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ON BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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ABSTRACT. The paper deals with boundary value problems associated to first-order differential inclusions in Banach spaces. The solvability is investigated in the (strong) Carathéodory sense on compact intervals. To this aim, we develop a general method that relies on degree arguments. This method is still combined with a bound sets technique for checking the behavior of trajectories in the neighborhood of a suitable parametric set of candidate solutions. On this basis, we obtain effective criteria for the existence of solutions of Floquet problems. The existence of entirely bounded solutions is also established by means of a sequence of solutions on compact increasing intervals.

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1. INTRODUCTION

Consider the first-order multivalued boundary value problem (b.v.p.)

$$\begin{cases} x' + A(t)x \in F(t,x), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a), \end{cases} \tag{1.1}$$

usually called *Floquet b.v.p.* We take x in a not necessarily separable or reflexive Banach space E satisfying the Radon–Nikodym property, and we denote by $\mathcal{L}(E)$ the space of all linear, bounded transformations $L : E \to E$. Throughout the paper, we always assume the following conditions:

(A1) $A : [a,b] \to \mathcal{L}(E)$ is Bochner integrable;

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(F1) $F : [a, b] \times E \multimap E$ is an upper-Carathéodory (shortly, u-Carathéodory) multivalued map (see Definition 2.1 in Section 2 below) with nonempty, compact and convex values;

(M) $M \in \mathcal{L}(E)$ is an invertible operator.

The notion of a solution is understood in a strong Carathéodory sense. Namely, by a *solution* of (1.1), we mean an absolutely continuous function $x : [a, b] \to E$ such that its derivative exists almost everywhere (a.e.) and is Bochner integrable.

Although the general theory of ordinary differential equations and inclusions in Banach spaces has been developed at a satisfactory level (cf. [22]), there are not many contributions to b.v.p.'s in infinite dimensional spaces (cf. [18], [19], [21] and [28]). The papers concerning multivalued problems are still more rare (cf. [1], [3], [14] and [20]).

Hence, in this paper, using degree arguments, we develop a general method for attacking such problems. In the particular case of semilinear problems (1.1), we are able to get effective criteria for their solvability. We will also prove the existence of entirely bounded solutions of a general class of differential inclusions. As a consequence of our approach, we obtain the localization of solutions in a given set.

One of conditions to be satisfied requires a fixed point free boundary of a set of candidate solutions w.r.t. the related representing operators. This will be checked by means of (Liapunov-type) bounding functions. Unfortunately, in the case of a u-Carathéodory right-hand side (r.h.s.), we are only able to guarantee the positive invariance of a convex parametric set of candidate solutions. On the other hand, if the r.h.s. is globally upper-semicontinuous (u.s.c.), then the bound sets approach enables us to consider also situations when some trajectories can escape from the mentioned set of parameters (candidate solutions).

The paper is organized as follows. We start with Preliminaries, where we recall suitable definitions and useful statements. Then, in Section 3, we develop, by means of topological degree arguments, a general method for the solvability of multivalued b.v.p.'s in E, in the form of a continuation principle (see Theorem 3.1 below). Section 4 is devoted to a bound sets technique. In Section 5, the general method is applied, jointly with a bound sets technique, for the investigation of Floquet problems (1.1) (see Theorem 5.2 below). The existence of entirely bounded solutions of differential inclusions is proved, in Section 6, by means of a sequence of solutions on compact increasing intervals. Finally, we supply concluding remarks.

2. **PRELIMINARIES**

In the entire text, all topological spaces will be at least metric and all multivalued maps $\varphi : X \multimap Y$ will have at least nonempty values, i.e. $\varphi : X \to 2^Y \setminus \{\emptyset\}$.

Let E be an arbitrary Banach space with the norm $|\cdot|$ and $\mathcal{L}(E)$ be the Banach space of all linear bounded transformations $L: E \to E$ endowed with the sup-norm $\|\cdot\|$, i.e. $\|L\| = \sup\{|Lx| : |x| = 1\}$. Given $C \subset E$ and $\epsilon > 0$, by B_C^ϵ we mean $B_C^\epsilon = C + \epsilon B$, where B is the standard open unit ball, i.e. $B = \{x \in E : |x| < 1\}$ and $\overline{B}$ its closure. When $C = \{x\}$ is a singleton, we simply put B_x^ϵ.

A *countably valued function* $x : [a, b] \to E$ with (no more than a countable number of nonzero) values x_k, and Lebesgue measurable sets $E_k = \{t \in [a, b] : x(t) = x_k\}$, $k = 1, 2, \ldots$, is said to be *Bochner integrable in* $[a, b]$ (see e.g. [13]) if the function $|x|$ is Lebesgue integrable on $[a, b]$. The Bochner integral is defined as

$$\int_a^b x(t)\,dt := \sum_{k=1}^{+\infty} x_k \lambda(E_k),$$

where $\lambda(E_k)$ denotes the Lebesgue measure of E_k.

A *function* $x : [a, b] \to E$ is said to be *Bochner integrable on* $[a, b]$ if x is the limit, on this interval, of an almost everywhere convergent sequence of countably valued functions $x_n(t)$ and $|x|$ is Lebesgue integrable on $[a, b]$. In this case,

$$\int_a^b x(t)\,dt := \lim_{n\to+\infty} \int_a^b x_n(t)\,dt.$$

It can be proved that the integral is independent of the choice of the sequence $\{x_n\}_n$ and enjoys all the usual properties of standard integrals. The space $L^1([a, b], E)$ will denote the *set of all Bochner integrable functions* $x : [a, b] \to E$.

Let $A : [a, b] \to \mathcal{L}(E)$ and $f : [a, b] \to E$ be Bochner integrable. For all $t, s \in [a, b]$, it is possible to define an operator $U = U(t, s)$ such that the unique solution of the linear initial value problem

$$x' + A(t)x = f(t), \qquad x(s) = x_s \in E, \tag{2.1}$$

is given (by definition) by

$$x(t) = U(t, s)x_s + \int_s^t U(t, \tau) f(\tau)\,d\tau; \tag{2.2}$$

U is said to be the *evolution operator* associated to the Cauchy problem (2.1) (see [13]). Moreover, for all $t, s \in [a, b]$, $U(t, s) \in \mathcal{L}(E)$ and it satisfies:

(i) $U(t, t) = \mathrm{Id}$;
(ii) $U(t, s)U(s, \tau) = U(t, \tau)$, for all $t, s, \tau \in [a, b]$;
(iii) $\|U(t, s)\| \le \mathrm{e}^{\int_s^t \|A(\tau)\|\,d\tau}$, for all $t, s \in [a, b]$, with $s < t$.

In the special case, when $A(t) \equiv A \in \mathcal{L}(E)$, $U(t, s)$ reduces to the exponential operator $\mathrm{e}^{-A(t-s)}$. We recall that, given $A \in \mathcal{L}(E)$ and $t \in \mathbb{R}$, $\mathrm{e}^{At} := \sum_{k=0}^{+\infty} \frac{A^k t^k}{k!}$, and since the numerical series $\sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!}$ converges, the definition is correct.

We say that a multivalued mapping $\tilde{F} : [a,b] \multimap E$ with closed values is a *step multivalued map* if there is a finite family of disjoint measurable subsets I_j, $j = 1, \ldots, n$, such that $[a,b] = \cup I_j$ and $\tilde{F}$ is constant (in the multivalued sense), on every I_j.

A multivalued mapping $F : [a,b] \multimap E$ with closed values is called *strongly measurable* if there exists a sequence $\{F_n\}_n$ of step multivalued maps such that $d_H\big(F_n(t), F(t)\big) \to 0$ as $n \to +\infty$, for a.a. $t \in [a,b]$, where d_H stands for the Hausdorff metric on bounded subsets of E.

Let X and Y be topological spaces. A multivalued mapping $F : X \multimap Y$ is said to be *upper-semicontinuous* (u.s.c.) if, for every open $U \subset Y$, the set $\{x \in X : F(X) \subset U\}$ is open in X.

Definition 2.1. Let $J \subseteq \mathbb{R}$ be an interval. A multivalued map $F : J \times E \multimap E$ with nonempty, compact, convex values is called an *upper-Carathéodory map* (shortly, u-Carathéodory) if:

(i) $F(\cdot, x)$ is strongly measurable, for every $x \in E$;
(ii) $F(t, \cdot)$ is u.s.c., for a.a. $t \in J$;
(iii) $|y| \leq r(t)(1 + |x|)$, for every $(t,x) \in J \times E$ and $y \in F(t,x)$, where $r \in L^1_{loc}(J, \mathbb{R})$.

The symbol $AC([a,b], E)$ will be reserved to denote the set of all absolutely continuous functions $x : [a,b] \to E$ having a derivative $x'(t)$, for a.a. $t \in [a,b]$ with $x' \in L^1([a,b], E)$. As it is known, an absolutely continuous function $x(t)$ with $t \in [a,b]$ need not admit, in general, a derivative $x' \in L^1([a,b], E)$ (see e.g. [12, Example 4.2]). It is so if the space E satisfies the Radon–Nikodym property; in particular, if E is reflexive. Let us note that, for all $x \in AC([a,b], E)$, the fundamental theorem of integral calculus is satisfied, i.e.

$$x(t) = x(\tau) + \int_\tau^t x'(s)\, ds, \qquad t, \tau \in [a,b].$$

Moreover, a (mild) solution x in (2.2) becomes a (strong, i.e. $x \in AC([a,b],\ E)$) Carathéodory solution of (2.1). For more details concerning the relationship between mild and strong (Caratheódory) solutions, see e.g. [23, Theorem 8.5]. The following convergence criterion in the space $AC([a,b], E)$ will be useful for us.

Lemma 2.2 (see e.g. [4, Lemma 1.30])**.** *Assume that a sequence $\{x_n\}_n \subset AC([a,b], E)$ satisfies the following conditions:*

(i) $\{x_n(t)\}_n$ *is relatively compact, for each* $t \in [a,b]$*;*
(ii) *there exists* $\alpha \in L^1([a,b], E)$ *such that* $|x_n'(t)| \leq \alpha(t)$*, for a.a.* $t \in [a,b]$*;*
(iii) $\{x_n'(t)\}_n$ *is weakly relatively compact, for a.a.* $t \in [a,b]$*.*

Then there exist a function $x \in AC([a,b],E)$ *and a subsequence, again denoted by* $\{x_n\}_n$*, such that* $x_n \to x$*, in* $C([a,b],E)$*, and* $x_n' \to x'$*, weakly in* $L^1([a,b],E)$*, as* $n \to +\infty$.

Given $U \subseteq E$, a function $f : U \to \mathbb{R}$ is said to be *Lipschitzian* with constant $L > 0$ if $|f(x_1) - f(x_2)| \le L|x_1 - x_2|$, for each $x_1, x_2 \in U$; f is said to be *locally Lipschitzian* if, for every $x \in U$, there is $\epsilon > 0$ such that $f|_{B_x^\epsilon}$, i.e. f restricted to B_x^ϵ, is Lipschitzian. Observe that, for all $x \in U$ and $w \in E$, the following limits

$$\liminf_{h\to 0^\pm} \frac{f(x+hw)-f(x)}{h}, \qquad \limsup_{h\to 0^\pm} \frac{f(x+hw)-f(x)}{h}$$

are real values.

Let $\mathcal{P}(X)$ denote the family of all nonempty subsets of an arbitrary set X.

Definition 2.3. Given a partially ordered set N, a function $\beta : \mathcal{P}(E) \to N$ is said to be a *measure of non-compactness* (m.n.c.) in E if $\beta(\overline{\text{co}}\Omega) = \beta(\Omega)$, for all $\Omega \subset E$, where $\overline{\text{co}}\Omega$ denotes the closed convex hull of Ω. A m.n.c. β is called:

(i) *monotone* if $\beta(\Omega_0) \le \beta(\Omega_1)$, for all $\Omega_0 \subset \Omega_1 \subset E$;
(ii) *nonsingular* if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for every $x \in E$ and $\Omega \subset E$;
(iii) *regular* when $\beta(\Omega) = 0$ if and only if Ω is relatively compact;
(iv) *algebraically semiadditive* if $\beta(\Omega_0 + \Omega_1) \le \beta(\Omega_0) + \beta(\Omega_1)$, for all $\Omega_0, \Omega_1 \subset E$.

The Hausdorff measure of non-compactness, defined as

$$\gamma(\Omega) := \inf\{\epsilon > 0 : \exists\, x_1, \dots x_n \in E : \Omega \subset \cup_{i=1}^n B_{x_i}^\epsilon\},$$

is a typical example of monotone, nonsingular, regular and algebraically semiadditive m.n.c. There are also useful properties combining the Hausdorff m.n.c. and linearity of an operator. In particular, if $L \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see e.g. [16, p. 35])

$$\gamma(L\Omega) \le ||L||_{\mathcal{L}(E)}\gamma(\Omega). \tag{2.3}$$

Moreover, letting $\{f_n\}_n \subset L^1([a,b],E)$, if there exist $\nu, c \in L^1[a,b]$ such that $|f_n(t)| \le \nu(t)$, for a.a. $t \in [a,b]$, and $n \in \mathbb{N}$ and $\gamma(\{f_n(t)\}_n) \le c(t)$, for a.a. $t \in [a,b]$, then (see [16, Corollary 4.2.5])

$$\gamma\Big(\Big\{\int_a^b f_n(t)dt\Big\}_n\Big) \le 2\int_a^b c(t)dt. \tag{2.4}$$

Finally, according to the definition of γ, it is easy to show that

$$\gamma(\cup_{\lambda\in[0,1]}\lambda\Omega) = \gamma(\Omega), \tag{2.5}$$

for all subsets Ω of E.

Another important example of monotone, nonsingular, semiadditive m.n.c. is the *Kuratowski measure of non-compactness*:

$$\alpha(\Omega) := \inf\{d > 0 : \Omega \text{ has a partition into a finite number of sets of diameter less than } d\},$$

where $\Omega \subset E$ is bounded.

In the sequel, we shall always employ the Hausdorff m.n.c., but since the following relation

$$\gamma(\Omega) \le \alpha(\Omega) \le 2\gamma(\Omega)$$

holds, for each bounded $\Omega \subset E$, the Hausdorff m.n.c. can be everywhere replaced by the Kuratowski m.n.c.

In a space of continuous functions, a further important example of monotone, nonsingular and algebraically semiadditive m.n.c. is the *modulus of equicontinuity*:

$$\mathrm{mod}_C(\Lambda) := \lim_{\delta\to 0} \sup_{x\in\Lambda} \max_{|t_1-t_2|\le\delta} |x(t_1) - x(t_2)|.$$

It is easy to see that the modulus of equicontinuity of a subset $\Lambda \subset E$ is equal to zero if and only if all the elements $x \in \Lambda$ of the set Λ are equicontinuous.

Given $X \subseteq E$, a multivalued mapping $F : X \multimap E$ with compact values (a family of multivalued maps $G : X \times [0,1] \multimap E$ with compact values) is called *condensing* with respect to a m.n.c. β (shortly, β-condensing) if, for every $\Omega \subseteq X$ such that

$$\beta\big(F(\Omega)\big) \ge \beta(\Omega) \quad (\beta\big(G(\Omega \times [0,1])\big) \ge \beta(\Omega)),$$

Ω is relatively compact.

Let $X \subseteq E$ be closed and convex. Let $F : X \mapsto X$ be a multivalued mapping with nonempty convex, compact values. Assume that F is β-condensing with respect to a monotone, nonsingular m.n.c. β. Let $D \subset X$ be open and the boundary ∂D be fixed point free for F. In this case, both the index: $\mathrm{ind}(F, X, D)$ (cf. [4]) as well as the topological degree: $\deg_E(\mathrm{Id} - F, \overline{D})$ (cf. [16]), satisfying usual properties, can be defined. Let $\mathrm{Fix}(F)$ denote the set of all fixed points of F, i.e.

$$\mathrm{Fix}(F) := \{x \in X : x \in F(x)\}. \tag{2.6}$$

In the following proposition, the main properties of the above mentioned index are stated. Since the degree can be defined here by the formula (cf. [4, p. 197])

$$\deg_E(\mathrm{Id} - F, \overline{D}) = \mathrm{ind}(F, X, D), \tag{2.7}$$

provided $\{x \in \overline{D} : 0 \in x - F(x)\} \cap \partial D = \emptyset$, analogous properties are valid for the degree.

Proposition 2.4 (see e.g. [2, Proposition 3.2]). *Under the above conditions imposed on X, F, β and D, it is possible to define* $\mathrm{ind}(F, X, D) \in \mathbb{Z}$ *in such a way that it satisfies the following properties:*

(i) *(Existence) If* $\mathrm{ind}(F,X,D) \neq 0$*, then* $\mathrm{Fix}(F) \neq \emptyset$.

(ii) *(Localization) If* $D_1 \subset D$ *are open subsets of* X *such that* $\mathrm{Fix}(F) \subset D_1 \subset D$*, then* $\mathrm{ind}(F,X,D) = \mathrm{ind}(F,X,D_1)$.

(iii) *(Additivity) If* D_j*,* $j = 1,\ldots,n$ *are open disjoint subsets of* D *and all fixed points of* $F|_D$ *are located in* $\cup_{j=1}^{m} D_j$*, then* $\mathrm{ind}(F,X,D_j)$*,* $j = 1,\ldots,n$*, are well-defined and satisfy*

$$\mathrm{ind}(F,X,D) = \sum_{j=1}^{n} \mathrm{ind}(F,X,D_j).$$

(iv) *(Homotopy) If there is a* β*-condensing homotopy* $\chi : X \times [0,1] \multimap X$ *with* $\chi(\cdot,0) = F$*,* $\chi(\cdot,1) = G$ *and the boundary* ∂D *of* D *is fixed point free w.r.t.* χ*, then*

$$\mathrm{ind}(F,X,D) = \mathrm{ind}(G,X,D).$$

(v) *(Normalization) If* $F \equiv \{a\} \not\subset \partial D$*, then*

$$\mathrm{ind}(F,X,D) = \begin{cases} 1, & \text{for } a \in D, \\ 0, & \text{for } a \notin D. \end{cases}$$

3. GENERAL METHOD

In this section, we shall develop a continuation principle (see Theorem 3.1) for the investigation of the following multivalued b.v.p.:

$$\begin{cases} x' \in P(t,x), & \text{for a.a. } t \in [a,b], \\ x \in S, \end{cases} \tag{3.1}$$

where $P : [a,b] \times E \multimap E$ is an upper-Carathéodory map and $S \subseteq AC([a,b],E)$.

We embed (3.1) into a family of related b.v.p.'s and we introduce a set $Q \subseteq C([a,b],E)$ of candidate solutions of (3.1). We denote by $T : Q \times [0,1] \multimap AC([a,b],E)$ the multivalued solution operator of this suitably given family. As usual in this context, we define these parameterized b.v.p.'s in such a way that all the fixed points of $T(\cdot,1)$ are solutions of (3.1). For each $\lambda \in [0,1]$, we put $T_\lambda := T(\cdot,\lambda)$, and we denote by $\mathrm{Fix}(T_\lambda)$ the set of fixed points of T_λ, as defined in (2.6). We impose suitable condensity conditions on the solution operator T, given in terms of a monotone and nonsingular m.n.c. In order to study the fixed points set of $T(\cdot,1)$, we can employ either the topological degree for condensing multivalued vector-fields in [16] or the fixed point index proposed in [4] (see also [2]). In fact, in a Banach space, the so called *pushing condition* (see e.g. condition (A_H) in [2, p. 25]) can be reduced to the assumption that the boundary is fixed point free.

We refer to [3] (see also [2, Theorem 4.4]) for a similar continuation principle, when t takes values in an unbounded interval, implying the set Q of candidate solutions to be a subset of a Fréchet space.

Theorem 3.1. *Consider problem* (3.1)*, where* $P : [a,b] \times E \multimap E$ *is an upper-Carathéodory map and* S *is a subset of absolutely continuous functions* $x : [a,b] \to E$*. Let* $H : [a,b] \times E \times E \times [0,1] \multimap E$ *be an upper-Carathéodory map such that*

$$H(t,c,c,1) \subset P(t,c), \quad \textit{for all } (t,c) \in [a,b] \times E. \tag{3.2}$$

Furthermore, assume that

(i) *there exists a closed and convex subset* $Q \subseteq C([a,b],E)$*, with* $Q \setminus \partial Q \neq \emptyset$*, and a closed subset* S_1 *of* S *such that the problem*

$$\begin{cases} x'(t) \in H(t,x(t),q(t),\lambda), & \textit{for a.a. } t \in [a,b], \\ x \in S_1 \end{cases}$$

is solvable with a convex set $T(q,\lambda)$ *of solutions, for each* $(q,\lambda) \in Q \times [0,1]$*;*

(ii) *the solution operator* T *is quasi-compact (i.e. it maps compact subsets onto relatively compact subsets) and* μ*-condensing with respect to a monotone and non-singular m.n.c.* μ *defined on* $C([a,b],E)$*;*

(iii) $T(Q \times \{0\}) \subset Q$*;*

(iv) *the map* T_λ *has no fixed points on the boundary* ∂Q *of* Q*, for every* $\lambda \in [0,1)$*.*

Then problem (3.1) *has a solution in* Q*.*

Proof. According to (3.2), every fixed point $x \in Q$ of the multivalued operator T_1, is a solution of (3.1). Thus, the proof reduces to the investigation of the fixed points of T_1. If $\mathrm{Fix}(T_1) \cap \partial Q \neq \emptyset$, there is nothing to prove; otherwise, (iv) can be reformulated as

$$\mathrm{Fix}(T_\lambda) \cap \partial Q = \emptyset, \quad \text{for every } \lambda \in [0,1]. \tag{3.3}$$

Firstly, we show that T has a closed graph Γ_T in the space $Q \times [0,1] \times C([a,b],E)$. Let $\{q_n, \lambda_n, x_n\}_n$ be a sequence in the graph Γ_T of T converging to $(q_0, \lambda_0, x_0) \in Q \times [0,1] \times C([a,b],E)$ as $n \to +\infty$. Since q_n and x_n are uniformly convergent on $[a,b]$, there exists a positive constant M such that

$$\|x_n(t)\| \leq M \text{ and } \|q_n(t)\| \leq M \text{ for all } t \in [a,b] \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Observe that $x_n(t) \to x(t)$, as $n \to +\infty$, implies that $\{x_n(t)\}_n$ is relatively compact in E, for all $t \in [a,b]$. Since H is u-Carathéodory and

$$x_n'(t) \in H(t, x_n(t), q_n(t), \lambda_n), \tag{3.5}$$

for a.a. $t \in [a,b]$, according to (3.4), there exists $k \in L^1([a,b],E)$ satisfying

$$\|x_n'(t)\| \leq k(t), \quad \text{for a.a. } t \in [a,b] \text{ and } n \in \mathbb{N}.$$

Thus, the sequence $\{x_n'\}_n$ is bounded and uniformly integrable in $L^1([a,b],E)$.

Now, we show that $\{x'_n(t)\}_n$ is also relatively compact, for a.e. $t \in [a, b]$. In fact, take $t \in [a, b]$ satisfying the inclusion (3.5), for all n. Since $H(t, \cdot)$ is u.s.c. in $E \times E \times [0, 1]$, given $\epsilon > 0$, we can find $\delta > 0$ such that $H(t, x, y, \lambda) \subseteq H(t, x_0(t), q_0(t), \lambda_0) + \epsilon B$, for all $(x, y, \lambda) \in E \times E \times [0, 1]$ satisfying $\|(x, y, \lambda) - (x_0(t), q_0(t), \lambda_0)\| \leq \delta$. Since $(x_n(t), q_n(t), \lambda_n) \to (x_0(t), q_0(t), \lambda_0)$, as $n \to +\infty$, this implies the existence of $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$H(t, x_n(t), q_n(t), \lambda_n) \subseteq H(t, x_0(t), q_0(t), \lambda_0) + \epsilon B, \quad \text{for all } n > n_0.$$

Consequently,

$$\{x'_n(t)\}_n \subseteq \bigcup_{n=1}^{n_0} H(t, x_n(t), q_n(t), \lambda_n) \cup \Big(H(t, x_0(t), q_0(t), \lambda_0) + \epsilon B\Big)$$

and since H is compact valued, the sequence $\{x'_n(t)\}_n$ is relatively compact in E. According to Lemma 2.2, we can extract a subsequence, again denoted by $\{x_n\}_n$, that converges to an absolutely continuous function $x : [a, b] \to E$ in the following sense: $x_n \to x$ in $C([a, b], E)$ and $x'_n \to x'$, weakly in $L^1([a, b], E)$. Since $x_n \to x_0$ in $C([a, b], E)$, as $n \to +\infty$, the uniqueness of the limit implies that $x = x_0$. Applying a classical closure theorem (see e.g. [2, Lemma 4.3] or [16, Lemma 5.1.1]), we can conclude that

$$x'_0(t) \in H(t, x_0(t), q_0(t), \lambda_0), \qquad \text{for a.a. } t \in [a, b].$$

Moreover, since S_1 is closed, then $x_0 \in S_1$, and consequently T has a closed graph.

Since, in particular, the set $T(q, \lambda)$ is closed, for all $(q, \lambda) \in Q \times [0, 1]$, and since T is quasi-compact, T has also compact values. This implies (see e.g. [16, Theorem 1.1.12]) that T is u.s.c. We can conclude that T is u.s.c. map with convex, compact values which is condensing on the closed set Q. These properties of the multivalued map T allow us to define both the topological degree (see e.g. [16]) as well as the fixed point index (see e.g. [2] or [4]) on open subsets of a Banach space, provided their boundaries are fixed point free. Moreover, both the degree and the index satisfy the standard properties (see Proposition 2.4 and cf. (2.7)). In particular, we obtain that the multivalued vector-fields $\Phi_0 := \mathrm{Id} - T_0$ and $\Phi_1 := \mathrm{Id} - T_1$ are homotopic and, according to (3.3), that T is an admissible homotopy. Consequently, $\deg_{C([a,b],E)}(\Phi_1, Q) = \deg_{C([a,b],E)}(\Phi_0, Q)$. Since $T_0(Q) \subseteq Q$ and $\mathrm{Fix}(T_0 \cap \partial Q) = \emptyset$, applying the localization property of the degree, (see [16, Property 3.2.2]), we obtain that $\deg_{C([a,b],E)}(\Phi_0, Q) = \deg_Q(\Phi_0, Q) = 1$. Therefore, by the existence property, $\emptyset \neq \mathrm{Fix}(T_1) \subseteq \big(Q \setminus \partial Q\big)$, which completes the proof. □

4. BOUND SETS APPROACH

The continuation principle proposed in Theorem 3.1 involves, in particular, a suitable set $Q \subseteq C([a, b], E)$ of candidate solutions which must satisfy the transversality condition (iv) in the quoted result. A quite natural way to construct Q is to assign the subset of E, where the functions $q \in Q$ take values. In this paper, we always assume $Q = C([a,b], \overline{K})$, where K is nonempty and open in E and $\overline{K}$ denotes its closure. In this way, we overcome the delicate point of checking the transversality condition by assuming that K is a bound set. We recall that a nonempty and open $K \subseteq E$ is a *bound set* for (3.1) (see e.g. [4, Definition 8.2]) if the trajectory of any solution of this problem entirely contained in $\overline{K}$ remains all the time inside K.

The theory of bound sets was initiated by Gaines and Mawhin [15] for the investigation of b.v.p.'s associated to differential equations. We refer to [4, 5, 6, 7] for its adoption in a multivalued setting and for several applications of this theory in the study of multivalued b.v.p.'s.

As already pointed out in [9] and [11], the theory of typical (global) guiding functions in arbitrary Banach spaces is not possible. Our approach here is, however, local. This explains why we are able to construct a bound set $K \subset E$ by means of a Liapunov-like function V, usually called a *bounding function*, as demonstrated below.

Proposition 4.1. *Let $P : [a,b] \times E \multimap E$ be a u-Carathéodory map and K be a nonempty open subset of E. Assume that there is a locally Lipschitzian function $V : E \to \mathbb{R}$ and an $\epsilon > 0$ such that*

(B1) $V/_{\partial K} = 0, \quad V/_{B^{\epsilon}_{\partial K} \cap \overline{K}} \leq 0$;
(B2) *for a.a.* $t \in (a,b),\ x \in B^{\epsilon}_{\partial K} \cap \overline{K},\ w \in P(t,x)$, *we have*

$$\liminf_{h \to 0^-} \frac{V(x+hw) - V(x)}{h} < 0.$$

Let $x(t)$ be a solution of $x' \in P(t,x)$ satisfying $x(t) \in \overline{K}$, for all $t \in [a,b]$. Then $x(t) \in K$, for all $t \in (a,b]$.

Proof. Let $x : [a,b] \to E$ be a solution of $x' \in P(t,x)$ satisfying $x(t) \in \overline{K}$, for all $t \in [a,b]$, and assume, by contradiction, the existence of $t_0 \in (a,b]$ such that $x_0 := x(t_0) \in \partial K$.

Since V is locally Lipschitzian, there exists an open $U \subseteq E$ with $x_0 \in U$ and $L > 0$ such that $V|_U$ is Lipschitzian with constant L. Let $k \in (0, t_0 - a]$ satisfying $x(t) \in U \cap B^{\epsilon}_{\partial K}$, for all $t \in [t_0 - k, t_0]$. It is easy to see that $g(t) := V(x(t))$ is absolutely continuous in $[t_0 - \overline{k}, t_0]$. Thus, $g'(t)$ exists, for a.a. $t \in [t_0 - k, t_0]$. If we prove that

$$g'(t) < 0, \quad \text{for a.a. } t \in [t_0 - k, t_0], \tag{4.1}$$

according to (B1), (B2), we obtain the following contradictory inequality

$$0 \leq -V(x(t_0 - k)) = g(t_0) - g(t_0 - k) = \int_{t_0 - k}^{t_0} g'(s)\, ds < 0.$$

So, it remains to prove (4.1). For this purpose, notice that such $x'(t)$ exists and satisfies $x'(t) \in P(t, x(t))$, for a.a. $t \in [t_0 - k, t_0]$; so take such a t. For a sufficiently small $h < 0$, put

$$\varphi(h) := x(t+h) - x(t) - x'(t)h,$$
$$\Delta(h) := \frac{V(x(t) + x'(t)h + \varphi(h)) - V(x(t) + x'(t)h)}{h}.$$

According to the Lipschitzianity of V and the definition of $\varphi(h)$, we have

$$|\Delta(h)| \leq \frac{L|\varphi(h)|}{|h|} \to 0, \quad \text{as } h \to 0.$$

Consequently, since

$$\frac{g(t+h) - g(t)}{h} = \frac{V(x(t) + x'(t)h) - V(x(t))}{h} + \Delta(h),$$

we obtain

$$\liminf_{h \to 0^-} \frac{g(t+h) - g(t)}{h} = \liminf_{h \to 0^-} \frac{V(x(t) + x'(t)h) - V(x(t))}{h} < 0$$

implying the validity of (4.1). □

Remark 4.2. If we replace in the previous proposition condition (B2) by

$$\text{(B2}'\text{)} \qquad \limsup_{h \to 0^+} \frac{V(x + hw) - V(x)}{h} > 0,$$

for a.a. $t \in (a, b), x \in B^{\epsilon}_{\partial K} \cap \overline{K}$, $w \in P(t, x)$ then, under all the other assumptions, every solution $x(t)$ of (3.1) such that $x(t) \in \overline{K}$, for all $t \in [a, b]$, satisfies $x(t) \in K$, for all $t \in [a, b)$.

Now, we are able to construct a bound set for the Cauchy as well as the Floquet problems. The proof of the following corollary is an immediate consequence of Proposition 4.1, so, it is left to the reader.

Corollary 4.3. *Let $P : [a, b] \times E \multimap E$ be a u-Carathéodory map, $K \subseteq E$ be a nonempty open set and $V : E \to \mathbb{R}$ be a locally Lipschitzian function satisfying (B1), for some $\epsilon > 0$.*

If (B2) holds, then K is a bound set for the initial (Cauchy) problem

$$\text{(4.2)} \qquad \begin{cases} x' \in P(t, x), & \text{for a.a. } t \in [a, b], \\ x(a) = x_0, \end{cases}$$

provided $x_0 \in K$.

If (B2′) holds, then K is a bound set for the terminal problem

$$\text{(4.3)} \qquad \begin{cases} x' \in P(t, x), & \text{for a.a. } t \in [a, b], \\ x(b) = x_1, \end{cases}$$

provided $x_1 \in K$.

Finally, if $M \in \mathcal{L}(E)$, $M\partial K = \partial K$ and either (B2) or (B2′) holds, then K is a bound set for the Floquet b.v.p.

$$\begin{cases} x' \in P(t,x), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a). \end{cases} \tag{4.4}$$

Notice that assumption (B2) in fact guarantees that K is also a positively invariant set for the inclusion $x' \in P(t,x)$, i.e. each trajectory $x(t)$ of this inclusion satisfying $x(a) \in K$ remains always inside K. We refer to [24] and [26] for an investigation of the positive invariance of given sets performed in a similar way by means of Liapunov-like functions.

On the other hand, the existence of an open positively invariant set K, for $x' \in P(t,x)$, is not enough in order to guarantee the absence of solutions of this inclusion on the boundary of $C([a,b],\overline{K})$. Hence, additional restrictions must be imposed on these solutions which take values in ∂K.

At last, if $P : [a,b] \times E \multimap E$ is globally u.s.c. in (t,x), the transversality condition can be localized only on the boundary ∂K of K, as showed by the following proposition.

Proposition 4.4. *Let $P : [a,b] \times E \multimap E$ be a u.s.c. multivalued map with nonempty, convex, compact values. Let $K \subseteq E$ be nonempty and open and $V : E \to \mathbb{R}$ be a locally Lipschitzian function satisfying (B1), for some $\epsilon > 0$. Assume that, for all $t \in (a,b)$, $x \in \partial K$, $w_1, w_2 \in P(t,x)$,*

$$0 \notin \left[\liminf_{h\to 0^+} \frac{V(x+hw_1)}{h},\ \limsup_{h\to 0^-} \frac{V(x+hw_2)}{h}\right]. \tag{4.5}$$

Let $x(t)$ be a solution of $x' \in P(t,x)$ such that $x(t) \in \overline{K}$, for all $t \in [a,b]$. Then $x(t) \in K$, for all $t \in (a,b)$.

Proof. Let $x(t)$ be a solution of $x' \in P(t,x)$ such that $x(t) \in \overline{K}$, for all $t \in [a,b]$. Assume, by contradiction, the existence of $t_0 \in (a,b)$ such that $x_0 := x(t_0) \in \partial K$. Take a sequence $\{h_n\}_n$ of positive real values satisfying $h_n \to 0^+$ as $n \to +\infty$.

Given $\delta > 0$, since P is globally u.s.c. and x is continuous, it is possible to find $\sigma = \sigma(\delta) > 0$ such that

$$P(t,x(t)) \subset P(t_0,x_0) + \delta B, \tag{4.6}$$

whenever $|t - t_0| \le \sigma$. Therefore, since $P(t_0,x_0)$ is convex, for all sufficiently large n,

$$\frac{x(t_0+h_n) - x(t_0)}{h_n} = \frac{1}{h_n}\int_{t_0}^{t_0+h_n} x'(s)\,ds \in P(t_0,x_0) + \delta\overline{B}. \tag{4.7}$$

Since δ was taken in an arbitrary way and since $P(t_0,x_0)$ is compact, this implies that

$$\left\{\frac{x(t_0+h_n) - x(t_0)}{h_n}, \quad n \in \mathbb{N}\right\}$$

is a relatively compact subset of E. Hence, by passing to a subsequence, again denoted as the sequence, we have

$$\frac{x(t_0+h_n)-x(t_0)}{h_n} \to y_1, \quad \text{as } n \to +\infty,$$

for some $y_1 \in E$. Condition (4.7) and the compactness of $P(t_0, x_0)$ then imply $y_1 \in P(t_0, x_0)$.

Now, let $\{\delta_n\}_n$ be such that $\delta_n \to 0$ as $n \to +\infty$ and $x(t_0+h_n) = x_0 + h_n(y_1+\delta_n)$, for all n. Let $U \subset E$ be open with $x_0 \in U$ and $L > 0$ be such that $V|_U$ is Lipschitzian with constant L. We can take, without any loss of generality, $x(t_0+h_n) \in U$, for all n. According to (B1),

$$0 \geq \frac{V(x(t_0+h_n))}{h_n} = \frac{V(x_0+h_n(y_1+\delta_n))}{h_n} = \frac{V(x_0+h_n y_1)}{h_n} + \Delta_n,$$

where

$$\Delta_n = \frac{V(x_0+h_n(y_1+\delta_n)) - V(x_0+h_n y_1)}{h_n},$$

and according to the Lipschitzianity of V in U, we have $|\Delta_n| \leq L|\delta_n| \to 0$ as $n \to +\infty$. Consequently,

$$\begin{aligned} 0 &\geq \liminf_{n\to+\infty} \frac{V(x(t_0+h_n))}{h_n} = \liminf_{n\to+\infty} \frac{V(x_0+h_n y_1)}{h_n} + \Delta_n \\ &\geq \liminf_{h\to 0^+} \frac{V(x_0+h y_1)}{h}. \end{aligned} \tag{4.8}$$

By a similar reasoning, it is also possible to get $y_2 \in P(t_0, x_0)$ such that, by passing to a subsequence,

$$\frac{x(t_0)-x(t_0-h_n)}{h_n} \to y_2 \quad \text{as } n \to +\infty.$$

Therefore, by the same argument as above, it is possible to find $\Omega_n \to 0$ as $n \to +\infty$ so that

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \frac{V(x(t_0-h_n))}{-h_n} = \limsup_{n\to+\infty} \frac{V(x_0-h_n y_2)}{-h_n} + \Omega_n \\ &\leq \limsup_{h\to 0^-} \frac{V(x_0+h y_2)}{h}. \end{aligned} \tag{4.9}$$

It can be easily seen that (4.8) and (4.9) are in contradiction with (4.5). This completes the proof. □

We are now able to obtain a bound set for (4.2), (4.3) and (4.4), when P is globally u.s.c.

Corollary 4.5. *Let $P : [a,b] \times E \multimap E$ be u.s.c. with nonempty, convex, compact values and $K \subseteq E$ be nonempty and open. Let $V : E \to \mathbb{R}$ be a locally Lipschitzian function satisfying (B1), for some $\epsilon > 0$, and (4.5). If*

$$\limsup_{h\to 0^-} \frac{V(x+hw)}{h} < 0, \quad \text{for } x \in \partial K \text{ and } w \in P(b,x), \tag{4.10}$$

then K is a bound set for the initial value problem (4.2), provided $x_0 \in K$. If

$$\liminf_{h\to 0^+} \frac{V(x+hw)}{h} > 0, \quad \text{for } x \in \partial K \text{ and } w \in P(a,x), \tag{4.11}$$

then K is a bound set for the terminal value problem (4.3), provided $x_0 \in K$. Finally, let $M \in \mathcal{L}(E)$ with $M\partial K = \partial K$ and assume

$$0 \notin \left[\liminf_{h\to 0^+} \frac{V(x+hw_1)}{h}, \limsup_{h\to 0^-} \frac{V(x+hw_2)}{h}\right], \tag{4.12}$$

for all $x \in \partial K$, $w_1 \in P(a,x)$ and $w_2 \in P(b,Mx)$. Then K is a bound set for (4.4).

Proof. Let $x : [a,b] \to \overline{K}$ be a solution of (4.4). Proposition 4.4 implies that $x(t) \in K$, for all $t \in (a,b)$. Reasoning as in the proof of the quoted proposition, it is also easy to see that, when (4.10) (respectively (4.11)) is satisfied, then K is a bound set for (4.2) (respectively (4.3)), provided that $x_0 \in K$.

Now, consider b.v.p. (4.4) and assume, by contradiction, that $x(a) \in \partial K$ or $x(b) \in \partial K$. Since M is invertible and $M\partial K = \partial K$, then both $x(a)$ and $x(b)$ belong to ∂K. Therefore, reasoning as in the proof of Proposition 4.4, it is possible to find $w_1 \in P(a, x(a))$ and $w_2 \in P(b, Mx(a))$ such that

$$\liminf_{h\to 0^+} \frac{V(x+hw_1)}{h} \le 0 \le \limsup_{h\to 0^-} \frac{V(x+hw_2)}{h}$$

which is a contradiction to (4.12). The proof is complete. □

Assume now that V is a C^1-function and let V'_x denote its derivative at the point $x \in E$. Condition (B2) ((B2′)) reduces to $V'_x(w) < 0$ (respectively $V'_x(w) > 0$), for a.a. $t \in (a,b)$), $x \in B^{\epsilon}_{\partial K} \cap \overline{K}$ and $w \in P(t,x)$.

Let us consider P to be u.s.c. Since P is convex valued, it is easy to see that (4.5) reduces to

$$V'_x(w) \neq 0, \quad \text{for } t \in (a,b),\ x \in \partial K \text{ and } w \in P(t,x), \tag{4.13}$$

whereas condition (4.12) becomes

$$0 \notin \left[V'_x(w_1), V'_x(w_2)\right], \text{ for } x \in \partial K, w_1 \in P(a,x), w_2 \in P(b,Mx). \tag{4.14}$$

Let us note that conditions (4.5) and (4.12) guarantee neither positive nor negative invariance of the bound set K as the following simple example shows.

Example 1. For $t \in [0,1]$, consider the b.v.p. $x' = 1$ and $x(1) = \pm x(0)$ with $x \in \mathbb{R}$. Take $K = (-1,1)$. It is easy to see that K is neither a positive invariant set for the problem nor a negative one. On the other hand, $V(x) = x^2 - 1$ is a bounding function for the problem, satisfying both (4.13) and (4.14). Thus, K is a bound set.

Example 2. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Given $r > 0$, put $K = rB$ and take $V(x) = |x|^2 - r^2$. It is easy to see that $V(x) \leq 0$, for all $x \in \overline{K}$, and $V(x) = 0$ if and only if $x \in \partial K$. Hence, V is a suitable candidate for a bounding function for K.

Moreover, V is Fréchet differentiable, for all x, and $V'_x : w \to 2\langle x, w \rangle$; so, V is a C^1-map. Indeed, for a fixed $x \in H$, the Fréchet derivative V'_x is namely a linear operator $V'_x : H \to \mathbb{R}$ defined by

$$\lim_{|h| \to \mathbf{0}} \frac{V(x+h) - V'_x(h) - V(x)}{|h|} = 0.$$

Hence, because of

$$\frac{V(x+h) - V'_x(h) - V(x)}{|h|} = \frac{|x+h|^2 - r^2 - V'_x(h) - |x|^2 + r^2}{|h|}$$
$$= \frac{\langle x+h, x+h \rangle - V'_x(h) - \langle x, x \rangle}{|h|} = \frac{2\langle x, h \rangle + \langle h, h \rangle - V'_x(h)}{|h|},$$

and since $\frac{\langle h, h \rangle}{|h|} = |h| \to 0$, we really obtain that $V'_x : h \to 2\langle x, h \rangle$, for all $h \in H$ and, in particular, that $V'_x : w \to 2\langle x, w \rangle$.

Assume $M\partial K = \partial K$ and consider the b.v.p. (4.4). Let P be a u-Carathéodory r.h.s. If there exists $r_0 \in (0, r)$ such that one of the following two conditions is satisfied

$$\langle x, w \rangle < 0, \quad \text{for a.a. } t \in (a, b),\ r_0 < |x| \leq r \text{ and } w \in P(t, x), \tag{4.15}$$

or

$$\langle x, w \rangle > 0, \quad \text{for a.a. } t \in (a, b),\ r_0 < |x| \leq r \text{ and } w \in P(t, x), \tag{4.16}$$

then, according to Corollary 4.3, K is a bound set for (4.4).

On the other hand, if P is globally u.s.c. and the following conditions are both satisfied

$$\langle x, w \rangle \neq 0, \quad \text{for all } t \in (a, b),\ |x| = r \text{ and } w \in P(t, x), \tag{4.17}$$

$$0 \notin \Big[\langle x, w_1 \rangle, \langle x, w_2 \rangle \Big], \quad \text{for all } |x| = r,\ w_1 \in P(a, x) \text{ and } w_2 \in P(b, Mx), \tag{4.18}$$

then, according to Corollary 2, K is a bound set for (4.4).

5. APPLICATION TO FLOQUET PROBLEMS

Combining the continuation principle in the form of Theorem 3.1 with the bound sets approach developed in Proposition 4.1, we are now able to give an effective criterium (see Theorem 5.2) for the solvability of the Floquet b.v.p. (1.1) in a Banach space E satisfying the Radon–Nikodym property. For this purpose, we need the following preliminary result.

Lemma 5.1. *Given $f \in L^1([a,b],E)$, consider the Floquet b.v.p.*

$$\begin{cases} x' + A(t)x = f(t), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a) \end{cases} \tag{5.1}$$

with $A(t)$ and M respectively satisfying (A1) and (M). Let $U(t,s)$ be the associated evolution operator, for $t,s \in [a,b]$. Then (5.1) has a unique solution if and only if the map $M - U(b,a)$ is invertible. In this case, the linear operator $G : L^1([a,b],E) \to AC([a,b],E)$ which associates to every f the unique solution of (5.1) is defined as follows

$$Gf(t) := U(t,a)\big(M - U(b,a)\big)^{-1}\int_a^b U(b,s)f(s)\,ds + \int_a^t U(t,s)f(s)\,ds.$$

Proof. Let us consider the linear operator

$$L : \{x \in AC([a,b],E) : x(b) = Mx(a)\} \to L^1([a,b],E)$$

defined by $Lx = x' + A(t)x$. It is easy to see that (5.1) is uniquely solvable, for every $f \in L^1([a,b],E)$, if and only if L is invertible and, in this case, $G = L^{-1}$. According to (A1), it is well known (see e.g. [13]) that, for each $f \in L^1([a,b],E)$, the linear equation $x' + A(t)x = f(t)$ has a one-parameter family of solutions $x \in AC([a,b],E)$ given by $x_c(t) = U(t,a)c + \int_a^t U(t,s)f(s)ds$ with $c = x(a)$ varying in E. Notice that L is invertible if and only if, for each f, there is exactly one $c \in E$ such that $x_c(b) = Mx_c(a)$, i.e. satisfying $U(b,a)c + \int_a^b U(b,s)f(s)\,ds = Mc$. For this is equivalent to require that, for each $f \in L^1([a,b],E)$, the equation $\big(M - U(b,a)\big)c = \int_a^b U(b,s)f(s)\,ds$ is solvable, for a unique c. The conclusion follows when observing that the linear operator $f \to \int_a^b U(b,s)f(s)\,ds$ is surjective in E. □

We are now able to state our main result concerning the solvability of the b.v.p. (1.1) in a Banach space E satisfying the Radon–Nikodym property.

Theorem 5.2. *Let us consider the b.v.p.* (1.1) *under conditions (A1), (F1) and (M). Assume, moreover, that*

(A2) *the map $M - U(b,a)$ is invertible;*

(F2) *$\gamma(F(t,\Omega)) \leq g(t)\gamma(\Omega)$, for a.a. $t \in [a,b]$ and each bounded $\Omega \subset E$, where g is a non-negative function in $L^1[a,b]$ and γ is the Hausdorff m.n.c. in E.*

Suppose that

$$\|g\|_{L^1([a,b],E)}\Big(\frac{e^{\int_a^b \|A(s)\|\, ds}}{\|M-U(b,a)\|}+1\Big)e^{\int_a^b \|A(s)\|\, ds}<1. \tag{5.2}$$

Finally, assume there exist a nonempty, open, bounded and convex $K\subset E$, *a locally Lipschitzian function* $V:E\to\mathbb{R}$ *and an* $\epsilon>0$ *such that* $M\partial K=\partial K$, $0\in K$, *(B1) holds and, for all* $\lambda\in(0,1)$, *(B2) is satisfied with* $P(t,x)=\lambda F(t,x)-A(t)x$. *Then* (1.1) *admits a solution with values in* $\overline{K}$.

Proof. Consider the subset of absolutely continuous functions $S=\{x\in AC([a,b],E): x(b)=Mx(a)\}$. Since M is continuous, S is closed. Take

$$\begin{aligned} H:[a,b]\times E\times E\times[0,1]&\multimap E\\ (t,x,y,\lambda)&\multimap -A(t)x+\lambda F(t,y). \end{aligned} \tag{5.3}$$

According to (A1) and (F1), H is a u-Carathéodory map. Since K is convex, the set $Q=C([a,b],\overline{K})$ is closed and convex. For each $q\in Q$ and $\lambda\in[0,1]$, let us denote by $T(q,\lambda)$ the solution set of the fully linearized problem

$$\begin{cases} x'+A(t)x\in\lambda F(t,q), & \text{for a.a. } t\in[a,b],\\ x(b)=Mx(a). \end{cases} \tag{5.4}$$

We show that (5.4) satisfies all the assumptions of the continuation principle, i.e. of Theorem 3.1, where now $P(t,x)=-A(t)x+F(t,x)$; this implies that (1.1) has a solution.

Firstly, we note that condition (3.2) is satisfied. As in Section 2, for all $\lambda\in[0,1]$ and $q\in Q$, we denote by $T(q,\lambda)$ the set of all solutions of (5.4). Since $0\in K$, from Lemma 5.1, we obtain that $T(Q\times\{0\})=\{0\}\subset \operatorname{int} Q$, by which condition (iii) is satisfied. Let us assume that $x\in Q$ is a fixed point of $T(\cdot,\lambda)$, for some $\lambda\in(0,1)$, i.e. $x(t)$ is a solution of

$$\begin{cases} x'+A(t)x\in\lambda F(t,x), & \text{for a.a. } t\in[a,b],\\ x(b)=Mx(a). \end{cases} \tag{5.5}$$

According to (B1) and (B2) (see Corollary 4.3), K is a bound set for each problem (5.5) with $\lambda\in(0,1)$. This implies that $x\notin\partial Q$, so also condition (iv) of Theorem 3.1 is satisfied.

We now prove that the multivalued map T has convex values. Given x_1 and $x_2\in T(q,\lambda)$, there exist f_1 and $f_2\in\lambda F(\cdot,q(\cdot))$ such that $x_i'+A(t)x_i=f_i(t)$, for $i=1,2$, and the convexity of $T(q,\lambda)$ follows from (F1). Thus, also condition (i) of Theorem 3.1 is verified.

Now, we prove that T is quasi-compact. Since $C([a,b],E)$ is a metric space, it is sufficient to prove the sequential quasi-compactness of T. Therefore, consider the sequences $\lambda_n \to \lambda$ in $[0,1]$ and $q_n \to q$ in Q as $n \to +\infty$ and, for each $n \in \mathbb{N}$, take $x_n \in T(q_n, \lambda_n)$. There exists $f_n \in F(\cdot, q_n(\cdot))$ such that

$$x_n'(t) + A(t)x_n(t) = \lambda_n f_n(t), \quad \text{for a.a. } t \in [a,b], \tag{5.6}$$

and $x_n(b) = Mx_n(a)$. Denote by $\Lambda : E \to E$ the bounded, linear operator

$$\Lambda := \big(M - U(b,a)\big)^{-1}.$$

According to Lemma 5.1,

$$x_n(t) = \lambda_n \Big(U(t,a)\Lambda \int_a^b U(b,s)f_n(s)\,ds + \int_a^t U(t,s)f_n(s)\,ds\Big). \tag{5.7}$$

Define $L := 1 + \max_{x\in\overline{K}} |x|$ and $D := e^{\int_a^b \|A(s)\|\,ds}$. Let us recall (see Section 2) that

$$\|U(t,s)\| \le D, \quad \text{for all } t,s \in [a,b]. \tag{5.8}$$

Due to (F1), we have

$$|f_n(t)| \le r(t)\big(1 + |q_n(t)|\big) \le Lr(t),$$

for a.a. $t \in [a,b]$ and all $n \in \mathbb{N}$. Therefore, $\{f_n\}_n$ is bounded and uniformly integrable in $L^1([a,b],E)$. As a consequence, the sequence $\{U(t,s)f_n(s)\}_n$, with t given in $(a,b]$, is also bounded and uniformly integrable on the interval $[a,t]$. In fact,

$$|U(t,s)f_n(s)| \le \|U(t,s)\||f_n(s)| \le LDr(s), \tag{5.9}$$

for a.a. $s \in [a,t]$ and all $n \in \mathbb{N}$. By virtue of (5.7), we have

$$\begin{aligned} |x_n(t)| &\le LD^2\|\Lambda\| \textstyle\int_a^b r(s)\,ds + LD \int_a^t r(s)\,ds \\ &\le LD\Big(D\|\Lambda\| + 1\Big)\|r\|_1 := J, \end{aligned}$$

for all $t \in [a,b]$. Thus, the sequence $\{x_n\}_n$ is bounded in $C([a,b],E)$. Consequently,

$$|x_n'(t)| \le |-A(t)x_n(t)| + |f_n(t)| \le J\|A(t)\| + Lr(t), \tag{5.10}$$

whence $\{x_n'\}_n$ is also bounded and uniformly integrable in $L^1([a,b],E)$.

Since the sequence $\{q_n\}_n$ is converging, according to (F2), we have

$$\gamma(\{f_n(t)\}_n) \le g(t)\gamma(\{q_n(t)\}_n) = 0, \quad \text{for a.a. } t \in [a,b],$$

implying that $\{f_n(t)\}_n$ is relatively compact. Therefore, given $t \in (a,b]$, we also obtain the relative compactness of $\{U(t,s)f_n(s)\}_n$, for a.a. $s \in [a,t]$. In fact, according to (2.3),

$$\gamma(\{U(t,s)f_n(s)\}_n) \le \|U(t,s)\|\gamma(\{f_n(s)\}_n) = 0. \tag{5.11}$$

We can now prove that $\{x_n(t)\}_n$ is a relatively compact sequence, for all $t \in [a, b]$. Indeed, due to (2.5), (5.7) and the monotonicity of the Hausdorff m.n.c., for all $t \in [a, b]$, we have

$$\begin{aligned}\gamma(\{x_n(t)\}_n) \\ &\leq \gamma\Big(\bigcup_{\lambda\in[0,1]} \lambda\Big\{U(t,a)\Lambda\int_a^b U(b,s)f_n(s)\,ds + \int_a^t U(t,s)f_n(s)\,ds\Big\}_n\Big) \\ &= \gamma\Big(\Big\{U(t,a)\Lambda\int_a^b U(b,s)f_n(s)\,ds + \int_a^t U(t,s)f_n(s)\,ds\Big\}_n\Big).\end{aligned}$$

Therefore, according to (2.4), (5.9) and (5.11), we have

$$\begin{aligned}\gamma(\{x_n(t)\}_n) \leq \|U(t,a)\|\|\Lambda\|\,\gamma\Big(\int_a^b U(b,s)f_n(s)\,ds\Big) \\ + \gamma\Big(\int_a^t U(t,s)f_n(s)\,ds\Big) = 0.\end{aligned}$$

This demonstrates that $\{x_n(t)\}_n$ is relatively compact, for all t, and w.r.t. (5.6) we also have that $\{x_n'(t)\}_n$ is relatively compact, for a.a. $t \in [a, b]$.

In view of (5.10), we can apply Lemma 2.2. Hence, we find $x \in AC([a, b], E)$ and a subsequence, again denoted as the sequence, such that $x_n \to x$ in $C([a, b], E)$ and $x_n' \to x'$, weakly in $L^1([a, b], E)$, as $n \to +\infty$. Applying, as in Theorem 3.1, a classical closure principle, we obtain that $x'(t) + A(t)x(t) \in \lambda F(t, q(t))$, for a.a. $t \in [a, b]$. We have so proved the quasi-compactness of T.

Now, it remains to show that T is condensing with respect to a monotone and nonsingular m.n.c. For this purpose, consider the monotone and nonsingular m.n.c. (see e.g. [16, Example 2.1.4]) defined for a bounded set $\Omega \subset C([a, b], E)$ as

$$\mu(\Omega) := \max_{\{w_n\}_n\subset\Omega}\Big(\sup_{t\in[a,b]}\gamma(\{w_n(t)\}_n), \mathrm{mod}_C(\{w_n\}_n)\Big), \tag{5.12}$$

where the ordering is induced by the positive cone in $\mathbb{R}^2$. Take $\Theta \subset Q$ such that

$$\mu(T(\Theta \times [0, 1])) \geq \mu(\Theta) \tag{5.13}$$

and let $\{x_n\}_n \subset T(\Theta \times [0, 1])$ be a sequence which realizes the maximum in (5.12), i.e. such that

$$\mu(T(\Theta \times [0, 1])) = \Big(\sup_{t\in[a,b]}\gamma(\{x_n(t)\}_n), \mathrm{mod}_C(\{x_n\}_n)\Big).$$

Define, for a bounded $\Omega \subset C([a, b], E)$,

$$\nu(\Omega) := \max_{\{w_n\}_n\subset\Omega}\sup_{t\in[a,b]}\gamma(\{w_n(t)\}_n). \tag{5.14}$$

According to (5.13), it follows that

$$\sup_{t\in[a,b]}\gamma(\{x_n(t)\}_n) \geq \nu(\Theta) \tag{5.15}$$

and

$$\operatorname{mod}_C(\{x_n\}_n)) \geq \max_{\{y_n\}_n \subset \Theta} \operatorname{mod}_C(\{y_n\}_n). \tag{5.16}$$

Since $\{x_n\}_n \subset T(\Theta \times [0,1])$, there exist $\{q_n\}_n \subset \Theta$, $f_n \in F(\cdot, q_n(\cdot))$ and $\{\lambda_n\}_n \subset [0,1]$ such that $x_n(\cdot)$ satisfies (5.7) in $[a,b]$, for all n. Thus, according to (F2), by a similar reasoning as before, it is possible to prove that

$$\gamma(\{U(t,s)f_n(s)\}_n) \leq \|U(t,s)\| g(s)\gamma(\{q_n(s)\}_n) \leq D\nu(\Theta)g(s),$$

for all $t \in (a,b]$ and a.a. $s \in [a,t]$. Consequently, by means of (2.3), (2.4) and (5.7), for all $t \in [a,b]$, it follows that

$$\begin{aligned}
&\gamma(\{x_n(t)\}_n) \\
&\leq \|U(t,a)\| \, \|\Lambda\| \gamma\left(\left\{\int_a^b U(b,s)f_n(s)\,ds\right\}_n\right) + \gamma\left(\left\{\int_a^t U(t,s)f_n(s)\,ds\right\}_n\right) \\
&\leq D\nu(\Theta)\left[D\|\Lambda\|\int_a^b g(s)\,ds + \int_a^t g(s)\,ds\right] \leq D\nu(\Theta)\big(D\|\Lambda\|+1\big)\|g\|_{L^1([a,b],E)}.
\end{aligned}$$

According to (5.15), we have that

$$\begin{aligned}
\nu(\Theta) &\leq \sup_{t\in[a,b]} \gamma(\{x_n(t)\}_n) \leq D\nu(\Theta)\big(D\|\Lambda\|+1\big)\|g\|_{L^1([a,b],E)} \\
&= \|g\|_{L^1([a,b],E)}\left(\frac{e^{\int_a^b \|A(s)\|\,ds}}{\|M-U(b,a)\|}+1\right)e^{\int_a^b \|A(s)\|\,ds}\nu(\Theta).
\end{aligned}$$

So condition (5.2) implies that $\nu(\Theta) = 0$. Consequently, $\gamma(\{y_n(t)\}_n) = 0$, for each $\{y_n\}_n \subset \Theta$ and $t \in [a,b]$, i.e. $\{y_n(t)\}_n$ is relatively compact in E.

Let us note that condition (5.10) does not depend on whether or not $\{q_n\}_n$ converges in Q; it simply holds for the sequence $\{x_n\}_n$ satisfying (5.15) and (5.16). Since (5.10) implies the equicontinuity of $\{x_n\}_n$, then $\operatorname{mod}_C(\{x_n\}_n) = 0$ (see Section 2). It follows from (5.16) that $\max_{\{y_n\}_n\subset\Theta} \operatorname{mod}_C(\{y_n\}_n) = 0$. Therefore, any sequence $\{y_n\}_n \subset \Theta$ is equicontinuous and the Ascoli-Arzelà lemma yields the relative compactness of Θ, which completes the proof. □

For $M = \mathrm{Id}$, the Floquet b.v.p. reduces to the investigation of periodic solutions. If in addition $A(t) \equiv A$ and $[a,b] = [0,T]$, condition (A2) is equivalent to assuming that

$$1 \in \rho(e^{-AT}), \tag{5.17}$$

where ρ denotes the resolvent of the operator e^{-AT}. This is, furthermore, equivalent to requiring that the only periodic solution of the homogeneous equation $x' + Ax = 0$ is the trivial one. According to the spectral mapping theorem (see e.g. [13]), inclusion (5.17) is the same as requiring that A is invertible.

Assume $A : [a,b] \to \mathcal{L}(E)$ to be continuous on $[a,b]$ and F globally u.s.c. in its variables (t,x) on all $[a,b] \times E$ and satisfying the growth condition (iii) in Definition

2.1. In view of Corollary 4.5, the transversality condition can be localized on the boundary ∂K of the set K. More precisely, in this case, the previous result is valid when (B2) is replaced by (4.5) and (4.12), and all the other conditions are satisfied.

Example 3. Let H be a real reflexive Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Consider problem (1.1) in H and assume that conditions (A1), (A2), (M), (F1), (F2) and (5.2) are satisfied. Let, for a given $r > 0$, either (4.15) or (4.16) hold, where $P(t,x) := \lambda F(t,x) - A(t)x$, $\lambda \in (0,1)$. If F is globally u.s.c., then instead of (4.15) or (4.16), assume only (4.17) and (4.18). Then, as in Example 2, conditions (B1) and (B2)(resp. (4.5) and (4.12)) are satisfied, by means of a bounding function $V(x) = |x|^2 - r^2$, with $K = rB$, and Theorem 5.2 (resp. its modification by means of Corollary 4.5) implies the existence of a solution of (1.1) with values in $\overline{K}$.

6. **ENTIRELY BOUNDED SOLUTIONS**

Given the u-Carathéodory map $P : \mathbb{R} \times E \multimap E$, where E is a Banach space satisfying the Radon–Nikodym property, let us consider again the differential inclusion

$$x' \in P(t,x), \quad \text{for a.a. } t \in \mathbb{R}. \tag{6.1}$$

As a final application of our method which combines the continuation principle in the form of Theorem 3.1 with a bound sets approach (see Section 4), we discuss the existence of entirely bounded solution of (6.1) (see Corollary 6.6). Our technique enables us to localize the bounded set $K \subset E$, where the solution values are located.

If P satisfies condition (F2), for all $t \in \mathbb{R}$ with $g \in L^1_{loc}(\mathbb{R})$, it is well known that the initial value problem associated to (6.1), i.e.

$$\begin{cases} x' \in P(t,x), \quad \text{for a.a. } t \in [-m,m], m \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{6.2}$$

is solvable, for each $x_0 \in E$ and $m \in \mathbb{N}$ (see e.g. [12, Theorem 9.2 and Remark 9.5.4]). Under similar conditions, the existence of a mild solution to (6.2) was given in [16, Theorem 5.2.2] (see also [16, Proposition 5.2.1]) when a linear term, which is the infinitesimal generator of a C_0-semigroup, is added to the r.h.s. Denote by x_m a solution of (6.2) on $[-m,m]$. It was proved in [2, Proposition 4.4 and Remark 4.6] that, whenever E is reflexive or separable and there exists a bounded closed $D \subset E$ satisfying $x_m(t) \in D$, for all $t \in [-m,m]$ and $m \in \mathbb{N}$, then (6.1) has a bounded solution with values in D. Unfortunately, in both the quoted results, the estimate of the norm of the solution depends on the interval $[-m,m]$, where it is defined. On the contrary, when assuming the existence of a bound set K, then we are able to localize each x_m in $\overline{K}$ by means of the following lemma.

Lemma 6.1. *Let us consider the initial value problem* (4.2)*, under conditions (F1) and (F2) (with P instead of F). Assume that there exist an open bounded $K \subset E$, a* *locally Lipschitzian function* $V : E \to \mathbb{R}$ *and an* $\epsilon > 0$ *such that (B1) holds and (B2) is satisfied. Then all solutions* x *of* (4.2) *such that* $x(a) = x_0 \in K$ *satisfy* $x(t) \in K$, *for all* $t \in [a, b]$.

Proof. It is well known that the Cauchy problem (4.2) is solvable (see e.g. [12, Theorem 9.2 and Remark 9.5.4]). Let $x(t)$ be a solution of (4.2). According to Corollary 4.3, there does not exist any $t_0 \in (a, b]$ such that $x(t_0) \in \partial K$. Therefore, $x(a) \in K$ implies $x(t) \in K$, for all $t \in [a, b]$. □

If P is u.s.c., Lemma 6.1 can be reformulated as follows.

Lemma 6.2. *Let us consider the initial value problem* (4.2), *where* P *is u.s.c. with nonempty, compact, convex values, satisfying (iii) in Definition* 2.1 *and (F2) (with* P *instead of* F*). Assume that there exist an open, bounded and convex* $K \subset E$, *a locally Lipschitzian function* $V : E \to \mathbb{R}$ *and* $\epsilon > 0$ *satisfying, (B1),* (4.5) *and* (4.10). *Then, for all* $x_0 \in K$, (4.2) *has a solution* x *with* $x(t) \in K$, *for all* $t \in [a, b]$.

Proof. Let

$$S = \{x \in AC([a, b], E) : x(a) = x_0\}, \quad H(t, x, y, \lambda) = \lambda P(t, y)$$

and $Q = C([a, b], \overline{K})$. It is easy to see that (3.2) is satisfied. Moreover, S is closed and, for each $(q, \lambda) \in Q \times [0, 1]$, the problem

$$\begin{cases} x' \in \lambda P(t, q(t)), & \text{for a.a. } t \in [a, b], \\ x(a) = x_0 \end{cases} \tag{6.3}$$

is solvable with a convex set $T(q, \lambda)$ of solutions. So condition (i) in Theorem 3.1 is valid. Since $x_0 \in K$, and according to (4.5) and (4.10), also conditions (iii) and (iv) in Theorem 3.1 are satisfied.

In order to apply Theorem 3.1, it remains to prove that T is quasi-compact and μ-condensing. For this purpose, take a positive constant L such that

$$l := \max_{t \in [a,b]} \int_a^t e^{-L(t-s)} g(s) ds < 1, \tag{6.4}$$

and consider the monotone, nonsingular and regular m.n.c. (see e.g. [16, Example 2.1.4])

$$\mu(\Omega) := \max_{\{w_n\}_n \subset \Omega} \big(\sup_{t \in [a,b]} e^{-Lt} \gamma(\{w_n\}_n), \operatorname{mod}_C \{w_n\}_n \big),$$

where the ordering is induced by the positive cone in $\mathbb{R}^2$. Consider $\Theta \subseteq Q$ and take $\{x_n\}_n \subseteq T(\Theta \times [0, 1])$. There exist $\{q_n\}_n \subseteq \Theta$, $\{\lambda_n\}_n \subset [0, 1]$ and $f_n \in P(\cdot, q_n(\cdot))$ such that

$$x_n(t) = x_0 + \lambda_n \int_a^t f_n(s)\, ds, \quad t \in [a, b].$$

Since K is bounded, according to (F1), (F2), $\{x_n\}_n$ is equicontinuous. Thus, $\mathrm{mod}_C T(\Theta \times [0,1]) = 0$. Moreover, defining for $\Omega \subset C([a,b],E)$ bounded

$$\nu(\Omega) := \max_{\{w_n\}_n \subset \Omega} \sup_{t\in[a,b]} e^{-Lt}\gamma(\{w_n(t)\}_n),$$

it follows that

$$e^{-Lt}\gamma(\{x_n(t)\}_n) \le e^{-Lt}\gamma\left(\left\{\int_a^t f_n(s)\,ds\right\}_n\right)$$

$$\le e^{-Lt}\int_a^t e^{Ls}g(s)e^{-Ls}\gamma(\{q_n(s)\}_n)ds \le \nu(\Theta)\int_a^t e^{-L(t-s)}g(s)\,ds \le l\nu(\Theta),$$

which implies $\nu(T(\Theta \times [0,1])) \le l\nu(\Theta)$. Hence,

$$\mu(T(\Theta \times [0,1])) \le l\mu(\Theta),$$

for all $\Theta \subseteq Q$, and since $l < 1$ and μ is regular, this yields both the quasi-compactness and the condensity of T.

Therefore, problem (4.2) has a solution x satisfying $x(t) \in \overline{K}$, for all $t \in [a,b]$. According to Proposition 4.4 and Corollary 4.5, we then obtain $x(t) \in K$, for all $t \in [a,b]$. □

Remark 6.3. Let us note that similar results can be stated for the terminal value problem (4.3). Specifically, when in Lemma 6.1 (B2) is replaced by (B2′) and every other condition remains unchanged, then all solutions x of the terminal value problem (4.3) such that $x(b) \in K$, satisfy $x(t) \in K$, for all $t \in [a,b]$.

Similarly, in the u.s.c. case, the same conclusion of Lemma 6.2 is true for (4.3), when (4.10) is replaced by (4.11).

Lemmas 6.1 and 6.2 deal with the well known viability problem for which we refer to [12] (see also [17] and [27], for some results in different contexts, and the references therein contained). If the r.h.s. is globally u.s.c. such a problem is usually formulated in terms of the Bouligand cone

$$T_D(x) := \left\{ y \in E : \liminf_{h\to 0^+} \frac{\mathrm{dist}(x+hy, D)}{h} = 0 \right\},$$

where D is a closed nonempty set in E. More precisely, the following result holds:

Theorem 6.4 ([12, Theorem 9.1 and Remark 9.5.4])**.** *Let D be a closed, nonempty subset of E with the Radon–Nikodym property. Let, furthermore, $P : [a,b] \times D \multimap E$ be a u.s.c. map with nonempty, compact, convex values satisfying (F2) (with P instead of F),*

(6.5) $$|P(t,x)| \le c(t)(1+|x|), \quad \text{on } [a,b]\times D, \text{ for } c \in L^1([a,b],R),$$

and

(6.6) $$P(t,x) \cap T_D(x) \ne \emptyset, \quad \text{on } [a,b]\times D.$$

Then (4.2) *has a solution, for every* $x_0 \in D$.

Let $K \subset E$ be as in Lemmas 6.1 and 6.2. It is easy to see that $\operatorname{dist}(x, \overline{K})$ is a Lipschitzian bounding function. In [5], we showed that condition (4.5) reduces to

$$P(t,x) \cap T_{\overline{K}}(x) = \emptyset \quad \text{or} \quad (-P(t,x)) \cap T_{\overline{K}}(x) = \emptyset, \tag{6.7}$$
$$\text{for } (t,x) \in (a,b) \times \partial K.$$

Indeed, in [5], we proved (6.7), for an autonomous P, where $x \in \mathbb{R}^n$, but the same reasoning repeats in this context. Moreover, it is easy to see that (4.10) becomes

$$\Big(-P(b,x)\Big) \cap T_{\overline{K}}(x) = \emptyset, \quad \text{for each } x \in \partial K. \tag{6.8}$$

Consequently, by means of Lemma 6.2, we are able to find a solution of (4.2) in $\overline{K}$, for all initial conditions x_0 in the open set K, when the Bouligand cone has an empty intersection with $P(t,x)$, on the whole boundary ∂K. In particular, we obtain the following result which is an immediate consequence of Lemma 6.2 and Theorem 6.4.

Corollary 6.5. *Let* $K \subset E$ *be open, bounded and convex. Let* $P : [a,b] \times \overline{K} \multimap E$ *be a u.s.c. map with nonempty, compact and convex values satisfying (F2) (with* P *instead of* F*) and* (6.5). *Let one of the following conditions be satisfied:*

(i) (6.6) *with* $\overline{K}$ *instead of* D*, for all* $x \in \partial K$ *and* $t \in [a,b]$*;*

(ii) (6.7)*, for all* $x \in \partial K$ *and* $t \in (a,b)$*, and* (6.8)*, for all* $x \in \partial K$*.*

Then, for all $x_0 \in K$*, there exists a solution of* (4.2) *such that* $x(t) \in \overline{K}$*, for all* $t \in [a,b]$*.*

We conclude this part with the investigation of entirely bounded solutions of inclusions with u-Carathéodory as well as of u.s.c. r.h.s.

Corollary 6.6. *Consider the u-Carathéodory multivalued mapping* $P : \mathbb{R} \times E \multimap E$ *satisfying (F2) (with* P *instead of* F*), for all* $t \in \mathbb{R}$*, with* $g \in L^1_{loc}(\mathbb{R})$*. Suppose that there exist an open, bounded and convex subset* K *of* E*, a locally Lipschitzian function* $V : E \to \mathbb{R}, \epsilon > 0$ *and* $t_0 \in \mathbb{R}$ *such that (B1) holds and (B2) and (B2′) are satisfied respectively, for all* $t > t_0$ *and* $t < t_0$*. Then, for each* $x_0 \in K$*, there exists a solution* x *of* $x' \in P(t,x)$ *such that* $x(t_0) = x_0$ *and* $x(t) \in K$*, for all* $t \in \mathbb{R}$*.*

Proof. Take $x_0 \in K$ and, for all $n \in \mathbb{N}$, consider problem (4.2) with $[a,b] = [t_0, t_0+n]$. Since all the assumptions of Lemma 6.1 are satisfied, we get, for each $n \in \mathbb{N}$, the existence of a solution x_n of $x' \in P(t,x)$ such that $x_n(t) \in \overline{K}$, for all $t \in [t_0, t_0+n]$. For every $n \in \mathbb{N}$, let us denote by $\tilde{x}_n$ the continuous extension of x_n to $[t_0, \infty)$ which is constant outside $[t_0, t_0+n]$. Given $m \in \mathbb{N}$, consider the sequence $\{\tilde{x}_n\}_{n \geq m}$. According to Definition 2.1, $\{\tilde{x}'_n\}_{n \geq m}$ is bounded in $L^1([t_0, t_0+m])$, yielding the equicontinuity of $\{\tilde{x}_n\}_{n\geq m}$ in $[t_0, t_0+m]$. Since $\tilde{x}_n(t_0) = x_0$, for all n, [4, Proposition III.1.36] implies that $\tilde{x}_n \to x_m \in C([t_0, t_0+m], \overline{K})$. Reasoning as in the proofs of Theorems 3.1 and 5.2, it is possible to prove firstly that $\{\tilde{x}'_n(t)\}_{n\geq m}$ is relatively compact, for a.a. $t \in [t_0, t_0+m]$, and then that x_m is a solution of the inclusion in $[t_0, t_0+m]$. Since m is arbitrary, from the uniqueness of the limit, we get the existence of a solution $\tilde{x}$ of $x' \in P(t,x)$ a.e. in $[t_0, \infty)$ with $\tilde{x}(t) \in \overline{K}$, for all t.

According to Remark 6.3, the hypotheses also assure the existence of a solution of the terminal value problem (4.3). Therefore, we get the existence of a solution x_n of $x' \in P(t,x)$ in $[t_0 - n, t_0]$ such that $x_n(t) \in \overline{K}$, for all t. By the same arguments as above we finally obtain the existence of a solution $\hat{x}$ of the inclusion in $(-\infty, t_0]$ with $\hat{x}(t) \in \overline{K}$, for all t. Finally, if there is a point $t_1 \in \mathbb{R}$ such that $x(t_1) \in \partial K$, then we would get a contradiction with Corollary 4.3 which completes the proof. □

Remark 6.7. Assume that P is u.s.c. in (t,x), on all $\mathbb{R} \times E$, and satisfies the growth condition (iii) in Definition 2.1. Let all the other assumptions of previous corollary be satisfied, with the exception of (B2) and (B2′) respectively replaced by (4.10), for $x \in \partial K$, $w \in P(t,x)$ and $t > t_0$, and (4.11), for $x \in \partial K$, $w \in P(t,x)$ and $t < t_0$. Then again, for each $x_0 \in K$, $x' \in P(t,x)$ has an entirely bounded solution x satisfying $x(t) \in K$, for all $t \in \mathbb{R}$.

7. CONCLUDING REMARKS

For noncompact operators, the degree arguments are usually related to closed convex subsets in a Banach [2], [16] (or Fréchet [3], [4]) space. For nonconvex subsets, many difficulties occur (see e.g. [8]). Such difficulties mainly depend on the application of the normalization property. That is also why our parameter set Q of candidate solutions was always assumed to be closed and convex.

In the single-valued case, for convex bound sets K such that $Q = C([a,b], \overline{K})$, bounding functions $V : E \to \mathbb{R}$ can always be taken smooth or even linear, as pointed out in [15]. Nevertheless, for the sake of a more convenient construction, we decided to employ locally Lipschitzian functions like those with absolute values, in the finite-dimensional case. On the other hand, it has not much meaning to consider here less regular than Lipschitzian bounding functions as in our former papers [5], [6], where the parameter set Q was not necessarily convex.

In more general than Hilbert spaces, the verification of bound sets conditions (B1), (B2) can be a difficult task (cf. [24], where positively invariant sets were considered in a similar way). A nice simple example of a bounding function in a general Banach space was constructed, for a positively invariant set, in [26].

An alternative approach, in terms of upper and lower solutions, can also be used (cf. e.g. [10], [25]), but difficulties related to a infinite-dimensional case obviously remain. For initial value problems, a discussion of such difficulties can be found e.g. in [10].

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