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ROBUST STABILIZATION OF STOCHASTIC SWITCHED DELAY SYSTEMS VIA STATE-DEPENDENT SWITCHING RULE

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ABSTRACT. This paper considers the robust stabilization of uncertain stochastic switched systems with constant time-delay. While most of the results on stability analysis of switched systems in literature assume all subsystems are stable, the results of this paper can deal with stochastic switched systems consisting of unstable subsystems. Assuming there exists a Hurwitz linear convex combination for the original system, it is shown that a state-dependent switching rule can be found to stabilize the stochastically perturbed system with both uncertainties and time-delay, provided that the perturbation, uncertainties, and time-delay are sufficiently small. An effort was made to give an explicit stability upper bound for the time-delay. The results are also extended to nonlinear systems. Numerical examples are presented to demonstrate the results.

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NOTATION

$\mathbb{R}^n$	the n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	the set of $n \times m$ matrices with real entries
I_n	the n-dimensional identity matrix
$\|\cdot\|$	$\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, for a vector $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$
	$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^m, \|x\| = 1\}$, for a matrix $A \in \mathbb{R}^{n\times m}$
$\mathrm{tr}\,[P]$	the trace of matrix P
$\|\cdot\|_{\mathbf{tr}}$	$\|A\|_{\mathbf{tr}} = \sqrt{\mathrm{tr}\,[A^T A]}$, i.e. the trace norm of a matrix A
$\mathcal{C}_h$	the Banach space of $\mathbb{R}^n$-valued continuous functions defined on the interval $[-h, 0]$
$\|\cdot\|_{\mathcal{C}_h}$	the norm on $\mathcal{C}_h$ defined by $\|\phi\| = \sup_{-h\le s\le 0} \|\phi(s)\|$, for $\phi \in \mathcal{C}_h$
$\lambda_{\max}(P)$	the maximal eigenvalue of matrix P, similarly for $\lambda_{\min}(P)$
$(\Omega, \mathcal{F}, P)$	a complete probability space
$\{\mathcal{F}_t, t \ge 0\}$	a standard filtration on a given complete probability space $(\Omega, \mathcal{F}, P)$

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$w(t)$	a Brownian motion with appropriate dimensions with respect to $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t, t \geq 0\}$
$\mathrm{E}[\cdot]$	the mathematical expectation
$L^{2,h}_{\mathcal{F}_t}$	the family of all $\mathcal{F}_t$ measurable $\mathcal{C}_h$-valued random variables $\phi = \{\phi(\theta) : -h \leq \theta \leq 0\}$ such that $\mathrm{E}\left[\|\phi\|^2_{\mathcal{C}_h}\right] < \infty$

1. INTRODUCTION

Stochastic dynamic modeling plays an essential role in numerous physics and engineering applications. It can be applied wherever random properties of a dynamical system have to be considered. Considerable emphasis has been placed on the stability analysis of the stochastic dynamical systems (see [3,8,15,21]). Moreover, in many applications, the physical or chemical processes are governed by more than one dynamics, in which the dynamic changes among a family of choices depending on the time t or the state x. Such processes are often described by switched systems, or more generally, hybrid systems, which have been studied extensively in recent years (see [18,27,32] and references therein). Due to its many applications in control of mechanical systems, automotive industry, aircraft and air traffic control, switching power converters, and many other fields, the stability analysis of switched systems has attracted a large number of researchers from mathematics, control engineering, and, more recently, computer science communities (see, e.g., [1,2,4–6,9–11,16,19,20,24–26,28–31,34,38–42]). While major advances on this topic have been made by various authors, many important questions related to the stability analysis of such systems still remain unanswered, even for linear systems (see a recent survey on this topic by [33]).

For switched systems, [17] and [33] summarized some basic problems related to their stability issues, among which we note, in particular, the problem of constructing stabilizing switching rule for a family of individually unstable systems. In [35] (see also [37]), the authors address the following problem:

> *Given two linear system $x' = A_1 x$ and $x' = A_2 x$, where A_1 and A_2 are not Hurwitz in that they both have some eigenvalues in the right half plane, determine if there exists a switching rule such that the resulting switched system is stable.*

It is established in [35] that, if there exists a Hurwitz convex linear combination of A_1 and A_2, i.e. there exists some $\alpha \in (0,1)$ such that $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz, then a stabilizing switching rule does exist. This result can be easily generalized to the case of finitely many subsystems. Namely, consider a family of linear systems with coefficient matrices A_1, A_2, $\ldots$, A_N and assume there exists a Hurwitz convex linear combination of these matrices, i.e. there exist real numbers $\alpha_i \in (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$ such that $\sum_{i=1}^N \alpha_i A_i$ is Hurwitz. Then a stabilizing switching rule can also be constructed. The idea of proof involves constructing a common quadratic Lyapunov function and the stability obtained is actually quadratic stability. Later, the work of [35] is extended in [36], by using piecewise quadratic Lyapunov functions as opposed to quadratic Lyapunov functions.

Following the treatment of [35], Kim *et al.* [12] (see also [13]) considered a class of switched systems with time-delay and established similar results for delayed switched systems. Actually, they considered a linear switched system which may consist of more than two subsystems, i.e.

$$\dot{x}(t) = A_i x(t) + B_i x(t-h),$$

where $i \in \{1, 2, \ldots, N\}$. The Hurwitz linear convex combination then became

$$\sum_{i=1}^{N} \alpha_i (A_i + B_i),$$

with $0 < \alpha_i < 1$ and $\sum_{i=1}^{N} \alpha_i = 1$. In [12], they also quantified the size of the stability bound for the time-delay.

While numerous researches have been done on switched systems and hybrid systems, much of the work has focused only on deterministic models that completely characterize the future of the system without allowing any uncertainty. Since nondeterministic factors are almost indispensable in practices, more suitable models should be uncertain and stochastic hybrid systems. Motivated by this fact, the main objective of this paper is to establish a stochastic version of the results in [12] and [35]. The following semi-linear stochastic switched system will be considered:

$$dx(t) = [(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))x(t-h)]dt + g_i(t, x(t), x(t-h))dw(t), \tag{1.1}$$

where $i \in \{1, 2, \ldots, N\}$ and both $\Delta A_i(t)$ and $\Delta B_i(t)$ represent the uncertainties, $g_i(t, x(t), x(t-h))$ the stochastic perturbation, and h the constant time-delay.

For nonlinear switched systems, we will consider the system given by

$$dx(t) = [f_i(t, x(t), x(t-h)) + \Delta f_i(t, x(t), x(t-h))]dt + g_i(t, x(t), x(t-h))dw(t), \tag{1.2}$$

where $i \in \{1, 2, \ldots, N\}$ and f_i can be nonlinear, Δf_i represent the uncertainties, g_i the stochastic perturbation, and h the time-delay.

It should be mentioned that the stability we considered here consists of two types. One is the exponential stability in mean square, and the other is almost sure exponential stability. Both types have been studied by many authors, e.g. [8,14,15,21,22], none of which, however, dealt with switched systems. It should also be noted that although stochastic switched systems were rarely studied, many authors studied the stochastic systems with Markov jump, or Markovian switching (e.g. [23]), where the switching rule is a continuous Markov process. In this sense, our switched systems should be regarded as systems with deterministic switching rules. A most natural question arises: *Is deterministic switching rule a special case of stochastic switching rule and thus our study on deterministically switched systems seems unnecessary?* The answer should be no. Actually, there are at least two reasons for this. First, the techniques dealing with Markovian switchings are different from those with deterministic switchings. Second, as far as stabilization by constrained switching rule is concerned, a deterministic rule would be more applicable than a stochastic one.

The organization of the rest of this paper is as follows. The main results are presented in Section 2, which consists of four subsections. Section 2.1 sets up the problem and proposes a switching rule for semi-linear stochastic switched systems. Section 2.2 proves that under the proposed switching rule and certain conditions on uncertainties, time-delay, and stochastic perturbation, our system is exponentially stable. As special cases, Section 2.3 considers deterministic switched systems, i.e. $g_i \equiv 0$ in (1.1) and (1.2), while Section 2.4 deals with non-switched systems, i.e., stochastic and deterministic differential equations. Section 3 extends the results in Section 2 to nonlinear systems. Section 4 presents numerical examples to demonstrate the main results in Sections 2 and 3. Finally, the paper is concluded with Section 5.

2. STABILITY OF SEMI-LINEAR STOCHASTIC SWITCHED SYSTEMS

2.1. **Problem statement and construction of the switching rule.** Consider the following stochastic uncertain switched system

$$\begin{aligned} dx(t) &= [(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))x(t-h)]dt \\ &\qquad + g_i(t, x(t), x(t-h))dw(t), \quad \sigma(t) = i,\ t \geq t_0, \\ x_{t_0} &= \phi, \end{aligned} \tag{2.1}$$

where $\sigma : [t_0, \infty) \to \{1, 2, \dots, N\}$ is the switching rule, $x \in \mathbb{R}^n$ is the state, h is the constant time-delay, $w(t)$ is an m-dimensional standard Wiener process, $\phi \in L^{2,h}_{\mathcal{F}_t}$ is the initial data, A_i, $B_i \in \mathbb{R}^{n\times n}$ are constant real matrices, and $\Delta A_i(t)$, $\Delta B_i(t)$ are bounded time-dependent uncertainties. We assume $g_i(t, x(t), x(t-h)) : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be locally Lipschitz continuous and satisfy the linear growth conditions as well, which guarantees that every single subsystem of (2.1) has a global unique solution (see [21]). It should be noted that although the switching rule σ here is state-dependent, we might not be able to denote it as a function of x. We shall explain this in detail when the switching rule is described. Moreover, based on the property of the switching rule, which will be explained later as well, we can see that the switched system (2.1) has a unique solution, which is denoted by $x(t; \phi)$ in this paper. Moreover, if we let x_t be defined by $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$, then $x_t \in \mathcal{C}_h$ for each sample ω. It can also be proved that $x_t \in L^{2,h}_{\mathcal{F}_t}$ for all $t \geq t_0$ (see [21]). We assume that $g(t, 0, 0) \equiv 0$ so that system (2.1) yields a trivial solution, whose stability is the main purpose of study in this paper.

Before we construct the switching rule, an essential assumption has to be presented, i.e.

Assumption 2.1. For system (2.1) defined above, there exists a Hurwitz linear convex combination of $A_i + B_i$, that is

$$H = \sum_{i=1}^{N} \alpha_i (A_i + B_i), \tag{2.2}$$

where $0 < \alpha_i < 1$ and $\sum_{i=1}^{N} \alpha_i = 1$.

Under Assumption 2.1, there exist positive definite symmetric matrices P and Q which satisfy

$$H^T P + PH = -Q. \tag{2.3}$$

Equation (2.3) leads immediately to the following lemma

Lemma 2.1 (Prop. 2 in Kim [12]). *Given P and Q in* (2.3)*, define*

$$\Omega_i = \{x \in \mathbb{R}^n : x^T[(A_i + B_i)^T P + P(A_i + B_i)]x \leq -x^T Qx\},$$

then $\mathbb{R}^n = \cup_{i=1}^{N} \Omega_i$.

From Lemma 2.1, we can see that if $x \in \Omega_i$, then the function $V(x) = x^T Px$ decreases along the trajectory of the original system $\dot{x}(t) = (A_i + B_i)x(t)$, which is reduced from system (2.1) when there are no uncertainties, stochastic perturbation, and time-delay. Based on this observation, a stabilizing switching rule is constructed in [35] and [12]. The idea of hysteresis switching [18] is important here to prevent chattering and maintain the property that two consecutive switching events are always separated by a time interval of positive length, i.e. the switching signal function $\sigma(t)$ is piecewise constant and has only a finite number of discontinuities on every bounded time interval.

To define the hysteresis switching, we first enlarge the region Ω_i a little bit to Ω_i' so that they have some overlaps near the boundaries. Ω_i' can be defined as [12]

$$\Omega_i' = \{x \in \mathbb{R}^n : x^T[(A_i + B_i)^T P + P(A_i + B_i)]x \leq -\frac{1}{\zeta} x^T Qx\},$$

where $\zeta > 1$ can be arbitrarily chosen. The switching rule $\sigma : [t_0, \infty) \to \{1, 2, \ldots, N\}$ now can be constructed as below:

(R1) (*Minimal Rule*) Starting from some $t = t_0$, let

$$\sigma(t_0) = \arg\min_i x^T[(A_i + B_i)^T P + P(A_i + B_i)]x,$$

where arg denotes the value of the argument i such that the minimal is attained;

(R2) Maintain $\sigma(t) = i$ as long as $x(t) \in \Omega_i'$ and $\sigma(t^-) = i$;

(R3) Once $x(t_1)$ hits the boundary of Ω_i' for some t_1, let $t_0 = t_1$ and start over according to (**R1**).

Remark 2.1. The designed overlapping regions serve as "buffering regions" and thus allow the switching rule to avoid chattering, i.e. the switching signal function $\sigma(t)$ is piecewise constant and has only a finite number of discontinuities, actually only discontinuities from the left, on every bounded time interval. This observation also enables us to easily construct the unique solution of system (2.1) step by step.

Remark 2.2. The observation that we might not be able to write σ as a function of x is based on the fact that the value of σ is not determined by the current value of x alone, but depends also on the previous value of σ according to (**R2**).

If $\Delta A_i(t) \equiv \Delta B_i(t) \equiv g_i(t, x, y) \equiv 0$, then the work of Wick [35] and Kim [12] guarantees the stability of system (2.1) under the switching rule σ constructed above. In the following section, we will show that the stability is preserved for the stochastic system provided $\Delta A_i(t)$, $\Delta B_i(t)$, and $g_i(t, x, y)$ are sufficiently small.

2.2. Exponential stabilization of stochastic switched systems.

Theorem 2.1. *Let Assumption* 2.1 *hold and P, Q be defined as thereafter. Assume also there exist positive constants β_j, $1 \le j \le 4$, such that*

$$\|\Delta A_i(t)\| \le \beta_1, \quad \|\Delta B_i(t)\| \le \beta_2, \tag{2.4}$$

and

$$\|g_i(t, x, y)\|_{\mathbf{tr}}^2 \le \beta_3 \|x\|^2 + \beta_4 \|y\|^2 , \tag{2.5}$$

for all $t \ge t_0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $i \in \{1, 2, \ldots, N\}$. If for some $\zeta > 1$,

$$h < \frac{a^2\sqrt{d^2 + acd}}{(d + \sqrt{d^2 + acd})(ac + d + \sqrt{d^2 + acd})},$$

where

$$a = \frac{1}{\zeta}\lambda_{\min}(Q) - \|P\| (2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) > 0,$$
$$c = 2 \max_{1 \le i \le N} \|PB_i\| \, (\max_{1 \le i \le N} \|A_i\| + \max_{1 \le i \le N} \|B_i\| + \beta_1 + \beta_2),$$
$$d = \max_{1 \le i \le N} \|PB_i\|^2 (\beta_3 + \beta_4),$$

then system (2.1) *is exponentially stable in mean square.*

Proof. For any given initial data ϕ, we write $x(t; \phi)$ as $x(t)$ for simplicity. By Itô's formula, we have

$$d[x^T(t)Px(t)] = 2x^T(t)P[(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))x(t-h)]dt$$

$$
\begin{aligned}
&+ \operatorname{tr}\left[g_i^T(t,x(t),x(t-h))Pg_i(t,x(t),x(t-h))\right]dt \\
&+ 2x^T(t)Pg_i(t,x(t),x(t-h))dw(t). \qquad (2.6)
\end{aligned}
$$

Note that here $\sigma(t) = i$ and thus we have $x(t) \in \Omega_i'$, which is exactly how we have constructed $\sigma(t)$. By the definition of Ω_i',

$$x^T[(A_i+B_i)^TP + P(A_i+B_i)]x \le -\frac{1}{\zeta}x^TQx.$$

Using this and the assumptions, we obtain that

$$
\begin{aligned}
2x^T(t)P[A_ix(t)+B_ix(t-h)] &= 2x^T(t)P[A_i+B_i]x(t) - 2x^T(t)PB_i[x(t)-x(t-h)] \\
&\le -\frac{1}{\zeta}x^T(t)Qx(t) - 2x^T(t)PB_i[x(t)-x(t-h)] \\
&\le -\frac{1}{\zeta}\lambda_{\min}(Q)\left\|x(t)\right\|^2 - 2x^T(t)PB_i[x(t)-x(t-h)],
\end{aligned}
$$

$$2x^T(t)P[\Delta A_i(t)x(t)+\Delta B_i(t)x(t-h)] \le \|P\|\left((2\beta_1+\beta_2)\left\|x(t)\right\|^2 + \beta_2\left\|x(t-h)\right\|^2\right),$$

and

$$\operatorname{tr}\left[g_i^T(t,x(t),x(t-h))Pg_i(t,x(t),x(t-h))\right] \le \|P\|\left(\beta_3\left\|x(t)\right\|^2 + \beta_4\left\|x(t-h)\right\|^2\right).$$

Substituting these into (2.6) gives

$$
\begin{aligned}
d[x^T(t)Px(t)] \le &-\left[\frac{1}{\zeta}\lambda_{\min}(Q) - \|P\|\left(2\beta_1+\beta_2+\beta_3\right)\right]\left\|x(t)\right\|^2 dt \\
&+ \|P\|\left(\beta_2+\beta_4\right)\left\|x(t-h)\right\|^2 dt - 2x^T(t)PB_i[x(t)-x(t-h)]dt \\
&+ 2x^T(t)Pg_i(t,x(t),x(t-h))dw(t). \qquad (2.7)
\end{aligned}
$$

Moreover, by the definition of stochastic integral, we have, for $t \ge h$,

$$
\begin{aligned}
x(t)-x(t-h) = &\int_{t-h}^t [(A_{\sigma(r)}+\Delta A_{\sigma(r)}(r))x(r) + (B_{\sigma(r)}+\Delta B_{\sigma(r)}(r))x(r-h)]dr \\
&+ \int_{t-h}^t g_{\sigma(r)}(r,x(r),x(r-h))dw(r).
\end{aligned}
$$

Note that regarding the switching signal as a function of time gives the simple expression above. Using this and the assumptions, we obtain, for $t \ge t_0 + h$,

$$
\begin{aligned}
&-2x^T(t)PB_i[x(t)-x(t-h)] \\
&\quad \le [h(\gamma_1+\gamma_2+\gamma_3+\gamma_4)+\gamma_5]\left\|x(t)\right\|^2 \\
&\qquad + \max_{1\le i\le N}\|PB_i\|^2\left(\gamma_1^{-1}\max_{1\le i\le N}\|A_i\|^2 + \gamma_3^{-1}\beta_1^2\right)\int_{t-h}^t \left\|x(r)\right\|^2 dr \\
&\qquad + \max_{1\le i\le N}\|PB_i\|^2\left(\gamma_2^{-1}\max_{1\le i\le N}\|B_i\|^2 + \gamma_4^{-1}\beta_2^2\right)\int_{t-h}^t \left\|x(r-h)\right\|^2 dr \\
&\qquad + \gamma_5^{-1}\max_{1\le i\le N}\|PB_i\|^2\left\|\int_{t-h}^t g_{\sigma(r)}(r,x(r),x(r-h))dw(r)\right\|^2,
\end{aligned}
$$

where $\gamma_i\,(1 \le i \le 5)$ can be chosen later to optimize the estimation. Substituting this into (2.7), we have
(i) for $t \ge t_0 + h$,

$$\begin{aligned} d[x^T(t)Px(t)] \le \Big[&\kappa_1 \|x(t)\|^2 + \kappa_2 \|x(t-h)\|^2 + \kappa_3 \int_{t-h}^{t} \|x(r)\|^2 dr \\ &+\kappa_4 \int_{t-h}^{t} \|x(r-h)\|^2 dr\Big] dt \\ &+ \kappa_5 \left\| \int_{t-h}^{t} g_{\sigma(r)}(r, x(r), x(r-h))dw(r)\right\|^2 dt \\ &+ 2x^T(t)Pg_i(t, x(t), x(t-h))dw(t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \kappa_1 &= -\frac{1}{\zeta}\lambda_{\min}(Q) + \|P\| (2\beta_1 + \beta_2 + \beta_3) + h(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + \gamma_5, \\ \kappa_2 &= \|P\| (\beta_2 + \beta_4), \\ \kappa_3 &= \max_{1\le i\le N} \|PB_i\|^2 (\gamma_1^{-1} \max_{1\le i\le N} \|A_i\|^2 + \gamma_3^{-1}\beta_1^2), \\ \kappa_4 &= \max_{1\le i\le N} \|PB_i\|^2 (\gamma_2^{-1} \max_{1\le i\le N} \|B_i\|^2 + \gamma_4^{-1}\beta_2^2), \\ \kappa_5 &= \gamma_5^{-1} \max_{1\le i\le N} \|PB_i\|^2 . \end{aligned}$$

(ii) for $t_0 \le t < t_0 + h$,

$$\begin{aligned} d[x^T(t)Px(t)] \le -&\left[\frac{1}{\zeta}\lambda_{\min}(Q) - \|P\| (2\beta_1 + \beta_2 + \beta_3) + 1\right] \|x(t)\|^2 dt \\ &+ \|P\| (\beta_2 + \beta_4) \|x(t-h)\|^2 dt + \max_{1\le i\le N} \|PB_i\|^2 \|x(t) - x(t-h)\|^2 dt \\ &+ 2x^T(t)Pg_i(t, x(t), x(t-h))dw(t). \end{aligned} \tag{2.9}$$

Now fix any $\varepsilon > 0$. It is clear that

$$d[e^{\varepsilon t}x^T(t)Px(t)] = \varepsilon e^{\varepsilon t}x^T(t)Px(t)dt + e^{\varepsilon t}d[x^T(t)Px(t)]. \tag{2.10}$$

Moreover, by the (vector form) Itô isometry (see [21]), we have

$$\begin{aligned} &\mathrm{E}\left[\left\|\int_{t-h}^{t} g_{\sigma(r)}(r, x(r), x(r-h))dw(r)\right\|^2\right] \\ &= \int_{t-h}^{t} \mathrm{E}\left[\left\|g_{\sigma(r)}(r, x(r), x(r-h))\right\|_{\mathbf{tr}}^2\right] dr. \end{aligned} \tag{2.11}$$

Substituting the previous estimates (2.8) and (2.9) for $d[x^T(t)Px(t)]$ into (2.10), integrating from t_0 to t, taking expectation from both sides, and by virtue of (2.11), we can obtain that, for all $t \ge t_0$ and $\varepsilon > 0$,

$$\mathrm{E}\left[e^{\varepsilon t}x^T(t)Px(t)\right]$$

$$
\begin{aligned}
&\le \mathrm{E}\left[\phi(0)P\phi(0)\right] + (\kappa_1 + \|P\|\,\varepsilon)\mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s}\,\|x(s)\|^2\,ds\right] \\
&\quad + \kappa_2 \mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s}\,\|x(s-h)\|^2\,ds\right] + \kappa_3 \int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(r)\|^2\right] dr ds \\
&\quad + \kappa_4 \int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(r-h)\|^2\right] dr ds \\
&\quad + \kappa_5 \int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\left\|g_{\sigma(r)}(r, x(r), x(r-h))\right\|_{\mathbf{tr}}\right]^2 dr ds + c_0 \\
&\le \mathrm{E}\left[\phi(0)P\phi(0)\right] + (\kappa_1 + \|P\|\,\varepsilon)\mathrm{E}\left[\int_{t-h}^{t} e^{\varepsilon s}\,\|x(s)\|^2\,ds\right] \\
&\quad + \kappa_2 \mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s}\,\|x(s-h)\|^2\,ds\right] \\
&\quad + (\kappa_3 + \kappa_5\beta_3) \int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(r)\|^2\right] dr ds \\
&\quad + (\kappa_4 + \kappa_5\beta_4) \int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(t-h)\|^2\right] dr ds + c_0, \qquad (2.12)
\end{aligned}
$$

where

$$
\begin{aligned}
c_0 &= -\left[\frac{1}{\zeta}\lambda_{\min}(Q) - \|P\|\left(2\beta_1 + \beta_2 + \beta_3 + \varepsilon\right) + 1\right] \mathrm{E}\left[\int_{t_0}^{h} e^{\varepsilon s}\,\|x(t)\|^2\,ds\right] \\
&\quad + \kappa_2 \mathrm{E}\left[\int_{t_0}^{h} e^{\varepsilon s}\,\|x(s-h)\|^2\,ds\right] \\
&\quad + \max_{1\le i\le N} \|PB_i\|^2\, \mathrm{E}\left[\int_{t_0}^{h} e^{\varepsilon s}\,\|x(s) - x(s-h)\|^2\,ds\right] \\
&\le -\left[\frac{1}{\zeta}\lambda_{\min}(Q) - \|P\|\left(2\beta_1 + \beta_2 + \beta_3 + \varepsilon\right) + 1 + \kappa_2 + 2\max_{1\le i\le N}\|PB_i\|^2\right] \\
&\quad \times h e^{\varepsilon h}\mathrm{E}\left[\|\phi\|_{\mathcal{C}_h}^2\right] + 2\max_{1\le i\le N}\|PB_i\|^2\, h e^{\varepsilon h}\mathrm{E}\left[\|x_{t_0+h}\|_{\mathcal{C}_h}^2\right] < \infty.
\end{aligned}
$$

To get a simpler estimation of the right hand side, we compute the integrals in the previous inequality as follows. Firstly, for $t \ge t_0 + h$,

$$
\begin{aligned}
\mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s}\,\|x(s-h)\|^2\,ds\right] &= \mathrm{E}\left[\int_{t_0}^{t-h} e^{\varepsilon(s+h)}\,\|x(s)\|^2\,ds\right] \\
&\le e^{\varepsilon h}\mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s}\,\|x(s)\|^2\,ds\right] + c_1,
\end{aligned}
$$

where $c_1 = he^{2\varepsilon h}\mathrm{E}\left[\|\phi\|_{\mathcal{C}_h}^2\right] < \infty$. Secondly,

$$
\begin{aligned}
\int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(r)\|^2\right] dr ds &\le \int_{t_0}^{t} \mathrm{E}\left[\|x(r)\|^2\right] \int_{r}^{r+h} e^{\varepsilon s} ds dr \\
&\le h e^{\varepsilon h}\mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon r}\,\|x(r)\|^2\,dr\right] + c_2,
\end{aligned}
$$

where $c_2 = hc_1 < \infty$. Thirdly,

$$\int_{t_0+h}^{t} e^{\varepsilon s} \int_{s-h}^{s} \mathrm{E}\left[\|x(r-h)\|^2\right] dr ds \le he^{\varepsilon h}\mathrm{E}\left[\int_{t_0}^{t} e^{\varepsilon r} \|x(r-h)\|^2 dr\right]$$

$$\le he^{2\varepsilon h}\mathrm{E}\left[\int_{t_0+h}^{t} e^{\varepsilon s} \|x(s)\|^2 ds\right] + c_3,$$

where $c_3 = c_1(h + he^{\varepsilon h}) < \infty$. Substituting all these into (2.12) and reorganizing the items, we finally get

$$\mathrm{E}\left[e^{\varepsilon t} x^T(t) P x(t)\right] \le \rho_1 + \rho_2 \mathrm{E}\left[\int_{t_0}^{t} e^{\varepsilon r} \|x(s)\|^2 ds\right], \tag{2.13}$$

where

$$\rho_1 = \mathrm{E}\left[\phi(0)P\phi(0)\right] + \kappa_2 c_1 + (\kappa_3 + \kappa_5\beta_3)c_2 + (\kappa_4 + \kappa_5\beta_4)c_3 + c_0 < \infty,$$

and

$$\rho_2 = \kappa_1 + \|P\|\,\varepsilon + \kappa_2 e^{\varepsilon h} + (\kappa_3 + \kappa_5\beta_3)he^{\varepsilon h} + (\kappa_4 + \kappa_5\beta_4)he^{2\varepsilon h}.$$

Our aim is to find some positive ε such that $\rho_2 = 0$. Since ρ_2 is increasing with respect to ε, it is clear that if

$$\kappa_1 + \kappa_2 + (\kappa_3 + \kappa_5\beta_3)h + (\kappa_4 + \kappa_5\beta_4)h < 0,$$

then there exists some positive ε such that ρ_2 becomes 0. Recall what we denote by $\kappa_i\,(1 \le i \le 4)$ and solve the previous inequality for h gives that

$$h < \frac{\frac{1}{\zeta}\lambda_{\min}(Q) - \|P\|\,(2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) - \gamma_5}{D},$$

where

$$D = (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + \max_{1\le i\le N} \|PB_i\|^2 \times$$
$$(\gamma_1^{-1} \max_{1\le i\le N} \|A_i\|^2 + \gamma_2^{-1} \max_{1\le i\le N} \|B_i\|^2 + \gamma_3^{-1}\beta_1^2 + \gamma_4^{-1}\beta_2^2 + \gamma_5^{-1}\beta_3 + \gamma_5^{-1}\beta_4).$$

Moreover, to relax the restriction on h, $\gamma_i\,(1 \le i \le 5)$ can be chosen to minimize the denominator D, that is

$$\gamma_1 = \max_{1\le i\le N} \|PB_i\| \max_{1\le i\le N} \|A_i\|,$$
$$\gamma_2 = \max_{1\le i\le N} \|PB_i\| \max_{1\le i\le N} \|B_i\|,$$
$$\gamma_3 = \max_{1\le i\le N} \|PB_i\|\,\beta_1,$$
$$\gamma_4 = \max_{1\le i\le N} \|PB_i\|\,\beta_2,$$

which immediately gives that

$$D = 2\max_{1\le i\le N} \|PB_i\|\,(\max_{1\le i\le N} \|A_i\| + \max_{1\le i\le N} \|B_i\| + \beta_1 + \beta_2) + \gamma_5^{-1} \max_{1\le i\le N} \|PB_i\|^2\,(\beta_3 + \beta_4).$$

Now, γ_5 is still not determined. To finish this, we observed that the function $f(x)$ on $(0, \infty)$ defined as

$$f(x) = \frac{a - x}{c + x^{-1}d}$$

for positive a, c, and d is increasing on $(0, \frac{ad}{d+\sqrt{d^2+acd}})$ and deceasing on $(\frac{ad}{d+\sqrt{d^2+acd}}, \infty)$. Hence if we use the notations

$$\begin{aligned} a &= \frac{1}{\zeta}\lambda_{\min}(Q) - \|P\|\,(2\beta_1 + 2\beta_2 + \beta_3 + \beta_4), \\ c &= 2 \max_{1\le i\le N} \|PB_i\|\,(\max_{1\le i\le N} \|A_i\| + \max_{1\le i\le N} \|B_i\| + \beta_1 + \beta_2), \\ d &= \max_{1\le i\le N} \|PB_i\|^2\,(\beta_3 + \beta_4), \end{aligned}$$

then $\gamma_5 = \frac{ad}{d+\sqrt{d^2+acd}}$ would maximize the upper bound of h and actually yields

$$h < \frac{a - \frac{ad}{d+\sqrt{d^2+acd}}}{c + \frac{d+\sqrt{d^2+acd}}{a}} = \frac{a^2\sqrt{d^2+acd}}{(d+\sqrt{d^2+acd})(ac+d+\sqrt{d^2+acd})},$$

which is our final restriction on h to guarantee that there exists some ε such that ρ_2 becomes 0. Hence we have for this ε that

$$\mathrm{E}\left[e^{\varepsilon t}x^T(t)Px(t)\right] \le \rho_1, \tag{2.14}$$

for $t \ge t_0 + h$, which eventually implies

$$\mathrm{E}\left[\|x(t)\|^2\right] \le \frac{\rho_1 e^{-\varepsilon t}}{\lambda_{\min}(P)},$$

for all $t \ge t_0 + h$. Hence

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathrm{E}\left[\|x(t)\|^2\right]) \le -\varepsilon,$$

which means system (2.1) is exponentially stable in mean square. □

Remark 2.3. The main techniques we use here in the proof are essentially the same as those in [22], with some modifications to make it suitable for switched systems and improve the results.

On the almost sure stability of system (2.1), we have the following theorem.

Theorem 2.2. *Under the same assumptions as in Theorem* 2.1*, system* (2.1) *is also almost surely exponentially stable.*

Proof. The proof is based on Doobs martingale inequality and the Borel-Cantelli lemma. For a complete proof one can refer to [22]. □

2.3. **Exponential stabilization of deterministic switched systems.** If there is no stochastic perturbation in system (2.1), as corollaries of Theorem 2.1 and Theorem 2.2, we can have some results on deterministic switched systems.

2.3.1. *The case* $g_i(t,x,y) \equiv 0$. In this case, system (2.1) reduces to

$$\dot{x}(t) = (A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))x(t-h), \quad \sigma(t) = i,\ t \geq t_0, \\ x_{t_0} = \phi. \tag{2.15}$$

For this system, we have the following corollary.

Corollary 2.1. *Let Assumption* 2.1 *hold and* P, Q *be defined as thereafter. Assume also there exist positive constants* β_j, $1 \leq j \leq 2$, *such that*

$$\|\Delta A_i(t)\| \leq \beta_1, \quad \|\Delta B_i(t)\| \leq \beta_2, \tag{2.16}$$

for all $t \geq t_0$ *and* $i \in \{1, 2, \dots, N\}$. *If for some* $\zeta > 1$,

$$h < \frac{\lambda_{\min}(Q) - 2\|P\|(\beta_1 + \beta_2)\zeta}{2\zeta \max\limits_{1\leq i\leq N} \|PB_i\| \left(\max\limits_{1\leq i\leq N} \|A_i\| + \max\limits_{1\leq i\leq N} \|B_i\| + \beta_1 + \beta_2\right)},$$

then system (2.15) *is exponentially stable in mean square and also almost surely exponentially stable.*

2.3.2. *The case* $g_i(t,x,y) \equiv 0$ *and* $\Delta A_i(t) \equiv \Delta B_i(t) \equiv 0$. Now system (2.1) further reduces to

$$\dot{x}(t) = A_i x(t) + B_i x(t-h), \quad \sigma(t) = i,\ t \geq t_0, \\ x_{t_0} = \phi, \tag{2.17}$$

which is actually the same system considered in [12]. For this system, we have the following corollary.

Corollary 2.2. *Let Assumption* 2.1 *hold and* P, Q *be defined as thereafter. If for some* $\zeta > 1$,

$$h < \frac{\lambda_{\min}(Q)}{2\zeta \max\limits_{1\leq i\leq N} \|PB_i\| \left(\max\limits_{1\leq i\leq N} \|A_i\| + \max\limits_{1\leq i\leq N} \|B_i\|\right)}, \tag{2.18}$$

then system (2.17) *is exponentially stable in mean square and also almost surely exponentially stable.*

Remark 2.4. Note that $\zeta > 1$ is arbitrarily chosen in the definition of Ω_i' in section 2.1 and it is useful to avoid chattering of the switched system via boundaries. However, it actually plays no role in the restrictions on the system. For example, (2.18) is mathematically equivalent to

$$h \leq \frac{\lambda_{\min}(Q)}{2 \max\limits_{1\leq i\leq N} \|PB_i\| \left(\max\limits_{1\leq i\leq N} \|A_i\| + \max\limits_{1\leq i\leq N} \|B_i\|\right)}.$$

2.4. **Exponential stabilization of stochastic and deterministic delay differential equations.** Although our main efforts have been devoted to establish results on switched systems, as a by-product, we can still have the following corollaries on the exponential stability of differential delay equations, which can also be regarded as an improvement of the results in [22].

Here, we first consider a stochastic differential equation

$$\begin{aligned} \dot{x}(t) &= [(A+\Delta A(t))x(t) + (B+\Delta B(t))x(t-h)]dt \\ &\qquad + g(t, x(t), x(t-h))dw(t), \quad t \geq t_0, \\ x_{t_0} &= \phi. \end{aligned} \tag{2.19}$$

For this equation, we have the following result.

Corollary 2.3. *Assume that there exists a pair of symmetric positive definite matrices P and Q such that*

$$P(A+B) + (A+B)^T P = -Q.$$

Assume also that there exist positive constants β_j, $1 \leq j \leq 4$, such that

$$\|\Delta A_i(t)\| \leq \beta_1, \quad \|\Delta B_i(t)\| \leq \beta_2, \tag{2.20}$$

and

$$\|g_i(t,x,y)\|_{\mathbf{tr}}^2 \leq \beta_3 \|x\|^2 + \beta_4 \|y\|^2, \tag{2.21}$$

for all $t \geq t_0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $i \in \{1, 2, \ldots, N\}$. If

$$h < \frac{a^2\sqrt{d^2+acd}}{(d+\sqrt{d^2+acd})(ac+d+\sqrt{d^2+acd})},$$

where

$$\begin{aligned} a &= \lambda_{\min}(Q) - \|P\|(2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) > 0, \\ c &= 2\|PB\|(\|A\| + \|B\| + \beta_1 + \beta_2), \\ d &= \|PB\|^2(\beta_3 + \beta_4), \end{aligned}$$

then system (2.19) *is exponentially stable in mean square and also almost surely exponentially stable.*

If $g_i(t,x,y) \equiv 0$, then (2.24) reduces to

$$\begin{aligned} \dot{x}(t) &= (A+\Delta A(t))x(t) + (B+\Delta B(t))x(t-h), \quad t \geq t_0, \\ x_{t_0} &= \phi. \end{aligned} \tag{2.22}$$

For this equation, we have the following result.

Corollary 2.4. *Assume that there exists a pair of symmetric positive definite matrices P and Q such that*

$$P(A+B)+(A+B)^TP=-Q.$$

Assume also that there exist positive constants β_j, $1 \le j \le 4$, such that

$$\|\Delta A_i(t)\| \le \beta_1, \quad \|\Delta B_i(t)\| \le \beta_2, \tag{2.23}$$

for all $t \ge t_0$. If

$$h < \frac{\lambda_{\min}(Q) - 2\|P\|(\beta_1+\beta_2)}{2\|PB\|(\|A\|+\|B\|+\beta_1+\beta_2)},$$

then system (2.22) *is exponentially stable in mean square and also almost surely exponentially stable.*

Furthermore, if $\Delta A(t) \equiv \Delta B(t) \equiv 0$, then equation (2.22) reduces to

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bx(t-h), \quad t \ge t_0, \\ &x_{t_0} = \phi. \end{aligned} \tag{2.24}$$

For this equation, we have the following result.

Corollary 2.5. *Assume that there exists a pair of symmetric positive definite matrices P and Q such that*

$$P(A+B)+(A+B)^TP=-Q.$$

If

$$h < \frac{\lambda_{\min}(Q)}{2\|PB\|(\|A\|+\|B\|)},$$

then system (2.24) *is exponentially stable in mean square and also almost surely exponentially stable.*

Remark 2.5. From Corollary 2.5, we can see that our results have improved those in [22]. We shall demonstrate this with some numerical results in the following section.

3. STABILITY OF NONLINEAR STOCHASTIC SWITCHED SYSTEMS

In this section, we extend the theory in [12, 35] and our previous section to nonlinear case, i.e. we try to find a suitable switching rule to stabilize switched systems arising from nonlinear stochastic differential equations.

Generally, we can consider the following nonlinear stochastic switched system

$$\begin{aligned} dx(t) &= [f_i(t,x(t),x(t-h)) + \Delta f_i(t,x(t),x(t-h))]dt \\ &\quad + g_i(t,x(t),x(t-h))dw(t), \quad \sigma(t)=i,\ t \ge t_0, \\ x_{t_0} &= \phi, \end{aligned} \tag{3.1}$$

where $\sigma : [t_0, \infty) \to \{1, 2, \ldots, N\}$ is the switching rule. We assume that f_i and Δf_i are locally Lipschitz continuous functions from $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the linear growth condition so that each subsystem (i.e. stochastic differential equation) has a unique solution (see [21]). Then we can have global existence and uniqueness result for system (3.1) by stepwise argument, i.e. the solution can be constructed and shown to be unique step by step. Also we assume $f_i(t, 0, 0) \equiv \Delta f_i(t, 0, 0) \equiv 0$ such that the system admits a trivia solution $x(t; 0) \equiv 0$. The following assumption is a counterpart to Assumption 2.1.

Assumption 3.1. For system (3.1) defined above, there exist a symmetric positive definite matrix P and positive constants λ and α_i, $1 \le i \le N$, such that $\sum_{i=1}^N \alpha_i = 1$ and

$$2\sum_{i}^{N} \alpha_i x^T P f_i(t, x, x) \le -\lambda \|x\|^2,$$

for all $t \ge t_0$ and $x \in \mathbb{R}^n$.

Lemma 3.1. *Define*

$$\Omega_i = \{x \in \mathbb{R}^n : x^T P f_i(t, x, x) + f_i^T(t, x, x) P x \le -\lambda \|x\|^2\}.$$

Then $\mathbb{R}^n = \cup_{i=1}^N \Omega_i$.

This lemma guarantees that we can construct a state-dependent switching rule for system (3.1) as we do in section 2. However, we do note that here Ω_i may depend on t. The idea is that for each fixed time t, we can divide the total state space $\mathbb{R}^n$ into N subspaces Ω_i, $1 \le i \le N$, such that for each state x (together with the associated time t), a unique switching signal can be determined as we do in section 2. This does not contradict that we write σ as function of t but not a function of x (see Remark 2.2). Nevertheless we can prove the following theorem.

Theorem 3.1. *Let Assumption* 3.1 *hold and P be the symmetric positive definite matrix. Assume also there exist positive constants δ_j, $1 \le j \le 3$, and β_k, $1 \le k \le 4$, such that*

$$\begin{aligned}
\|f_i(t, x, x) - f_i(t, x, y)\| &\le \delta_1 \|x - y\|, \\
\|f_i(t, x, y)\| &\le \delta_2 \|x\| + \delta_3 \|y\|, \\
\|\Delta f_i(t, x, y)\| &\le \beta_1 \|x\| + \beta_2 \|y\|, \\
\|g_i(t, x, y)\|_{\mathbf{tr}}^2 &\le \beta_3 \|x\|^2 + \beta_4 \|y\|^2,
\end{aligned}$$

for all $t \ge t_0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $i \in \{1, 2, \ldots, N\}$. If for some $\zeta > 1$,

$$h < \frac{a^2\sqrt{d^2 + acd}}{(d + \sqrt{d^2 + acd})(ac + d + \sqrt{d^2 + acd})},$$

where

$$a = \frac{1}{\zeta}\lambda - \|P\|\,(2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) > 0,$$
$$c = 2\delta_1\,\|P\|\,(\delta_2 + \delta_3 + \beta_1 + \beta_2),$$
$$d = \delta_1\,\|P\|^2\,(\beta_3 + \beta_4),$$

then system (3.1) *is exponentially stable in mean square and also almost surely exponentially stable.*

Proof. The proof is essentially similar to the proof for Theorem 2.1 and is omitted. □

Now we turn to consider nonlinear deterministic systems. If $g_i(t, x, y) \equiv 0$, then system (3.1) reduces to a deterministic switched system, i.e.

$$\dot{x}(t) = f_i(t, x(t), x(t-h)) + \Delta f_i(t, x(t), x(t-h)), \quad \sigma(t) = i,\ t \geq t_0, \qquad (3.2)$$
$$x_{t_0} = \phi.$$

The following is a corollary of Theorem 2.2.

Corollary 3.1. *Let Assumption* 3.1 *hold and* P *be the symmetric positive definite matrix. Assume also there exist positive constants* δ_j, $1 \leq j \leq 3$, *and* β_k, $1 \leq k \leq 2$, *such that*

$$\|f_i(t, x, x) - f_i(t, x, y)\| \leq \delta_1\,\|x - y\|\,,$$
$$\|f_i(t, x, y)\| \leq \delta_2\,\|x\| + \delta_3\,\|y\|\,,$$
$$\|\Delta f_i(t, x, y)\| \leq \beta_1\,\|x\| + \beta_2\,\|y\|\,,$$

for all $t \geq t_0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, *and* $i \in \{1, 2, \ldots, N\}$. *If for some* $\zeta > 1$,

$$h < \frac{\lambda - 2\,\|P\|\,(\beta_1 + \beta_2)\zeta}{2\delta_1\zeta\,\|P\|\,(\delta_2 + \delta_3 + \beta_1 + \beta_2)},$$

then system (3.2) *is exponentially stable in mean square and also almost surely exponentially stable.*

Furthermore, if $\Delta f_i(t, x, y) \equiv 0$, system (3.2) further reduces to

$$\dot{x}(t) = f_i(t, x(t), x(t-h)), \quad \sigma(t) = i,\ t \geq t_0, \qquad (3.3)$$
$$x_{t_0} = \phi.$$

For this system, we have the following result.

Corollary 3.2. *Let Assumption* 3.1 *hold and* P *be the symmetric positive definite matrix. Assume also there exist positive constants* δ_j, $1 \leq j \leq 3$, *such that*

$$\|f_i(t, x, x) - f_i(t, x, y)\| \leq \delta_1\,\|x - y\|\,,$$
$$\|f_i(t, x, y)\| \leq \delta_2\,\|x\| + \delta_3\,\|y\|\,,$$

for all $t \geq t_0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, *and* $i \in \{1, 2, \ldots, N\}$. *If for some* $\zeta > 1$,

$$h < \frac{\lambda}{2\delta_1 \zeta \|P\| (\delta_2 + \delta_3)},$$

then system (3.3) *is exponentially stable in mean square and also almost surely exponentially stable.*

To close this section, let us point out that, as byproducts, Theorem 3.1, Corollaries 3.1 and 3.2 can be applied to nonlinear stochastic and deterministic differential equations, i.e. the cases when $N = 1$ and there is only one subsystem in the switched system. We have similar corollaries listed in Section 2.4 and hence we omit them to avoid redundancy.

Remark 3.1. When dealing with nonlinear stochastic and deterministic differential equations, Corollaries 3.1 and 3.2 also improved the results in [22].

4. NUMERICAL EXAMPLES

4.1. Stabilization of switched system via state-dependent switching rule.

Example 4.1. Consider the switched system given by

$$\begin{aligned} dx(t) = {} & [(A_1 + \Delta A_1(t))x(t) + (B_1 + \Delta B_1(t))x(t-h)]dt \\ & + g_1(t, x(t), x(t-h))dw(t), \quad \sigma(t) = 1, \end{aligned} \tag{4.1}$$

$$\begin{aligned} dx(t) = {} & [(A_2 + \Delta A_2(t))x(t) + (B_2 + \Delta B_2(t))x(t-h)]dt \\ & + g_2(t, x(t), x(t-h))dw(t), \quad \sigma(t) = 2, \end{aligned} \tag{4.2}$$

where

$$A_1 = \begin{bmatrix} -2 & 2 \\ -20 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & -7 \\ 23 & 6 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2 & 10 \\ -4 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 & -5 \\ 1 & -8 \end{bmatrix},$$

$$\|\Delta A_i(t)\| \leq 0.1, \quad \|\Delta B_i(t)\| \leq 0.1, \quad i = 1, 2,$$

and

$$\|g_i(t, x, y)\|_{\mathbf{tr}}^2 \leq 0.1 \|x\|^2 + 0.1 \|y\|^2, \quad i = 1, 2,$$

for all $t \geq 0$ and $x = [x_1 \ x_2]^T$ and $y = [y_1 \ y_2]^T \in \mathbb{R}^2$. Let

$$H = 0.52(A_1 + B_1) + 0.48(A_2 + B_2)$$

and $Q = I_2$, we can find

$$P = \begin{bmatrix} 0.8247 & -0.0429 \\ -0.0429 & 0.1870 \end{bmatrix}$$

such that $PH + H^T P = -Q$. It is easy to compute that

$$\|P\| = 0.8276, \ \max_{1\le i\le 2} \|PB_i\| = 7.0148, \ \max_{1\le i\le 2} \|A_i\| = 20.1803, \ \max_{1\le i\le 2} \|B_i\| = 23.9390.$$

Applying Theorem 2.1 with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.1$, we can conclude that the system switching between (4.1) and (4.2) via the switching rule σ is exponentially stable for $h < 5.6895 \times 10^{-4}$. If there are no stochastic perturbations and uncertainties, i.e. $g_i(t,x,y) \equiv 0$ and $\Delta A_i(t) \equiv \Delta B_i(t) \equiv 0$, then the system reduces to Example 1 in [12]. Corollary 2.2 shows that the system is exponentially stable for $h < 0.001615$, which improves the delay upper bound of 0.001573 in [12]. Typical paths of the evolution and the switching rule σ for the above system are shown in Figures 1 and 2 for $h = 0.0005$, $\zeta = 1.1$,

$$\Delta A_1(t) = \Delta B_1(t) = 0.1\sin(10t)I_2,$$
$$\Delta A_2(t) = \Delta B_2(t) = 0.1\cos(10t)I_2,$$

and

$$g_1 = g_2 = \frac{\sqrt{0.1}}{2}\begin{bmatrix} x_1\sin(x_1y_1) + x_2\sin(x_2y_2) & 0 \\ 0 & y_1\cos(x_1y_1) + y_2\cos(x_2y_2) \end{bmatrix}.$$

Also, Figures 3 and 4 show that each subsystem is unstable even without uncertainties and stochastic perturbations.

Next we consider a switched system comprised by nonlinear subsystems.

Example 4.2. Consider the switched system given by

$$\begin{aligned} dx(t) &= [f_1(t,x(t),x(t-h)) + \Delta f_1(t,x(t),x(t-h))]dt \\ &\quad + g_1(t,x(t),x(t-h))dw(t), \quad \sigma(t) = 1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} dx(t) &= [f_2(t,x(t),x(t-h)) + \Delta f_2(t,x(t),x(t-h))]dt \\ &\quad + g_2(t,x(t),x(t-h))dw(t), \quad \sigma(t) = 2, \end{aligned} \tag{4.4}$$

where

$$f_1(t,x,y) = \begin{Bmatrix} 0.5x_1 + 0.5y_1 - x_2\sin(t) \\ x_1\cos(t) - x_2 - y_2 \end{Bmatrix},$$
$$f_2(t,x,y) = \begin{Bmatrix} -x_1 - y_1 - x_2\cos(t) \\ x_1\sin(t) + 0.1x_2 + 0.9y_2 \end{Bmatrix},$$

for all $t \ge 0$ and $x,\ y \in \mathbb{R}^2$, and the uncertainties Δf_i and stochastic perturbations g_i are assumed to satisfy

$$\|\Delta f_i(t,x,y)\| \le 0.1\|x\| + 0.1\|y\|,$$
$$\|g_i(t,x,y)\|_{\mathbf{tr}}^2 \le 0.1\|x\|^2 + 0.1\|y\|^2,$$

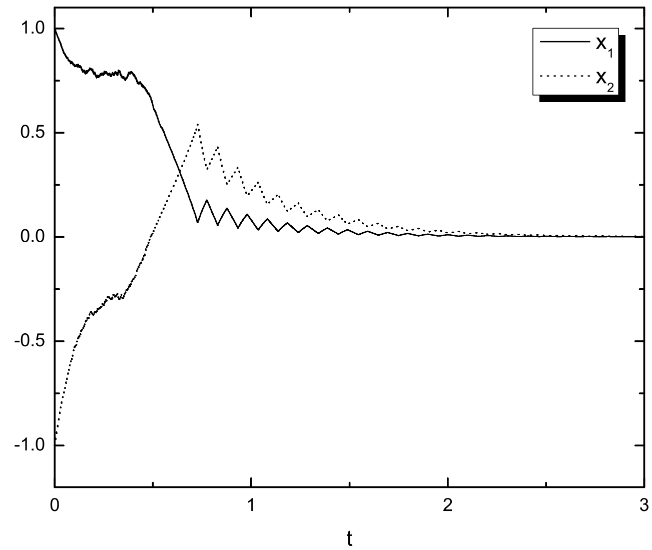

FIGURE 1. The solution of the system switching between (4.1) and (4.2) according to the switching signal $\sigma(t)$. The time-delay is set to be $h = 0.0016$ and the initial data given by $x = [1 \ -1]^T$.

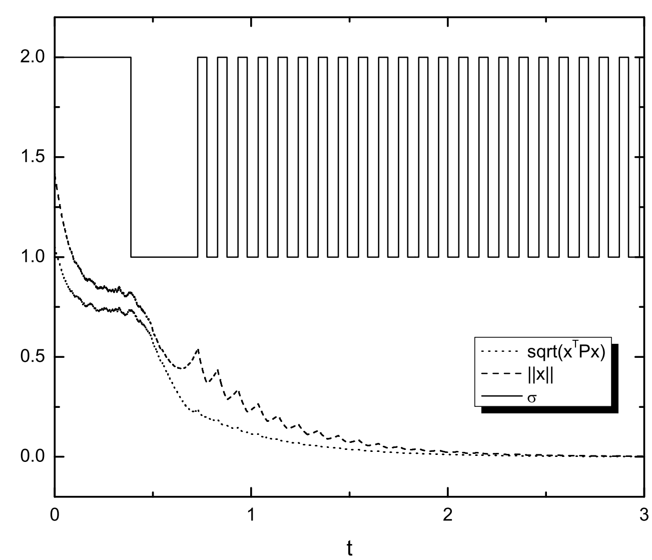

FIGURE 2. The Euclidian norm $\|x\|$ and quadratic norm $\sqrt{x^T P x}$ vs. the constructed switching signal $\sigma(t)$.

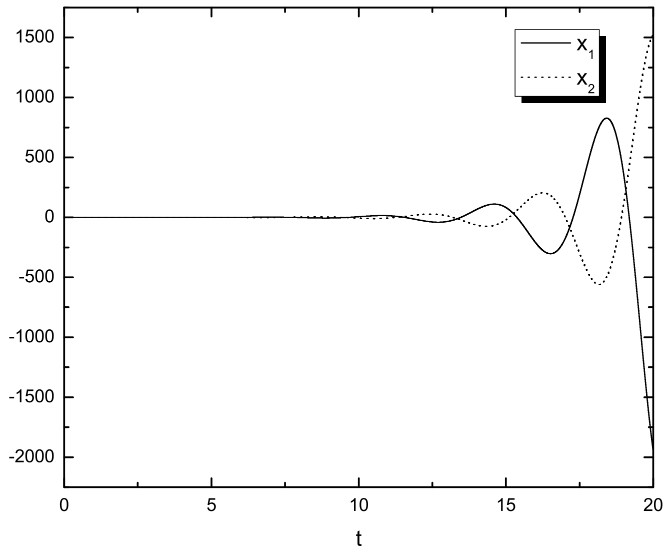

FIGURE 3. The solution of subsystem (4.1) without uncertainty and stochastic perturbation. The time-delay is set to be $h = 0.0005$ and the initial data given by $x = [0.01\ 0.01]^T$.

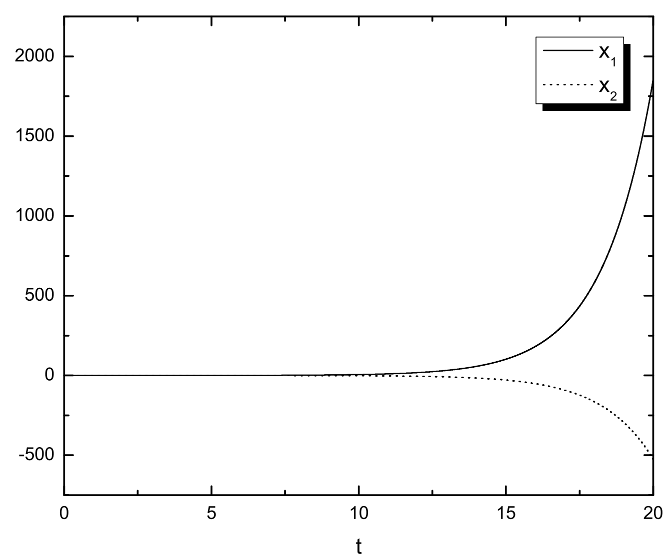

FIGURE 4. The solution of subsystem (4.2) without uncertainty and stochastic perturbation. The time-delay is set to be $h = 0.0005$ and the initial data given by $x = [0.01\ 0.01]^T$.

for $i = 1, 2$, $t \geq 0$, and $x, y \in \mathbb{R}^2$. It is easy to show that

$$2[\frac{1}{2}x^T f_1(t,x,x) + \frac{1}{2}x^T f_2(t,x,x)] \leq -\|x\|^2,$$
$$\|f_i(t,x,x) - f_i(t,x,y)\| \leq \|x-y\|,$$
$$\|f_i(t,x,y)\| \leq 2\|x\| + \|y\|,$$

for $i = 1, 2$, $t \geq 0$, and $x, y \in \mathbb{R}^2$. To apply Theorem 2.2 with $\zeta = 1.1$, $\delta_1 = 1$, $\delta_2 = 2.24$, $\delta_3 = 1.74$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.1$, $P = I_2$, and $\lambda = 1$, we can compute that

$$a = \frac{1}{\zeta}\lambda - \|P\|(2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) = 0.3091,$$
$$c = 2\delta_1\|P\|(\delta_2 + \delta_3 + \beta_1 + \beta_2) = 6.4,$$
$$d = \delta_1\|P\|^2(\beta_3 + \beta_4) = 0.2.$$

Thus Theorem 2.2 gives that, for

$$h < \frac{a^2\sqrt{d^2 + acd}}{(d + \sqrt{d^2 + acd})(ac + d + \sqrt{d^2 + acd})} = 0.0258,$$

(2.20) is exponentially stable. Typical paths of the evolution and the switching rule σ for the above system are shown in Figures 5 and 6 for $h = 0.02$, $\zeta = 1.1$,

$$\Delta f_1 = \Delta f_2 = \begin{Bmatrix} 0.1x_1\sin(y_1y_2) + 0.1y_1\cos(x_1x_2) \\ 0.1x_2\sin(y_1y_2) + 0.1y_2\cos(x_1x_2) \end{Bmatrix},$$

and

$$g_1 = g_2 = \frac{\sqrt{0.1}}{2}\begin{bmatrix} x_1\sin(x_1y_1) + x_2\sin(x_2y_2) & 0 \\ 0 & y_1\cos(x_1y_1) + y_2\cos(x_2y_2) \end{bmatrix}.$$

Also, Figures 7 and 8 show that each subsystem is unstable even without uncertainties and stochastic perturbations.

4.2. **Stability analysis of delay differential equations.**

Example 4.3. Consider the following equation

$$\begin{aligned} dx(t) = {} & [(A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-h)]dt \\ & + g(t, x(t), x(t-h))dw(t), \quad t \geq 0, \end{aligned} \tag{4.5}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},$$
$$\|\Delta A(t)\| \leq 0.1, \quad \|\Delta B(t)\| \leq 0.1,$$

and

$$\|g(t,x,y)\|^2_{\mathbf{tr}} \leq 0.1\|x\|^2 + 0.1\|y\|^2,$$

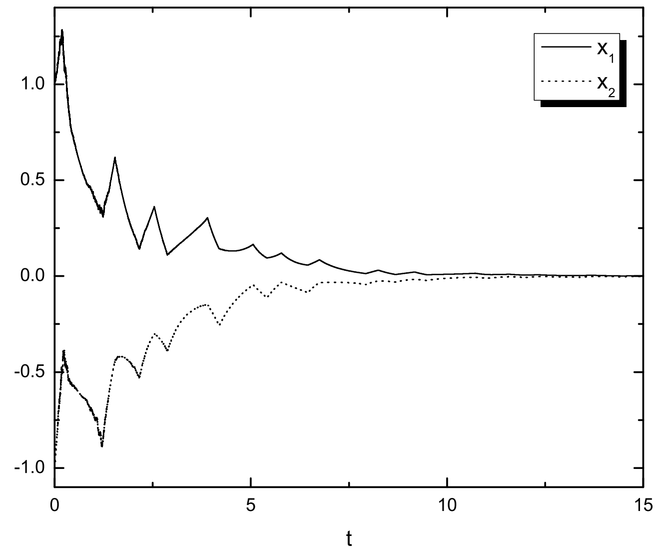

FIGURE 5. The solution of the system switching between (4.3) and (4.4) according to the switching signal $\sigma(t)$. The time-delay is set to be $h = 0.02$ and the initial data given by $x = [1\ \ -1]^T$.

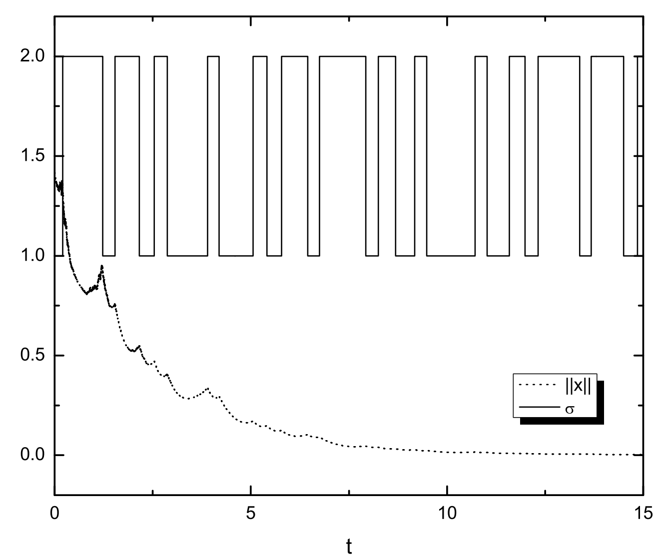

FIGURE 6. The Euclidian norm of x vs. the constructed switching signal $\sigma(t)$. Note that in this case $P = I_2$ and the quadratic norm $\sqrt{x^T P x}$ coincides with the Euclidian norm.

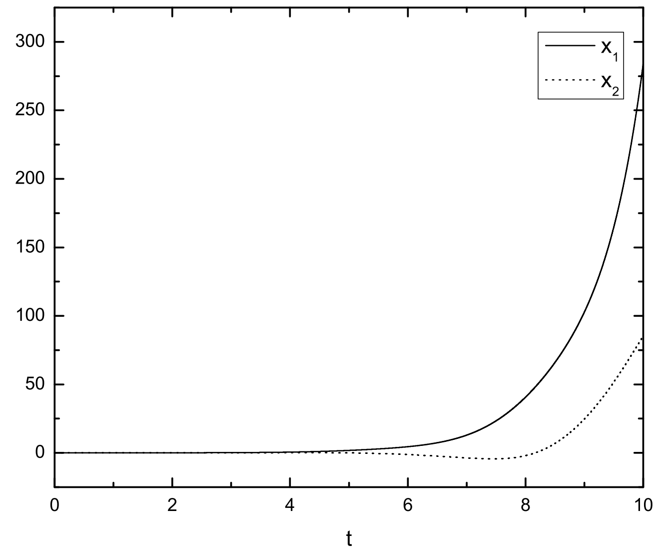

FIGURE 7. The solution of subsystem (4.3) without uncertainty and stochastic perturbation. The time-delay is set to be $h = 0.02$ and the initial data given by $x = [0.01\ 0.01]^T$.

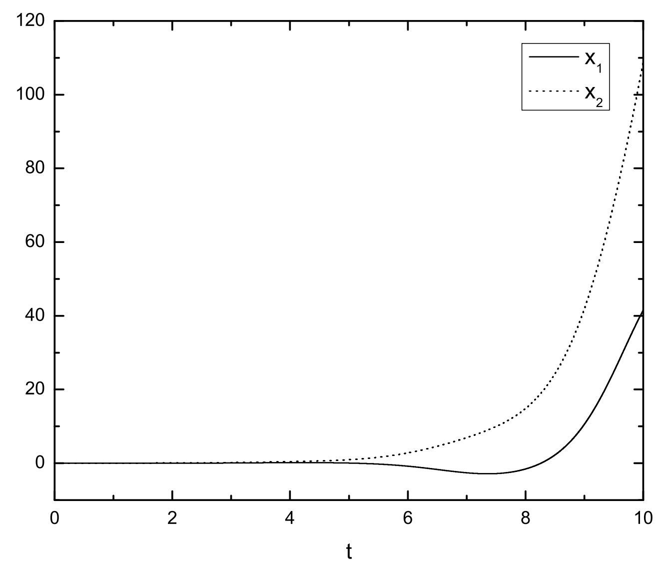

FIGURE 8. The solution of subsystem (4.4) without uncertainty and stochastic perturbation. The time-delay is set to be $h = 0.02$ and the initial data given by $x = [0.01\ 0.01]^T$.

for all $t \geq 0$ and $x, y \in \mathbb{R}^2$. Let $H = A + B$ and $Q = I_2$, we can find

$$P = \begin{bmatrix} 0.1708 & 0.0250 \\ 0.0250 & 0.2500 \end{bmatrix}$$

such that $PH + H^T P = -Q$. It is easy to compute that

$$\|P\| = 0.2572, \quad \|PB\| = 0.3191, \quad \|A\| = 2.2882, \quad \|B\| = 1.2808.$$

To apply Corollary 2.3 with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.1$, we can compute that

$$\begin{aligned} a &= \lambda_{\min}(Q) - \|P\| (2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) = 0.8457, \\ c &= 2 \|PB\| (\|A\| + \|B\| + \beta_1 + \beta_2) = 2.4054, \\ d &= \|PB\|^2 (\beta_3 + \beta_4) = 0.0204. \end{aligned}$$

Thus Corollary 2.3 shows that (2.20) is exponentially stable for

$$h < \frac{a^2\sqrt{d^2 + acd}}{(d + \sqrt{d^2 + acd})(ac + d + \sqrt{d^2 + acd})} = 0.2879,$$

which improves the delay upper bound of 0.175 in [22]. If there is no stochastic perturbation in (4.5), i.e. $g(t, x, y) \equiv 0$, then Corollary 2.4 shows that (4.5) is exponentially stable for $h < 0.3729$, which also improves the delay upper bound of 0.189 in [22].

Example 4.4 ([22, Example 5.2])**.** Consider the following equation

$$\begin{aligned} dx(t) = {} & [f(t, x(t), x(t - h)) + \Delta f(t, x(t), x(t - h))]dt \\ & + g(t, x(t), x(t - h))dw(t), \end{aligned} \tag{4.6}$$

where

$$f(t, x, y) = \begin{Bmatrix} -0.5x_1 - 0.5y_1 + x_2 \sin(x_1 x_2) \\ -x_1 \sin(x_1 x_2) - 0.6x_2 - 0.4y_2 \end{Bmatrix}$$

for $t \geq 0$ and $x, y \in \mathbb{R}^2$, and the uncertainty Δf and stochastic perturbation g are assumed to satisfy

$$\begin{aligned} \|\Delta f(t, x, y)\| &\leq 0.1 \|x\| + 0.1 \|y\|, \\ \|g(t, x, y)\|_{\mathbf{tr}}^2 &\leq 0.1 \|x\|^2 + 0.1 \|y\|^2, \end{aligned}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}^2$. It is easy to show that

$$\begin{aligned} 2x^T f(t, x, x) &\leq -2 \|x\|^2, \\ \|f(t, x, x) - f(t, x, y)\| &\leq 0.5 \|x - y\|, \\ \|f(t, x, y)\| &\leq 2.2 \|x\| + 0.64 \|y\|. \end{aligned}$$

To apply Theorem 2.2 with $\zeta = 1$, $\delta_1 = 0.5$, $\delta_2 = 2.2$, $\delta_3 = 0.64$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.1$, $P = I_2$ and $\lambda = 2$, we can compute that

$$a = \lambda - \|P\| (2\beta_1 + 2\beta_2 + \beta_3 + \beta_4) = 1.4,$$

$$c = 2\delta_1 \|P\| (\delta_2 + \delta_3 + \beta_1 + \beta_2) = 3.04,$$
$$d = \delta_1 \|P\|^2 (\beta_3 + \beta_4) = 0.1.$$

Thus Theorem 2.2 gives that for

$$h < \frac{a^2\sqrt{d^2 + acd}}{(d + \sqrt{d^2 + acd})(ac + d + \sqrt{d^2 + acd})} = 0.3393,$$

(2.20) is exponentially stable, which improves the delay upper bound of 0.21 in [22]. Note that we have maintained the estimate $\|f(t, x, y)\| \leq 2.2 \|x\| + 0.64 \|y\|$ as in [22], for the sake of comparison. Actually, it is easy to derive another estimate as

$$\|f(t, x, y)\| \leq 1.6 \|x\| + 0.5 \|y\| .$$

By virtue of this estimate, we can conclude that (2.20) is exponentially stable for $h < 0.4287$.

5. CONCLUSION

The robust stabilization of a class of uncertain stochastic switched systems with time-delay via a state-dependent switching rule is investigated in this paper. Assuming there exists a Hurwitz linear convex combination for the original system, it has been shown that under a certain state-dependent switching rule, the stochastically perturbed system with both uncertainties and time-delay is still exponentially stable, provided that the perturbation, uncertainties, and time-delay are sufficiently small. Moreover, the results are extended to nonlinear systems. We have also quantified the stability upper bound for the time-delay. Numerical results show that our upper bound improves some results in literature.

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