb{R} \to Comp\mathbb{R}$ admits a convex and locally Lipschitz selection.*

Proof. By Proposition 3.4 of [3] the upper separated set-valued function F admits a convex selection $f : \mathbb{R} \to \mathbb{R}$. Let us notice that such f is continuous ([5] Proposition 1.19). By Proposition 1.6 of [5] it follows that a convex and continuous function is locally Lipschitz. □

We will give one more useful property of a convex selection of an upper separated set-valued function. In order to do that we define the following class of functions.

Definition 2.3. ([2]) A function $f : \mathbb{R} \to \mathbb{R}$ belongs to the antiderivative class (AD) if f is absolutely continuous and f' admits a cádlág version, that is $f' = h \;\; dx - a.e.$, for some

$$h \in D = \{h : \mathbb{R} \to \mathbb{R} : \quad h \text{ is right continuous with left limits}\}.$$

Proposition 2.4. *Let $F : \mathbb{R} \to Comp\mathbb{R}$ be an upper separated set-valued function. Then it admits a selection $f \in AD$.*

Proof. By Lemma 2.2 of [7] it suffices to show that f is continuous and its right derivative is cádlág. The upper separated set-valued function F admits a convex and continuous selection f. Convexity of the selection f provides that the right derivative of f is cádlág by [5, Th. 1.16]. □

3. SET-VALUED INTEGRALS

Let $I = [0,T]$ and let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathbf{F}_t\}_{t\in I}$ is an increasing and right continuous family of σ-subalgebras of $\mathbf{F}$ and $\mathbf{F}_0$ contains all P-null sets. The stochastic process x is $\{\mathbf{F}_t\}_{t\in I}$-adapted or shortly adapted if $x(t)$ is $\mathbf{F}_t$-measurable for each $t \in I$. A stochastic process x is called cádlág if its almost all sample paths are right continuous, with left limits. A stochastic adapted and cádlág process x is called a continuous semimartingale, if it can be decomposed into a sum $x = m+v$, where m is a continuous local martingale with respect to $\{\mathbf{F}_t\}_{t\in I}$ while v is a continuous FV-process, i.e., an adapted process with paths of a finite variation.

We define set-valued integrals, which will be used in the next Section.

Definition 3.1. Let a set-valued function $G : \mathbb{R} \to Comp\mathbb{R}$ be given. For a given semimartingale x and an FV-process a we denote by $S(G\circ x, da)$ the set of a-integrable selections of $G \circ x$, i.e.,

$$S(G \circ x, da) := \{h : h(s) \in G(x(s)), \quad s \in I, \quad a.s.,$$

$$\text{and} \int_0^T |h(s)||da(s)| < \infty\}.$$

If the set $S(G \circ x, da)$ is nonempty, we define the set-valued integral

$$\int_0^t G(x(s))da(s) = \left\{ \int_0^t h(s)da(s) : h \in S(G \circ x, da) \right\}.$$

Remark 3.2. If a set-valued function G is upper separated, x is a continuous semimartingale and a is a continuous FV-process, it follows from Proposition 2.2 that the set $S(G \circ x, da)$ is nonempty.

Definition 3.3. A set-valued function $A : \mathbb{R} \to 2^{\mathbb{R}}$ is maximal monotone if

$$\forall s,t \in I \ \ \forall u \in A(s), v \in A(t) \ \ (u-v)(s-t) \geq 0 \ \text{ and } \mathrm{Range}(Id + A) = \mathbb{R},$$

where Id denotes the identity operator. By $A^0(x)$ we denote the minimal norm element from the set $A(x)$.

Let $x = m + v$ be a continuous semimartingale.

Remark 3.4. Since $h(s) = A^0(x(s))$ is a composition of a nondecreasing function $A^0(\cdot)$ and continuous function $x(\cdot,\omega)$ on I we deduce that the set

$$S(A \circ x, d[m,m]) := \{h : h(s) \in A(x(s)), \quad s \in I, \quad a.s.,$$

$$\text{and} \int_0^t |h(s)|d[m,m](s) < \infty \text{ for each } t \in I\}$$

is nonempty.

The single valued Stratonovich integral of a semimartingale h with respect to a continuous semimartingale z is meant as

$$\int_0^t h(s) \circ dz(s) = \int_0^t h(s)_- dz(s) + \frac{1}{2}[h,z]^c(t),$$

where $[h,z]^c(t)$ denotes the path by path continuous part of $[h,z]$. Now we are able to define a set-valued Stratonovich integral.

Definition 3.5. Let a set-valued function $F : \mathbb{R} \to Conv\mathbb{R}$ be given. For a continuous semimartingale x we denote by $S(F \circ x, dz)$ the set of Stratonovich integrable selections of $F \circ x$, i.e.,

$$S(F \circ x, dz) = \{h : h(s) \in F(x(s)), \quad s \in I, \text{ a.s.}, \\ \text{and } \int_0^t h(s) \circ dz(s) < \infty \text{ for each } t \in I\}.$$

If the set $S(F \circ x, dz)$ is nonempty, we define the set-valued integral

$$\int_0^t F(x(s)) \circ dz(s) = \{\int_0^t h(s) \circ dz(s) : \quad h \in S(F \circ x, dz)\}.$$

Remark 3.6. Let us notice that an upper separated set-valued function F admits a convex selection f by Proposition 2.2. Since convex functions preserve semimartingales (e.g., [6, Th. IV.47]), then $h(s) = f(x(s))$ is a semimartingale, too. Therefore, the set $S(F \circ x, dz)$ is nonempty and $\int_0^t F(x(s)) \circ dz(s)$ is well defined.

4. STOCHASTIC STRATONOVICH INCLUSION

We prove the existence of strong solutions to the Stratonovich stochastic inclusion related to set-valued stochastic integrals defined in the previous section.

Let set-valued functions $F, G : \mathbb{R} \to Comp\mathbb{R}$ and $A : \mathbb{R} \to Conv\mathbb{R}$, continuous semimartingales x and z, a continuous FV-process a and $x_0 \in \mathbb{R}$ be given. Let $[m,m]$ denote a quadratic variation process of a local martingale part of z.

Definition 4.1. By the Stratonovich stochastic inclusion we denote the following relation

(SI) $x(t) \in x_0 + \int_0^t F(x(s)) \circ dz(s) + \int_0^t G(x(s))da(s) - \int_0^t A(x(s))d[m,m](s)$

and we say that process x is a strong solution to the inclusion (**SI**) up to explosion time $\widetilde{T} \leq T$ if there exist $f \in S(F \circ x, dz)$, $g \in S(G \circ x, da)$, $h \in S(A \circ x, d[m,m])$ and x is a continuous semimartingale satisfying

$$x(t) = x_0 + \int_0^t f(s)dz(s) + \frac{1}{2}[f,z](t) + \int_0^t g(s)da(s) - \int_0^t h(s)d[m,m](s)$$

for every $t \in [0, \widetilde{T}]$ a.s.

Let us mention that upper separated set-valued functions need not satisfy the linear growth condition. Therefore, one can expect that solutions of inclusion (SI) may have explosions. Recall, a stopping time $\widetilde{T}$ is an explosion time for a solution process x if x is a solution to inclusion (**SI**) on $[0,\widetilde{T})$, $x(\widetilde{T}) = +\infty$ P.1 on $\{\widetilde{T} \le T\}$ and $\widetilde{T} = \lim_{n\to\infty} T_n$, where $T_n := \inf\{t \in [0,T] : |x(t)| > n\}$, for $n \ge 1$. If $P(\widetilde{T} > T) = 1$, then a process x is a nonexploding solution of a Stratonovich inclusion.

For convenience of the reader we recall definitions and lemmas, which will be used in the proof of our main result.

Definition 4.2 ([1])**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be arbitrary functions. We denote $f <^* g$ if for all x there exists $\delta > 0$ such that if $|x-y| < \delta$ and $|x-y'| < \delta$, then $f(y) \le g(y')$.

Lemma 4.3 ([7, Lemma 4.6])**.** *Let m be a continuous local martingale and let a be a continuous, FV-process. Assume $f : \mathbb{R} \to \mathbb{R}$ is locally bounded and satisfies*

$$\forall \Lambda \subseteq \mathbb{R} \textit{ compact} \quad \sup_{(x,y)\in\Lambda\times\Lambda,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

Let h_1 and h_2 be locally bounded and satisfy $h_1 <^* h_2$. If $x_i(t)$, $i = 1,2$ are solutions of

$$x_i(t) = x_0 + \int_0^t f(x_i(s))dm(s) + \int_0^t h_i(x_i(s))da(s),$$

then $x_1(t) \le x_2(t)$ for all $t \in I$ a.s.

Definition 4.4. Process x is called a minimal solution of a stochastic equation, if any other solution y of the equation satisfies $x(t) \le y(t)$ for all $t \in I$ a.s.

Lemma 4.5 ([7, Lemma 4.12])**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be a lower semicontinuous function bounded by M. Then if $h_n(x) = \inf_y\{h(y) + n|x-y|\} - \frac{1}{2^n}$, we have the following:*

1° $\forall x, n\ |h_n(x)| \le M + 1$.
2° $h_n(x) \nearrow h(x)$.
3° $\forall x, y\ |h_n(x) - h_n(y)| \le n|x-y|$.
4° $h_n <^* h_{n+1}$
5° *If h is continuous at x and $x_n \to x$, then $h_n(x_n) \to h(x)$.*

Now, we are ready to prove the main result of the paper.

Theorem 4.6. *Let z be a continuous adapted semimartingale, $[m,m]$ be a quadratic variation process of a local martingale part of z and let "a" be a continuous, FV-process. Let $F, G : \mathbb{R} \to Comp\mathbf{R^1}$ be upper separated set-valued functions and let $A : \mathbb{R} \to Conv\mathbb{R}$ be a maximal monotone set-valued function. Then, there exists a strong solution up to explosion time to Stratonovich stochastic inclusion (SI).*

Proof. Step 1: By Proposition 2.2 there exist convex and continuous selections f and g for set-valued functions F and G respectively. Additionally, by Proposition 2.4 we know that $f \in AD$. From properties of a maximal monotone set-valued function it follows that a set $A(x)$ is closed and convex for each x. Hence there exists the unique minimal norm element $A^0(x)$ in the set $A(x)$. This implies that a cádlág version of a nondecreasing function $A^0(\cdot)$ (denoted also by A^0) is also a selection of a set-valued function A. For the existence of solutions we need the Lipschitz and boundedness property. Let

$$f_k(u) := \begin{cases} f(u), & u \in [-k, k] \\ f(k), & u > k \\ f(-k), & u < -k \end{cases}$$

and

$$(A^0)_k(u) := \begin{cases} A^0(u), & u \in [-k, k] \\ A^0(k), & u > k \\ A^0(-k), & u < -k \end{cases},$$

for $k \geq 1$. Similarly, a function g_k is defined. By Proposition 2.2 functions f and g are convex and locally Lipschitz. Therefore, functions f_k and g_k are globally Lipschitz. Since $f \in AD$, the derivative of f admits a cádlág version. We denote by f_k' the following function

$$f_k'(u) := \begin{cases} f'(u), & u \in [-k, k) \\ 0, & u < -k \text{ lub } u \geq k \end{cases}$$

Since the derivative f_k' coincides on the interval $[-k, k)$ with the right derivative of a function f, by the proof of Proposition 2.4 it turns out that f_k' is a cádlág function. Additionally, the convexity of a function f implies that its right derivative is a nondecreasing function. It means that f_k' is a nondecreasing and right continuous function on $[-k, k)$. Thus and from the fact that f_k' is equal to zero outside $[-k, k)$ we deduce that f_k' is a globally bounded function. We will prove that for every fixed $k \geq 1$ there exists a unique minimal strong solution $\underline{x}_k$ to the equation

$$\begin{aligned} x(t) = x_0 + \int_0^t f_k(x(s))dz(s) + \frac{1}{2}\int_0^t f_k'(x(s))f_k(x(s))d[z,z](s) \\ + \int_0^t g_k(x(s))da(s) - \int_0^t (A^0)_k(x(s))d[m,m](s), \end{aligned} \tag{4.1}$$

which is also a solution to the equation

$$\begin{aligned} x(t) = x_0 + \int_0^t f_k(x(s))dz(s) + \frac{1}{2}[f_k(x), z](t) \\ + \int_0^t g_k(x(s))da(s) - \int_0^t (A^0)_k(x(s))d[m,m](s). \end{aligned}$$

Since $z = m+v$ is the canonical decomposition of z into a local continuous martingale m and a continuous FV-process v, then equation (4.1) can be rewritten as

$$x(t) = x_0 + \int_0^t f_k(x(s))dm(s) + \int_0^t f_k(x(s))dv(s) + \int_0^t g_k(x(s))da(s)$$
$$+ \frac{1}{2}\int_0^t f_k'(x(s))f_k(x(s))d[m,m](s) + \int_0^t -(A^0)_k(x(s))d[m,m](s).$$

Since its second and third components have similar properties i.e., $\pm f_k$ and $\pm g_k$ are Lipschitz and bounded while v and a are continuous FV-processes, we will consider them in the same manner. Therefore, we can restrict our study to the following equation

$$x(t) = x_0 + \int_0^t f_k(x(s))dm(s) + \frac{1}{2}\int_0^t f_k'(x(s))f_k(x(s))d[m,m](s)$$
$$+ \int_0^t b_k(x(s))du(s) + \int_0^t -(A^0)_k(x(s))d[m,m](s), \tag{4.2}$$

with a continuous increasing and adapted process u, and Lipschitz and bounded function b_k. Let us notice that coefficients $f_k'(x(s))f_k(x(s))$ and $-(A^0)_k(x(s))$ need not be Lipschitz. However, using Lemma 4.5 which holds true for lower semicontinuous and bounded functions we can prove the existence of a unique and strong solution to equation (4.2). Since $f_k'(x(s))f_k(x(s))$ are not lower semicontinuous we need to define the following version of the derivative of f_k

$$\widetilde{f_k'}(x) = \begin{cases} f_k'(x-) \wedge f_k'(x), & \text{if } f_k(x) \geq 0 \\ f_k'(x-) \vee f_k'(x), & \text{if } f_k(x) < 0. \end{cases}$$

Let us denote $\rho_k(x) = \frac{1}{2} f_k(x)\widetilde{f_k'}(x)$. The function ρ_k is lower semicontinuous and moreover, $\liminf_{y\to x} \rho_k(y) = \rho_k(x)$. The nonincreasing function $-(A^0)_k$ is also lower semicontinuous because it is cádlág and moreover,

$$\liminf_{y\to x} (-(A^0)_k(y)) = \lim_{y\downarrow x} (-(A^0)_k(y)) = (-(A^0)_k(x)).$$

Step 2: We will prove that the equation

$$x(t) = x_0 + \int_0^t f_k(x(s))dm(s) + \frac{1}{2}\int_0^t \widetilde{f_k'}(x(s))f_k(x(s))d[m,m](s)$$
$$+ \int_0^t b_k(x(s))du(s) + \int_0^t -(A^0)_k(x(s))d[m,m](s), \tag{4.3}$$

possesses a unique minimal solution. For this we adapt the method used by J. San Martin in [7]. Without any loss of generality we assume additionally that $[m,m]$ is uniformly bounded by some constant C. This condition can be easy relaxed by the usual localization method. Let us consider the following functions:

$$\rho_k^r(x) = \inf_y\{\rho_k(y) + r|x-y|\} - 2^{-r}$$

$$b_k^r(x) = \inf_y \{b_k(y) + r|x-y|\} - 2^{-r}$$

$$A_k^r(x) = \inf_y \{(-(A^0)_k(y)) + r|x-y|\} - 2^{-r}.$$

Since functions f_k, b_k, and $-(A^0)_k$ are globally bounded, then there exists a constant also denoted by C, which bounds all of them. Therefore, $\rho_k \leq \frac{1}{2}C^2$. Since these functions are lower semicontinuous, then by Lemma 4.5 we deduce

1° $\forall x \in R\ \ |\rho_k^r(x)| \leq C_1,\ \ |b_k^r(x)| \leq C_1$ and $|A_k^r(x)| \leq C_1$, where $C_1 = (\frac{1}{2}C^2 \vee C)+1$.
2° $\rho_k^r(\cdot) \nearrow \rho_k(\cdot),\ \ b_k^r(\cdot) \nearrow b_k(\cdot),\ \ A_k^r(\cdot) \nearrow (-(A^0)_k(\cdot))$, with $r \to \infty$.
3° $\rho_k^r,\ \ b_k^r$ i A_k^r are Lipschitz with constant r.
4° $\rho_k^r <^* \rho_k^{r+1},\ \ b_k^r <^* b_k^{r+1},\ \ A_k^r <^* A_k^{r+1}$.
5° If ρ_k and $-(A^0)_k$ are continuous at some point p and $x_r \to p$, then $\rho_k^r(x_r) \to \rho_k(p)$ and $A_k^r(x_r) \to (-(A^0)_k(p))$, as $r \to \infty$, each $k = 1, 2, \ldots$.

For each $r = 1, 2, \ldots$ we consider the equation

$$\begin{aligned} y(t) = x_0 &+ \int_0^t f_k(y(s))dm(s) + \int_0^t b_k^r(y(s))du(s) \\ &+ \int_0^t \rho_k^r(y(s))d[m,m](s) + \int_0^t A_k^r(y(s))d[m,m]s. \end{aligned} \tag{4.4}$$

Since all coefficients of (4.4) are Lipschitz, this equation has a unique, strong solution $y_k^r(t)$. The sequence of solutions to equations (4.4) is increasing in r for every fixed k and $t \in [0,T]$ a.s. by Lemma 4.3.

Let $\underline{x}_k(t) = \lim_{r\to\infty} y_k^r(t) \leq \infty$. We claim that $\underline{x}_k(t) < \infty$. Really, let us take a process $z_k^r(t) = y_k^r(t) - x_0$, which for every fixed $t \in [0,T]$ is also increasing in r. Using the Doob's inequality and estimating separatelly coefficients of 4.4 we obtain

$$E\left(\sup_{t\in[0,T]} |\int_0^t f_k(y_k^r(s))dm(s)|^2\right) \leq 4E\left(\int_0^T |f_k(y_k^r(s))|^2 d[m,m](s)\right) \leq 4C^3$$

$$E\left(\sup_{t\in[0,T]} |\int_0^t \rho_k^r(y_k^r(s))d[m,m](s)|^2\right) \leq (C_1)^2C^2,$$

$$E\left(\sup_{t\in[0,T]} |\int_0^t A_k^r(y_k^r(s))d[m,m](s)|^2\right) \leq (C_1)^2C^2$$

$$E\left(\sup_{t\in[0,T]} |\int_0^t b_k^r(y_k^r(s))da(s)|^2\right) \leq (C_1)^2|u(T)-u(0)|^2 \leq (C_1)^2C^2.$$

Moreover,

$$\begin{aligned} E\left(\sup_{t\in[0,T]} |z_k^r(t)|^2\right) = E\Bigg(\sup_{t\in[0,T]} |\int_0^t f_k(y_k^r(s))dm(s) + \int_0^t b_k^r(y_k^r(s))du(s) \\ + \int_0^t \rho_k^r(y_k^r(s))d[m,m](s) + \int_0^t A_k^r(y_k^r(s))d[m,m](s)|^2\Bigg) \leq C_2 \end{aligned}$$

where $C_2 = 4(4C^3+3(C_1)^2C^2)$ does not depend on r. By the Dominated Convergence Theorem we conclude

$$\lim_{r\to\infty} E\left(\sup_{t\in[0,T]} |z_k^r(t)|^2\right) = E\left(\sup_{t\in[0,T]} [(\underline{x}_k(t) - x_0) \vee 0]^2\right) \le C_2 < \infty.$$

Thus $\underline{x}_k(t)$ is finite a.s. for every fixed $t \in [0,T]$. To show that $\underline{x}_k(t)$ is a solution to equation (4.3) we have to prove that for each fixed k

(a) $\int_0^t f_k(y_k^r(s))dm(s) \to \int_0^t f_k(\underline{x}_k(s))dm(s)$
(b) $\int_0^t b_k^r(y_k^r(s))du(s) \to \int_0^t b_k(\underline{x}_k(s))du(s)$
(c) $\int_0^t \rho_k^r(y_k^r(s))d[m,m](s) \to \int_0^t \rho_k(\underline{x}_k(s))d[m,m](s)$
(d) $\int_0^t A_k^r(y_k^r(s))d[m,m](s) \to \int_0^t(-(A^0)_k(\underline{x}_k(s)))d[m,m](s)$,

where the convergence is meant in the sense of ucp as $r \to \infty$. Since f_k is uniformly continuous and for every fixed $t \in [0,T]$ $y_k^r(t) \to \underline{x}_k(t)$, then by the property 5° of Lemma 4.5 we get $f_k(y_k^r(s)) \to f_k(\underline{x}_k(s))$, as $r \to \infty$. Using again the Doob's inequality we obtain

$$\begin{aligned} &E\left(\sup_{t\in[0,T]} |\int_0^t f_k(y_k^r(s))dm(s) - \int_0^t f_k(\underline{x}_k(s))dm(s)|^2\right) \\ &\quad\le 4E\int_0^T |f_k(y_k^r(s)) - f_k(\underline{x}_k(s))|^2 d[m,m](s) \to_{r\to\infty} 0. \end{aligned}$$

Therefore, for some subsequence of $(y_k^r(s))_{r\ge1}$, again denoted by $(y_k^r(s))_{r\ge1}$, we get

$$\sup_{t\in[0,T]} |\int_0^t f_k(y_k^r(s))dm(s) - \int_0^t f_k(\underline{x}_k(s))dm(s)| \to_{r\to\infty} 0 \ \text{ a.s.}$$

Similarly, we obtain

$$\sup_{t\in[0,T]} |\int_0^t [b_k^r(y_k^r(s)) - b_k(\underline{x}_k(s))]du(s)| \to_{r\to\infty} 0.$$

Conditions (c) and (d) are still left to be proved. Thus one should show that

$$\begin{aligned} &\rho_k^r(y^r(s)) \to_{r\to\infty} \rho_k(\underline{x}_k(s)) \\ &A_k^r(y_k^r(s)) \to_{r\to\infty} (-(A^0)_k(\underline{x}_k(s))) \end{aligned} \tag{4.5}$$

a.e. relative to the random measure μ associated with $[m,m]$. The above is clear only for points of continuity of ρ_k and $-(A^0)_k$ respectively. Let us denote

$$\begin{aligned} y_k^r(t) = x(0) &+ \int_0^t f_k(y_k^r(s))dm(s) + \int_0^t b_k^r(y_k^r(s))du(s) \\ &+ \int_0^t [\rho_k^r(y_k^r(s)) + A_k^r(y_k^r(s))]d[m,m](s) = x(0) + S_1 + S_2 + S_3 \end{aligned}$$

Since y_k^r, S_1 and S_2 are convergent, so S_3 also has to converge. Let us denote its limit by $l(t)$. Hence for $t < t'$ we get

$$|l(t) - l(t')| \leq \lim_{r\to\infty} (\int_t^{t'} (\mid \rho_k^r(y_k^r(s)) \mid + \mid A_k^r(y_k^r(s)) \mid) d[m,m](s)) \\ \leq 2C_1 | \, [m,m](t) - [m,m](t') \, |.$$

Therefore, $l(t)$ is continuous and has paths of finite variation. We have shown that

$$\underline{x}_k(t) = x_0 + \int_0^t f_k(\underline{x}_k(s))dm(s) + \int_0^t b_k(\underline{x}_k(s))du(s) + l(t) \tag{4.6}$$

Thus $\underline{x}_k(t)$ is a continuous semimartingale. Since $y_k^r(t)$ and $\underline{x}_k(t)$ are continuous and $y_k^r(t)$ is an increasing in r and convergent to $\underline{x}_k(t)$ for every fixed t, then by the Dini's theorem we conclude that for every ω, $y_k^r(t) \to_{r\to\infty} \underline{x}_k(t)$ uniformly on compact sets. Denote by D_1 a set of points, in which a function ρ_k is discontinuous. Let us notice that $\mu(D_1) = 0$ because D_1 (with μ taken as a measure on s) coincides with the set of points of discontinuity of a cádlág function f_k'. Let D_2 denotes the set of points of discontinuity of a function $(A^0)_k$. Since $(A^0)_k$ is monotone on $[-k,k]$ and constant outside this interval, then $\mu(D_2) = 0$. Let $D = D_1 \cup D_2$. For $\alpha > 0$ we define sets

$$D^\alpha = \{x : (\rho_k(x) \text{ or } (A^0)_k(x) \text{ are discontinuous) and } |f_k(x)| \geq \alpha > 0\}.$$

Let us note that $D = \bigcup_{\alpha>0} D^\alpha$ is a set of points of discontinuity of functions ρ_k or $(A^0)_k$, for which $f_k \neq 0$. Since $D^\alpha \subset D$, then $\mu(D^\alpha) = 0$. To verify conditions (4.5) we will show that $P\{\omega : \mu\{s : \underline{x}_k(s) \in D\} = 0\} = 1$. Let

$$I = E\left(\int_0^t 1_{D^\alpha}(\underline{x}_k(s))d[m,m](s)\right) \\ \leq E\left(\int_0^t 1_{D^\alpha}(\underline{x}_k(s))\alpha^{-2} f_k^2(\underline{x}_k(s))d[m,m](s)\right).$$

By properties of the quadratic covariation process ([6, Th. II.29]Th.II.29) we obtain

$$[\underline{x}_k, m](s) = [x_0 + \int_0^\cdot f_k(\underline{x}_k(q))dm(q) + \int_0^\cdot b_k(\underline{x}_k(q))du(q) + l(\cdot), m](s) \\ = [\int_0^\cdot f_k(\underline{x}_k(q))dm(q), m](s) = \int_0^s f_k(\underline{x}_k(q))d[m,m](q).$$

Hence $d[\underline{x}_k, m](s) = f_k(\underline{x}_k(s))d[m,m](s)$. Then

$$I \leq \alpha^{-2} E(\sup_{t\in[0,T]} \int_0^t 1_{D^\alpha}(\underline{x}_k(s)) f_k(\underline{x}_k(s)) d[\underline{x}_k, m](s)),$$

where 1_A denotes a characteristic function of the set A. Using the Kunita-Watanabe inequality ([6, Th. II.25]) we get

$$I \le \alpha^{-2} E\left(\left(\int_0^T d[m,m](s)\right)^{\frac{1}{2}} \left(\int_0^T 1_{D^\alpha}(\underline{x}_k(s)) f_k^2(\underline{x}_k(s)) d[\underline{x}_k, \underline{x}_k](s)\right)^{\frac{1}{2}}\right)$$

$$\le C^{\frac{1}{2}} \alpha^{-2} E\left(\int_0^T 1_{D^\alpha}(\underline{x}_k(s)) f_k^2(\underline{x}_k(s)) d[\underline{x}_k, \underline{x}_k](s)\right)^{\frac{1}{2}}.$$

By Corollary 1 to Theorem IV.51 of [6] we obtain

$$I \le C^{\frac{1}{2}} \alpha^{-2} E\left(\int_R \int_0^T 1_{D^\alpha}(q) f_k^2(q) L(q, ds) dq\right)^{\frac{1}{2}},$$

where $L(p,s) = L_s^p(\underline{x}_k)$ is the local time of the process $\underline{x}_k$. Considering a global boundedness of a function f_k we get

$$I \le C^{\frac{3}{2}} \alpha^{-2} E\left(\int_R \int_0^T 1_{D^\alpha}(q) L(q, ds) dq\right)^{\frac{1}{2}} \le C^{\frac{3}{2}} \alpha^{-2} E\left(\int_R \int_0^T 1_{D}(q) L(q, ds) dq\right)^{\frac{1}{2}}.$$

Since the set $\mu(D) = 0$, then

$$\int_R \int_0^T 1_D(q) L(q, ds) dq = 0 \quad \text{a.s.}$$

Therefore, $I = 0$ a.e. In particular $P\{\omega : \mu\{s : \underline{x}_k(s) \in D^\alpha\} = 0\} = 1$. Thus $P\{\omega : \mu\{s : \underline{x}_k(s) \in D\} = 0\} = 1$. Let us denote by D^c a complement of the set D. The functions ρ_k and $-(A^0)_k$ are continuous on D^c. Therefore, there exists a set $B \in F$, $P(B) = 1$ such that for every $\omega \in B$

$$\rho_k^r(y_k^r(s)) \to_{r\to\infty} \rho_k(\underline{x}_k(s)) \quad d\mu(s)\text{-a.e.}$$

and

$$A^r(y_k^r(s)) \to_{r\to\infty} (-(A^0)_k(\underline{x}_k(s))) \quad d\mu(s)\text{-a.e.}$$

By the Dominated Convergence Theorem we obtain the convergence property of integrals

$$\int_0^t \rho_k^r(y^r(s)) d[m,m](s) \to \int_0^t \rho_k(\underline{x}_k(s)) d[m,m](s)$$

$$\int_0^t A_k^r(y^r(s)) d[m,m](s) \to \int_0^t (-(A^0)_k(\underline{x}_k(s))) d[m,m](s).$$

Then we have proved that $\underline{x}_k$ is a solution to equation (4.3).

Step 3: Now we will prove that $\underline{x}_k$ is a unique minimal solution to equation (4.2). Let us claim that

$$J(t) = |\int_0^t f_k(\underline{x}_k(s)) \widetilde{f}_k'(\underline{x}_k(s)) - f_k(\underline{x}_k(s)) f_k'(\underline{x}_k(s)) d[m,m](s)| = 0 \quad \text{a.s.}$$

Really, similarly as in the step 2 we get

$$J(t) = |\int_0^t \widetilde{f}_k'(\underline{x}_k(s)) - f_k'(\underline{x}_k(s))d[\underline{x}_k, m](s)|$$
$$\leq \left(\int_0^t d[m,m](s)\right)^{\frac{1}{2}} \left(\int_0^t (\widetilde{f}_k'(\underline{x}_k(s)) - f_k'(\underline{x}_k(s)))^2 d[\underline{x}_k, \underline{x}_k](s)\right)^{\frac{1}{2}}.$$

Using again Corollary 1 to Theorem IV.51 of [6] we obtain

$$J(t) \leq ([m,m](t))^{\frac{1}{2}} \left(\int_R \int_0^t 1_D(q)|\widetilde{f}_k'(q) - f_k'(q)|^2 L(q,ds)dq\right)^{\frac{1}{2}}$$
$$\leq 2C([m,m](t))^{\frac{1}{2}} \left(\int_R \int_0^t 1_D(q)L(q,ds)dq\right)^{\frac{1}{2}} = 0 \text{ a.s.}$$

Therefore, $\underline{x}_k$ is a solution to (4.2) which is in fact equation (4.1). We will prove that $\underline{x}_k$ is a unique minimal solution. Let y_k be any other solution to (4.2). Such an y_k is also a solution to (4.3). By Lemma 4.5 $y_k^r(t) \leq y_k(t)$ for every $t \in [0,T]$, from which we deduce that $\underline{x}_k(t) = \lim_{r\to\infty} y_k^r \leq y_k(t)$ for every $t \in [0,T]$. Then the minimal solution $\underline{x}_k$ is unique.

Step 4: We will prove that $\underline{x} = \lim_{k\to\infty} \underline{x}_k$ is a solution to the equation

$$x(t) = x_0 + \int_0^t f(x(s))dz(s) + \int_0^t g(x(s))da(s)$$
$$+ \frac{1}{2}[f(x), z](t) + \int_0^t -A^0(x(s))d[m,m](s), \tag{4.7}$$

Let us define stopping times $S_k := \inf\{t \in [0,T] : |\underline{x}_k(t)| > k\}$. Then for k and $k+1$ solutions $\underline{x}_k$ and $\underline{x}_{k+1}$ satisfy equations

$$\underline{x}_k(t \wedge S_k) = x_0 + \int_0^{t\wedge S_k} f_k(\underline{x}_k(s))dz(s) + \frac{1}{2}\int_0^{t\wedge S_k} f_k(\underline{x}_k(s))f_k'(\underline{x}_k(s))d[m,m](s)$$
$$+ \int_0^{t\wedge S_k} g_k(\underline{x}_k(s))da(s) - \int_0^{t\wedge S_k} (A^0)_k(\underline{x}_k(s))d[m,m](s)$$

and

$$\underline{x}_{k+1}(t \wedge S_{k+1}) = x_0 + \int_0^{t\wedge S_{k+1}} f_{k+1}(\underline{x}_{k+1}(s))dz(s)$$
$$+ \frac{1}{2}\int_0^{t\wedge S_{k+1}} f_{k+1}(\underline{x}_{k+1}(s))f_{k+1}'(\underline{x}_{k+1}(s))d[m,m]s$$
$$+ \int_0^{t\wedge S_{k+1}} g_{k+1}(\underline{x}_{k+1}(s))da(s)$$
$$- \int_0^{t\wedge S_{k+1}} (A^0)_{k+1}(\underline{x}_{k+1}(s))d[m,m](s).$$

Let us note that $f(u) = f_k(u) = f_{k+1}(u)$ for $|u| \leq k$ and $k = 1, 2, dots$. A similar property holds for functions g, g_k, g_{k+1}, A^0, $(A^0)_k$, $(A^0)_{k+1}$, and multiplications $f'f$,

$f_k' f_k$, $f_{k+1}' f_{k+1}$. Since $|\underline{x}_k(t \wedge S_k)| \le k$, then

$$\begin{aligned}\underline{x}_k(t \wedge S_k) &= x_0 + \int_0^{t\wedge S_k} f(\underline{x}_k(s))dz(s) + \frac{1}{2}\int_0^{t\wedge S_k} f(\underline{x}_k(s))f'(\underline{x}_k(s))d[m,m](s) \\ &\quad + \int_0^{t\wedge S_k} g(\underline{x}_k(s))da(s) - \int_0^{t\wedge S_k} A^0(\underline{x}_k(s))d[m,m](s) \\ &= x_0 + \int_0^{t\wedge S_k} f_{k+1}(\underline{x}_k(s))dz(s) + \frac{1}{2}\int_0^{t\wedge S_k} f_{k+1}(\underline{x}_k(s))f_{k+1}'(\underline{x}_k(s))d[m,m](s) \\ &\quad + \int_0^{t\wedge S_k} g_{k+1}(\underline{x}_k(s))da(s) - \int_0^{t\wedge S_k} (A^0)_{k+1}(\underline{x}_k(s))d[m,m](s).\end{aligned}$$

By the uniqueness of minimal, strong solution we deduce that $\underline{x}_k = \underline{x}_{k+1}$ on $[0, S_k]$. Moreover, $S_k < S_{k+1}$ on $\{S_k < T\}$. Since a sequence of stopping times is increasing, we can define a predictable stopping time $S := \lim_{k\to\infty} S_k$ and the process $\underline{x}$ on the interval $[0, S]$ such that $\underline{x} = \underline{x}_k$ on $[0, S_k]$. The process $\underline{x}$ satisfies equation (4.1) on $[0, S_k]$ for $k = 1, 2, \ldots$, so $\underline{x}$ satisfies also equation (4.6) on $[0, S)$. The stopping time S is the explosion time, which was mentioned in the statement of the Theorem. If $P(S > T) = 1$, then we have a nonexploding solution. Since f, g and A^0 are selectors of F, G and A, respectively, we deduce that $f \circ \underline{x}$, $g \circ \underline{x}$ and $A^0 \circ \underline{x}$ are selections desired in Definition 4.1 and that means that $\underline{x}$ is a solution to inclusion (SI). □

Example 4.7. Let A be a subset of the interval $[0, 1]$ such that $\mu(A) = 1/2$ and for every interval $[a, b] \subset [0, 1]$ $0 < \mu(A \cap [a, b]) < b - a$. Then the measure of the set $A' = [0, 1] \setminus A$ equals $1/2$ and $0 < \mu(A' \cap [a, b]) < b - a$ also. Let us notice that sets A and A' should be dense in $[0, 1]$. Define a set-valued function $F : [0, 1] \to 2^{\mathbb{R}}$ by the formula

$$F(x) = \begin{cases} [1, e^{|x|} + 3], & x \in A \\ [2, e^{|x|} + 4], & x \in ([0, 1] \setminus A) \end{cases}$$

It is clear that the set-valued function F is not Lipschitz continuous, nor lower semicontinuous, nor upper semicontinuous in any point. It does not satisfy any of monotone type conditions either. However, F is upper separated.

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