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POSITIVE SOLUTIONS FOR DISCRETE BOUNDARY VALUE PROBLEMS

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ABSTRACT. Existence results for positive solutions are established for a discrete Dirichlet boundary value problem. Various fixed point techniques including the Guo-Krasnosels'kii theorem and Schauder's fixed point theorem are used. Examples illustrating the results are included.

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1. INTRODUCTION

This paper discusses the discrete Dirichlet boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = f\left(k, u(k)\right), & k \in D = \{1, 2, \ldots, T\}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where T is a positive integer, and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. We let $D^+ = \{0, 1, 2, \ldots, T+1\}$ and let $\mathcal{B}$ be the Banach space defined by

$$\mathcal{B} = \{u \mid u : D^+ \to \mathbb{R}\}$$

with the norm

$$\|u\|_0 = \max_{k \in D^+} |u(k)|.$$

Discrete boundary value problems have been discussed extensively in the literature; see, for example, Agarwal et al. [1], Anderson et al. [5], and Avery et al. [6], as well as the references cited therein. In [1], the authors use critical point theory to establish the existence of multiple solutions of some regular as well as singular discrete boundary value problems. Solvability of a nonlinear, second-order difference equation with discrete Neumann boundary conditions is studied in [5]. The methods used there involve new inequalities on the right-hand side of the difference equation and Schaefer's fixed point theorem in the finite-dimensional space setting.

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In [6], fixed point theorems of cone expansion and compression of norm type are generalized by replacing the norms with two functionals satisfying certain conditions. The authors then give an application yielding the existence of a positive solution to a discrete second-order conjugate boundary value problem.

In this paper, by using some of the ideas in [4], together with some techniques used in [7] for ordinary differential equations, we present some new existence results for positive solutions of the discrete problem (1.1).

2. PRELIMINARIES

In this section, we present some results that will be needed in Section 3. Our first lemma is due to Agarwal and O'Regan.

Lemma 2.1 ([2]). *Let $v \in \mathcal{B}$ satisfy $v(k) \geq 0$ for $k \in D^+$. If $u \in \mathcal{B}$ satisfies*

$$\begin{cases} -\Delta^2 u(k-1) = v(k), & k \in D, \\ u(0) = u(T+1) = 0, \end{cases} \tag{2.1}$$

then

$$u(k) \geq q(k)\|u\|_0 \quad \textit{for} \quad k \in D^+,$$

where

$$q(k) = \min\left\{\frac{T+1-k}{T+1}, \frac{k}{T}\right\}.$$

Remark 2.2. From the definition of $q(k)$, we have $q(k) \geq \frac{1}{T+1}$ for $k \in D$.

It is easy to prove the following lemma.

Lemma 2.3. *The Green function $G(k,j)$ associated to the problem* (1.1) *is given by*

$$G(k,j) = \begin{cases} \frac{j(T+1-k)}{T+1} & \textit{if} \quad 0 \leq j \leq k-1, \\ \\ \frac{k(T+1-j)}{T+1} & \textit{if} \quad k \leq j \leq T+1. \end{cases} \tag{2.2}$$

Under suitable assumptions on the nonlinear function f, we shall prove the existence of a positive solution to the boundary value problem (1.1). The proof relies on the Guo-Krasnosel'skii fixed point theorem in cones ([9], [10]). We begin by recalling the following definition.

Definition 2.4. A nonempty subset $\mathcal{C}$ of a Banach space X is called a cone if $\mathcal{C}$ is convex, closed, and satisfies

(i) $\alpha u \in \mathcal{C}$ for all $u \in \mathcal{C}$ and any real positive number α, and
(ii) $u, -u \in \mathcal{C}$ imply $u = 0$.

Next, we state the Guo-Krasnosel'skii Theorem.

Theorem 2.5 ([10])**.** *Let X be a Banach space and $\mathcal{C} \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets in X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{C}$ be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$ *for* $u \in \mathcal{C} \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for* $u \in \mathcal{C} \cap \partial\Omega_2$*, or*
(ii) $\|Au\| \geq \|u\|$ *for* $u \in \mathcal{C} \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for* $u \in \mathcal{C} \cap \partial\Omega_2$*.*

Then A has at least one fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the sequel, we take $X = \mathcal{B}$ and define the cone $\mathcal{C} \subset \mathcal{B}$ by

$$\mathcal{C} = \{u \in \mathcal{B} : u(k) \geq q(k)\|u\|_0 \quad \text{for} \;\; k \in D^+\}.$$

We define the operator $A : \mathcal{B} \longrightarrow \mathcal{B}$ by

$$Au(k) = \sum_{j=1}^{T} G(k,j) f\left(j, u(j)\right).$$

Our final lemma shows that the operator satisfies the covering assumptions of Theorem 2.5

Lemma 2.6 ([3])**.** *Assume that $f : D^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous. Then A is well defined and maps $\mathcal{C}$ into $\mathcal{C}$. Moreover, A is continuous and completely continuous.*

3. EXISTENCE OF POSITIVE SOLUTIONS

First, we will use Theorem 2.5 together with some ideas in [4] to obtain new results for the existence of a positive solution of the Dirichlet discrete boundary value problem (1.1).

Theorem 3.1. *Suppose that $f(k,u) = p(k)\left(g(u) + h(u)\right)$ where $p : D^+ \longrightarrow \mathbb{R}^+$, $g : \mathbb{R} \longrightarrow \mathbb{R}$, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying the following conditions:*

$$g(u) > 0 \text{ and is nonincreasing for } u \in (0,\infty); \tag{3.1}$$

$$h(u) \geq 0 \text{ for } u \in [0,\infty); \tag{3.2}$$

$$\frac{h}{g} \text{ is nondecreasing for } u \in (0,\infty); \tag{3.3}$$

there exists a constant $K_0 > 0$ such that

$$g(uv) \leq K_0 g(u) g(v) \text{ for all } u, v \geq 0; \tag{3.4}$$

there exists a constant $r > 0$ with

$$\frac{r}{a_0 K_0 (g(r) + h(r))} \geq 1, \tag{3.5}$$

where

$$a_0 = \max_{k \in D} \sum_{j=1}^{T} G(k,j)p(j)g(q(j));$$

and there exists a constant $R > r$ with

$$\frac{Rg(\frac{R}{T+1})}{b_0 g(R)\left(g(\frac{R}{T+1}) + h(\frac{R}{T+1})\right)} \leq 1, \tag{3.6}$$

where

$$b_0 = \min_{k \in D} \sum_{j=1}^{T} G(k,j)p(j).$$

Then the boundary value problem (1.1) *has a solution $u \in \mathcal{B}$ satisfying $r \leq \|u\|_0 \leq R$.*

Proof. Let

$$\Omega_1 = \{u \in \mathcal{B} \, : \, \|u\|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in \mathcal{B} \, : \, \|u\|_0 < R\}.$$

We will first show that

$$\|Au\|_0 \leq \|u\|_0 \quad \text{for} \quad u \in \mathcal{C} \cap \partial\Omega_1.$$

Let $u \in \mathcal{C} \cap \partial\Omega_1$, i.e., $u \in \mathcal{C}$ and $\|u\|_0 = r$. Then, we have

$$\begin{aligned}
0 \leq Au(k) &= \sum_{j=1}^{T} G(k,j)p(j)\left(g(u(j)) + h(u(j))\right) \\
&= \sum_{j=1}^{T} G(k,j)p(j)g(u(j))\left(1 + \frac{h(u(j))}{g(u(j))}\right) \\
&\leq \sum_{j=1}^{T} G(k,j)p(j)g(q(j)\|u\|_0)\left(1 + \frac{h(\|u\|_0)}{g(\|u\|_0)}\right) \\
&= \sum_{j=1}^{T} G(k,j)p(j)g(q(j)r)\left(1 + \frac{h(r)}{g(r)}\right) \\
&\leq \left(1 + \frac{h(r)}{g(r)}\right) K_0 g(r) \sum_{j=1}^{T} G(k,j)p(j)g(q(j)) \\
&\leq \left(1 + \frac{h(r)}{g(r)}\right) K_0 g(r) \max_{k \in D} \sum_{j=1}^{T} G(k,j)p(j)g(q(j)) \\
&= \left(1 + \frac{h(r)}{g(r)}\right) K_0 g(r) a_0 \\
&\leq r = \|u\|_0.
\end{aligned}$$

It follows that

$$\|Au\|_0 \leq \|u\|_0 \quad \text{for } u \in \mathcal{C} \cap \partial\Omega_1.$$

Next, we show that

$$\|Au\|_0 \geq \|u\|_0 \quad \text{for } u \in \mathcal{C} \cap \partial\Omega_2.$$

Let $u \in \mathcal{C} \cap \partial\Omega_2$, i.e., $u \in \mathcal{C}$ and $\|u\|_0 = R$. We have $u(k) \geq q(k)\|u\|_0$, and so

$$\begin{aligned}
Au(k) &= \sum_{j=1}^{T} G(k,j)p(j)\left(g(u(j)) + h(u(j))\right) \\
&= \sum_{j=1}^{T} G(k,j)p(j)g(u(j))\left(1 + \frac{h(u(j))}{g(u(j))}\right) \\
&\geq \sum_{j=1}^{T} G(k,j)p(j)g(R)\left(1 + \frac{h(q(j)\|u\|_0)}{g(q(j)\|u\|_0)}\right) \\
&\geq \sum_{j=1}^{T} G(k,j)p(j)g(R)\left(1 + \frac{h(\frac{R}{T+1})}{g(\frac{R}{T+1})}\right) \\
&= g(R)\left(1 + \frac{h(\frac{R}{T+1})}{g(\frac{R}{T+1})}\right)\sum_{j=1}^{T} G(k,j)p(j) \\
&\geq g(R)\left(1 + \frac{h(\frac{R}{T+1})}{g(\frac{R}{T+1})}\right)\min_{k\in D}\sum_{j=1}^{T} G(k,j)p(j) \\
&= b_0 g(R)\left(1 + \frac{h(\frac{R}{T+1})}{g(\frac{R}{T+1})}\right) \\
&\geq R = \|u\|_0.
\end{aligned}$$

Hence, we obtain

$$\|Au\|_0 \geq \|u\|_0 \quad \text{for } u \in \mathcal{C} \cap \partial\Omega_2.$$

By Theorem 2.5, the problem (1.1) admits a positive solution u with $r \leq \|u\|_0 \leq R$. □

Next, we will use Theorem 2.5 together with some ideas in [7] to obtain some additional new existence results for the problem (1.1).

Theorem 3.2. *Let $f : D^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function and suppose there exist continuous functions $a, b : D^+ \longrightarrow \mathbb{R}^+$ and a constant $p > 0$ such that*

$$f(k,u) \leq a(k) + b(k)u^p \quad \textit{for all} \quad (k,u) \in D^+ \times \mathbb{R}^+. \tag{3.7}$$

In addition, assume that

$$\left(\frac{1}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{1}{pM_2}\right)^{\frac{p}{p-1}} - M_1 \geq 0 \quad \textit{if} \quad p > 1, \tag{3.8}$$

or

$$\left(\frac{1}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{1}{pM_2}\right)^{\frac{p}{p-1}} - M_1 \leq 0 \quad \textit{if} \quad 0 < p < 1, \tag{3.9}$$

where

$$M_1 :\ = \max_{k\in D}\sum_{j=1}^{T} G(k,j)a(j), \quad M_2 :\ = \max_{k\in D}\sum_{j=1}^{T} G(k,j)b(j),$$

and

$$M_2 \neq 1 \quad \textit{if} \quad p = 1.$$

If there exist $\beta > 0$ and $k_0 \in D$ such that

$$\min_{\substack{k\in D\\ u\in\left[\frac{\beta}{T+1},\beta\right]}} f(k,u) \geq \beta\left[\sum_{j=1}^{T} G(k_0,j)\right]^{-1}, \tag{3.10}$$

then the boundary value problem (1.1) *has at least one positive solution $u \in \mathcal{B}$.*

Proof. Let

$$\Omega_1 = \{u \in \mathcal{B} \,:\, \|u\|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in \mathcal{B} \,:\, \|u\|_0 < R\},$$

where the constants $0 < r < R$ are real positive numbers to be selected later. Consider a function $u \in \mathcal{C} \cap \partial\Omega_1$.

First, we consider the case $p \neq 1$. We have

$$\begin{aligned} 0 \ &\leq\ Au(k) = \sum_{j=1}^{T} G(k,j)f(j,u(j)) \leq \sum_{j=1}^{T} G(k,j)\left[a(j) + b(j)|u(j)|^p\right] \\ &\leq\ \sum_{j=1}^{T} G(k,j)a(j) + \|u\|^p \sum_{j=1}^{T} G(k,j)b(j) \leq (M_1 + M_2\|u\|^p) \leq M_1 + M_2 r^p \leq r, \end{aligned}$$

where the last inequality follows from conditions (3.8) and (3.9) after an application of elementary calculus on the function $F(x) = x - M_2x^p - M_1$. Thus, we have proved that $\|Au\|_0 \leq \|u\|_0$.

Now, if $u \in \mathcal{C} \cap \partial\Omega_2$, then taking $\|u\| = R = \beta$ where β is defined in condition (3.10), we see that $u(k) \geq \frac{\beta}{T+1}$ by Remark 2.2. Hence, for any $k \in D$, $\frac{\beta}{T+1} \leq u(k) \leq \beta$. Furthermore, we have

$$Au(k_0) = \sum_{j=1}^{T} G(k_0,j)f(j,u(j)) \geq \left[\min_{\substack{k\in D\\ u\in\left[\frac{\beta}{T+1},\beta\right]}} f(k,u)\right]\sum_{j=1}^{T} G(k_0,j) \geq \beta.$$

Consequently, $Au(k_0) \geq \beta$, that is, $\|Au\|_0 \geq \|u\|_0$ for any $u \in \mathcal{C} \cap \partial\Omega_2$.

The case $p = 1$ is easy to prove. In view of Theorem 2.5, the operator A has a fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and so problem (1.1) admits a positive solution u in the cone $\mathcal{C}$. $\square$

Our final result has a sum condition on the nonlinear term; it makes use of the following Schauder fixed point theorem.

Theorem 3.3 ([8], [12]). *Let X be a Banach space and let K be a bounded, closed, and convex subset of X. Let $A : K \longrightarrow K$ be a completely continuous operator. Then A has a fixed point in K.*

Our existence result is as follows.

Theorem 3.4. *Let $f : D^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function and suppose there is a continuous function $H : D^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that is nondecreasing in its second argument and satisfying*

$$f(k,u) \leq H(k,|u|) \quad \text{for all} \quad k \in D^+. \tag{3.11}$$

If there exists a constant $c_* > 0$ such that

$$\max_{k\in D} \sum_{j=1}^{T} G(k,j)H(j,c_*) \leq c_*, \tag{3.12}$$

then the boundary value problem (1.1) has at least one positive solution.

Proof. Let K be the closed convex subset of $\mathcal{B}$ defined by

$$K = \{u \in \mathcal{B} \,:\, 0 \leq u(k) \leq c_* \quad \text{for all} \quad k \in D^+\}.$$

Using the above conditions, we see that A maps K into itself. Indeed,

$$\begin{aligned} 0 \leq Au(k) = \sum_{j=1}^{T} G(k,j)f\left(j,u(j)\right) &\leq \sum_{j=1}^{T} G(k,j)H\left(j,|u(j)|\right) \leq \sum_{j=1}^{T} G(k,j)H\left(j,\|u\|_0\right) \\ &\leq \sum_{j=1}^{T} G(k,j)H\left(j,c_*\right) \leq \max_{k\in D}\sum_{j=1}^{T} G(k,j)H\left(j,c_*\right) \leq c_*. \end{aligned}$$

Schauder's fixed point theorem guarantees that the problem (1.1) has a positive solution. □

4. EXAMPLES

In this section of the paper we give some examples to illustrate our results.

Example 4.1. Consider the discrete Dirichlet boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \frac{1}{4T(T+1)^{\alpha+1}}\Big(u^{-\alpha} + u^{\beta}\Big), & k \in D = \{1,2,\dots,T\}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1 < \beta$ and T is a positive integer. Here $p(k) = \frac{1}{4T(T+1)^{\alpha+1}}$, $g(u) = u^{-\alpha}$, and $h(u) = u^{\beta}$. Notice that $K_0 = 1$ and $G(k,j) \leq T+1$ for $k,j \in D$. Moreover, $q(j) \geq 1/(T+1)$ so $g(q(j)) \leq (T+1)^{\alpha}$. We have

$$a_0 = \max_{k\in D^+} \sum_{j=1}^{T} G(k,j)p(j)g(q(j)) \leq \frac{1}{4},$$

so if we take $r = 1$, then

$$\frac{r}{a_0 K_0(g(r) + h(r))} = 2 > 1.$$

Taking the limit,

$$\lim_{R\to+\infty} \frac{Rg(\frac{R}{T+1})}{b_0 g(R)\left(g(\frac{R}{T+1}) + h(\frac{R}{T+1})\right)} = \lim_{R\to+\infty} \frac{R^{\alpha+1}}{b_0\left(1 + (\frac{1}{T+1})^{\alpha+\beta} R^{\alpha+\beta}\right)} = 0 \le 1$$

for any b_0, and we see that there exists $R > r = 1$ such that

$$\frac{Rg(\frac{R}{T+1})}{b_0 g(R)\left(g(\frac{R}{T+1}) + h(\frac{R}{T+1})\right)} \le 1.$$

Thus, by Theorem 3.1, the problem (4.1) has a solution u satisfying $r \le \|u\|_0 \le R$.

Example 4.2. Consider the boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \dfrac{T}{3} + \dfrac{2T}{3} u^{1/2}, & k \in D = \{1, 2, \dots, T\}, \\ \\ u(0) = u(T+1) = 0. \end{cases} \tag{4.2}$$

Notice that $M_2 = 2M_1$, and since $1/(T+1) \le G(k,j) \le T+1$ for k, $j \in D$, (3.10) holds with $\beta = T^4/3(T+1)^3$. It is easy to see that (3.9) holds, so the problem (4.2) has at least one positive solution by Theorem 3.2.

Example 4.3. Consider the boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \dfrac{\sqrt{2}}{T(T+1)} + \dfrac{1}{\sqrt{2}\,T(T+1)} u^2, & k \in D = \{1, 2, \dots, T\}, \\ \\ u(0) = u(T+1) = 0. \end{cases} \tag{4.3}$$

Here, $M_2 = M_1/2$, and moreover, $M_1 \le 1/\sqrt{2}$, so (3.8) holds. With $\beta = (T+1)^4$, we see that (3.10) also holds. The problem (4.3) then has at least one positive solution by Theorem 3.2.

Example 4.4. Consider the boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \frac{1}{2(T+1)^2} u^{1/3} & k \in D = \{1, 2, \dots, T\}, \\ \\ u(0) = u(T+1) = 0. \end{cases} \tag{4.4}$$

It is easy to see that condition (3.12) holds with $c_* = 1$, so problem (4.4) has at least one positive solution by Theorem 3.4.

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