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ON THE STABILITY OF HIGHER GENERALIZED RING DERIVATIONS

YONG-SOO JUNG AND KYOO-HONG PARK

Department of Mathematics, Sun Moon University
Asan, Chungnam 336-708, Korea
Department of Mathematics Education, Seowon University
Chungbuk, Cheongju 361-742, Korea

ABSTRACT. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability and the Bourgin-type superstability of a functional inequality corresponding to the following functional equation:

$$f_n(xy) = \sum_{i=0}^{n} f_{n-i}(x)g_i(y).$$

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, let $\mathcal{R}$ and $\mathcal{Q}$ be associative rings and we assume that $\mathcal{A}$ and $\mathcal{B}$ are algebras over the real or complex field $\mathbb{F}$. An additive map $g : \mathcal{R} \to \mathcal{Q}$ will be said to be a *generalized ring homomorphism* associated with h if there exists a ring homomorphism $h : \mathcal{R} \to \mathcal{Q}$ such that $g(xy) = g(x)h(y)$ for all $x, y \in \mathcal{R}$. In particular, if $g = h$, then h is a usual *ring homomorphism*. Let $\alpha, \beta : \mathcal{R} \to \mathcal{Q}$ be additive maps. An additive map $f : \mathcal{R} \to \mathcal{Q}$ is called a (α, β)-*generalized ring derivation* associated with d if there exists an additive map $d : \mathcal{R} \to \mathcal{Q}$ such that $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in \mathcal{R}$. If $\mathcal{R} = \mathcal{Q}$ and $\alpha = \beta$ is an identity map on $\mathcal{R}$, then f is a generalized ring derivation on $\mathcal{R}$ associated with d. Furthermore, if $f = d$, then f is just a ring derivation on $\mathcal{R}$.

Let $\mathbb{N}$ be the set of the natural numbers. For $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp. $H = \{h_0, h_1, \ldots, h_n, \ldots\}$) of additive operators from $\mathcal{R}$ into $\mathcal{Q}$ is called a *higher ring derivation* of rank m (resp. infinite rank) from $\mathcal{R}$ into $\mathcal{Q}$ if the functional equation

$$h_n(xy) = \sum_{i=0}^{n} h_{n-i}(x)h_i(y)$$

holds for each $n = 0, 1, \ldots, m$ (resp. $n = 0, 1, \ldots$) and for all $x, y \in \mathcal{R}$ (see [8] and [9]).

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Here we consider the following map:

By a *higher generalized ring derivation* of rank m (resp. infinite rank) from $\mathcal{R}$ into $\mathcal{Q}$ associated with a sequence G we mean a sequence $F = \{f_0, f_1, \ldots, f_m\}$ (resp. $F = \{f_0, f_1, \ldots, f_n, \ldots\}$) of additive maps from $\mathcal{R}$ into $\mathcal{Q}$ such that there exists a sequence $G = \{g_0, g_1, \ldots, g_m\}$ (resp. $G = \{g_0, g_1, \ldots, g_n, \ldots\}$) of additive maps from $\mathcal{R}$ into $\mathcal{Q}$ if the functional equation

$$f_n(xy) = \sum_{i=0}^{n} f_{n-i}(x)g_i(y)$$

holds for each $n = 0, 1, \ldots, m$ (resp. $n = 0, 1, \ldots$) and for all $x, y \in \mathcal{R}$. Of course, if $F = G$, then F is a usual higher ring derivation.

Suppose that F and G are sequences of additive maps on $\mathcal{R}$. Then we will say that the higher generalized ring derivation F associated with the sequence G is *strong* if $f_0 = g_0$ is an identity map on $\mathcal{R}$.

Let $F = \{f_0\}$ and $G = \{g_0\}$, where f_0 and g_0 are additive maps from $\mathcal{R}$ into $\mathcal{Q}$. (resp. $F = \{f_0\}$ and $G = \{g_0\}$, where f_0 and g_0 are additive maps on $\mathcal{R}$). Then the higher generalized ring derivation F of rank 0 associated with G (resp. the strongly higher generalized ring derivation F of rank 1) associated with G is a generalized ring homomorphism from $\mathcal{R}$ into $\mathcal{Q}$ (resp. a generalized ring derivation on $\mathcal{R}$). Note that a higher generalized ring derivation is a generalization of both a generalized ring homomorphism and a generalized ring derivation.

Remark 1.1. Let $\mathcal{R}$ and $\mathcal{Q}$ be rings, $\alpha : \mathcal{R} \to \mathcal{Q}$ a ring homomorphism and $a \in \mathcal{Q}$. Defining a map $\beta : \mathcal{R} \to \mathcal{Q}$ by $\beta(x) = a\alpha(x)$ for all $x \in \mathcal{R}$, we see that β is a generalized ring homomorphism associated with α. Let $b \in \mathcal{Q}$ and $d : \mathcal{R} \to \mathcal{Q}$ an additive map defined by $d(x) = b\alpha(x) - \alpha(x)b$ for all $x \in \mathcal{R}$. If we define a map $f : \mathcal{R} \to \mathcal{Q}$ by $f(x) = b\alpha(x) - \beta(x)b$ for all $x \in \mathcal{R}$, then it follows that f is a generalized (α, β)-ring derivation associated with d.

Set $f_0 = \beta$, $g_0 = \alpha$, $f_n = g_n = 0$ $(1 \le n \le m-1)$, $f_m = f$ and $g_m = d$. Now it is easy to see that $F = \{f_0, f_1, \ldots, f_m\}$ and $G = \{g_0, g_1, \ldots, g_m\}$ are sequences of additive maps from $\mathcal{R}$ into $\mathcal{Q}$ such that

$$f_n(xy) = \sum_{i=0}^{n} f_{n-i}(x)g_i(y)$$

for all $x, y \in \mathcal{R}$. That is, F is a higher generalized ring derivation of rank m from $\mathcal{R}$ into $\mathcal{Q}$ associated with G.

Here it is natural to ask that there exists an approximately higher generalized ring derivation which is not an exactly higher generalized ring derivation. The following remark is a slight modification of an example due to [11].

Remark 1.2. Let A be a compact Hausdorff space and $C(A)$ the commutative Banach algebra of complex-valued continuous functions on A under pointwise operations and the supremum norm $\|\cdot\|_\infty$. Let $\alpha : C(A) \to C(A)$ be an additive map which is continuous. Assume that $\beta : C(A) \to C(A)$ is a continuous generalized ring homomorphism associated with α. We define $f : C(A) \to C(A)$ (resp. $d : C(A) \to C(A)$) by

$$f(x)(a) = \begin{cases} \beta(x)(a)\log|\beta(x)(a)| & \text{if } \beta(x)(a) \neq 0, \\ 0 & \text{if } \beta(x)(a) = 0 \end{cases}$$

$$\left(\text{resp. } d(x)(a) = \begin{cases} \alpha(x)(a)\log|\alpha(x)(a)| & \text{if } \alpha(x)(a) \neq 0, \\ 0 & \text{if } \alpha(x)(a) = 0 \end{cases}\right)$$

for all $x \in C(A)$ and $a \in A$. It is easy to see that $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in C(A)$. Let $f_0 = \beta$, $g_0 = \alpha$, $f_n = g_n = 0$, $1 \le n \le m-1$, $f_m = f$ and $g_m = d$. Then we see that the sequences $F = \{f_0, f_1, \dots, f_m\}$ and $G = \{g_0, g_1, \dots, g_m\}$ satisfy the relation

$$f_n(xy) = \sum_{i=0}^{n} f_{n-i}(x)g_i(y)$$

for each $n = 0, 1, \dots, m$ and all $x, y \in C(A)$. From the same method as in [11, Example 1.1], it follows that for all $u, v \in \mathbb{C} \setminus \{0\}$ with $u + v \neq 0$, where $\mathbb{C}$ is a complex field,

$$\big|(u+v)\log|u+v| - u\log|u| - v\log|v|\big| \le |u| + |v|$$

which gives

$$\|g_n(x+y) - g_n(x) - g_n(y)\|_\infty \le \|\alpha\|(\|x\|_\infty + \|y\|_\infty)$$

and

$$\|f_n(x+y) - f_n(x) - f_n(y)\|_\infty \le \|\beta\|(\|x\|_\infty + \|y\|_\infty)$$

for each $n = 0, 1, \dots, m$ and all $x, y \in C(A)$. Hence we may regard F as an approximately higher generalized ring derivation of rank m on $C(A)$ associated with G.

In connection with Remark 2, it will be of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation. The study of stability problems originated from a famous talk given by S. M. Ulam [17] in 1940: "*Under what condition does there exists a homomorphism near an approximate homomorphism?*" In the next year 1941, D.H. Hyers [6] was answered affirmatively the question of Ulam for Banach spaces. A generalized version of the theorem of Hyers for approximately additive maps was given by Th. M. Rassias [13] in 1978 as follows: *if there exist a* $\theta \ge 0$ *and* $0 \le p < 1$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{X}$, then there exists a unique additive map $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in \mathcal{X}$. The result of Th. M. Rassias has provided a lot of influence in the development of what is known as Hyers-Ulam-Rassias stability of functional equations (see [7, 10, 14, 15, 16]). In 1992, a generalization of the Rassias theorem, the so-called generalized Hyers-Ulam-Rassias stability, was obtained by P. Găvruţă [4].

In 1949, D. G. Bourgin [3] proved the following result, which is sometimes called the superstability of ring homomorphisms: *suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with unit. If $f : \mathcal{A} \to \mathcal{B}$ is a surjective mapping such that*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon,$$
$$\|f(xy) - f(x)f(y)\| \le \delta$$

for some $\varepsilon \ge 0$, $\delta \ge 0$ and all $x, y \in \mathcal{A}$, then f is a ring homomorphism.

Recently, R. Badora [1] gave a generalization of the above Bourgin's result and he [2] also examined the Hyers-Ulam stability and the Bourgin-type superstability of ring derivations. On the ther hand, M.S. Moslehian [12] investigated the Hyers-Ulam-Rassias stability of generalized derivations.

The main purpose of the present paper is to investigate the stability problem of higher generalized ring derivations, i.e., the generalized Hyers-Ulam-Rassias stability for approximately higher generalized ring derivations. Moreover, we are going to show the Bourgin-type superstability [3] for approximately higher generalized ring derivations.

2. MAIN RESULTS

For each $n = 0, 1, 2, \dots$, we denote by $\varphi_n, \phi_n : \mathcal{R} \times \mathcal{R} \to [0, \infty)$ functions such that

$$\psi_n(x, y) := \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varphi_n(2^k x, 2^k y)}{2^k} < \infty,$$

$$\lim_{k\to\infty} \frac{\phi_n(2^k x, y)}{2^k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\phi_n(x, 2^k y)}{2^k} = 0 \tag{2.1}$$

for all $x, y \in \mathcal{R}$.

By following the similar way as in [1], we obtain the next theorem.

Theorem 2.1. *Let $\mathcal{R}$ be a ring and $\mathcal{B}$ a Banach algebra. Suppose that $G = \{g_0, g_1, \dots, g_n, \dots\}$ is a sequence of maps from $\mathcal{R}$ into $\mathcal{B}$ such that for each $n = 0, 1, \dots,$*

$$\|g_n(x + y) - g_n(x) - g_n(y)\| \le \varphi_n(x, y) \tag{2.2}$$

for all $x, y \in \mathcal{R}$. If $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ is a sequence of maps from $\mathcal{R}$ into $\mathcal{B}$ such that for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{R}$,

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \leq \varphi_n(x,y) \tag{2.3}$$

and

$$\left\| f_n(xy) - \sum_{i=0}^{n} f_{n-i}(x) g_i(y) \right\| \leq \phi_n(x,y). \tag{2.4}$$

Then there exists a unique higher generalized ring derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank associated with a sequence $D = \{d_0, d_1, \ldots, d_n, \ldots\}$ of additive maps such that for each $n = 0, 1, \ldots$ and all $x \in \mathcal{R}$,

$$\|g_n(x) - d_n(x)\| \leq \psi_n(x,x) \tag{2.5}$$

and

$$\|f_n(x) - h_n(x)\| \leq \psi_n(x,x). \tag{2.6}$$

Moreover,

$$\sum_{i=0}^{n} \{f_{n-i}(y) - h_{n-i}(y)\} d_i(x) = 0 \quad \text{and} \quad \sum_{i=0}^{n} h_{n-i}(x)\{g_i(y) - d_i(y)\} = 0 \tag{2.7}$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{R}$.

Proof. First, from Găvruţă's theorem [4] with (2.2) and (2.3), we see that for each $n = 0, 1, \ldots$, there exist unique additive maps $d_n, h_n : \mathcal{R} \to \mathcal{B}$ satisfying (2.5) and (2.6). The additive maps are given by

$$d_n(x) = \lim_{k \to \infty} \frac{1}{2^k} g_n(2^k x) \tag{2.8}$$

and

$$h_n(x) = \lim_{k \to \infty} \frac{1}{2^k} f_n(2^k x) \tag{2.9}$$

for all $x \in \mathcal{R}$. Next, we need to show that the sequence $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ is a higher generalized ring derivation of any rank from $\mathcal{R}$ into $\mathcal{B}$ associated with the sequence $D = \{d_0, d_1, \ldots, d_n, \ldots\}$, i.e., the identity

$$h_n(xy) = \sum_{i=0}^{n} h_{n-i}(x) d_i(y)$$

holds for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{R}$. Let a map $\Delta_n : \mathcal{R} \times \mathcal{R} \to \mathcal{B}$ be defined by

$$\Delta_n(x,y) = f_n(xy) - \sum_{i=0}^{n} f_{n-i}(x) g_i(y) \tag{2.10}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. From (2.1) and (2.4), it follows that the map Δ_n satisfies the relation

$$\lim_{k\to\infty} \frac{1}{2^k}\Delta_n(x, 2^k y) = 0 \tag{2.11}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. Now, using (2.8), (2.10) and (2.11), we have

$$\begin{aligned} h_n(xy) &= \lim_{k\to\infty} \frac{1}{2^k} f_n(2^k(xy)) = \lim_{k\to\infty} \frac{1}{2^k} f_n(x(2^k y)) \\ &= \lim_{k\to\infty} \frac{1}{2^k}\left\{\sum_{i=0}^{n} f_{n-i}(x) g_i(2^k y) + \Delta_n(x, 2^k y)\right\} \\ &= \lim_{k\to\infty} \sum_{i=0}^{n} f_{n-i}(x)\frac{1}{2^k} g_i(2^k y) + \lim_{k\to\infty} \frac{1}{2^k}\Delta_n(x, 2^k y) \\ &= \sum_{i=0}^{n}\left\{\lim_{k\to\infty} f_{n-i}(x)\frac{1}{2^k} g_i(2^k y)\right\} = \sum_{i=0}^{n} f_{n-i}(x) d_i(y) \end{aligned}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. That is, we obtain that

$$h_n(xy) = \sum_{i=0}^{n} f_{n-i}(x) d_i(y) \tag{2.12}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. Let $k \in \mathbb{N}$ be fixed. Applying (2.12) and the additivity of each h_n, $n = 0, 1, \dots$, we get

$$\begin{aligned} \sum_{i=0}^{n} f_{n-i}(2^k x) d_i(y) &= h_n((2^k x)y) = h_n(x(2^k y)) \\ &= \sum_{i=0}^{n} f_{n-i}(x) d_i(2^k y) = 2^k \sum_{i=0}^{n} f_{n-i}(x) d_i(y) \end{aligned}$$

Hence we get

$$\sum_{i=0}^{n} f_{n-i}(x) d_i(y) = \sum_{i=0}^{n} \frac{1}{2^k} f_{n-i}(2^k x) d_i(y) \tag{2.13}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. Taking $k \to \infty$ in (2.13), it follows that

$$\sum_{i=0}^{n} f_{n-i}(x) d_i(y) = \sum_{i=0}^{n} h_{n-i}(x) d_i(y) \tag{2.14}$$

holds for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{A}$. Combining (2.12) with (2.14), we now see that the relation

$$h_n(xy) = \sum_{i=0}^{n} h_{n-i}(x) d_i(y). \tag{2.15}$$

is valid for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. That is, H is a higher generalized ring derivation from $\mathcal{R}$ into $\mathcal{B}$ associated with D.

Moreover, by (2.1) and (2.4), we have

$$\lim_{k\to\infty}\frac{1}{2^k}\Delta_n(2^k x, y) = 0 \tag{2.16}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. Hence, from (2.9), (2.10) and (2.16), we deduce that

$$\begin{aligned} h_n(xy) &= \lim_{k\to\infty}\frac{1}{2^k}f_n(2^k(xy)) = \lim_{k\to\infty}\frac{1}{2^k}f_n((2^k x)y) \\ &= \lim_{k\to\infty}\frac{1}{2^k}\Big\{\sum_{i=0}^{n} f_{n-i}(2^k x)g_i(y) + \Delta_n(2^k x, y)\Big\} \\ &= \lim_{k\to\infty}\sum_{i=0}^{n}\frac{1}{2^k}f_{n-i}(2^k x)g_i(y) + \lim_{k\to\infty}\frac{1}{2^k}\Delta_n(2^k x, y) \\ &= \sum_{i=0}^{n}\Big\{\lim_{k\to\infty}\frac{1}{2^k}f_{n-i}(2^k x)g_i(y)\Big\} = \sum_{i=0}^{n} h_{n-i}(x)g_i(y) \end{aligned}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. Namely, we get

$$h_n(xy) = \sum_{i=0}^{n} h_{n-i}(x)g_i(y) \tag{2.17}$$

for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{R}$. With the aid of (2.12), (2.15) and (2.17), we can obtain (2.7) which completes the proof of the theorem. □

Remark 2.2. Let $\mathcal{A}$ be an algebra and let $\varphi_n, \phi_n : \mathcal{A}\times\mathcal{A} \to [0,\infty)$ be functions such that

$$\psi_n(x, y) := 2\sum_{k=0}^{\infty} 2^k\varphi_n(2^{-k}x, 2^{-k}y) < \infty$$

and

$$\lim_{k\to\infty} 2^k\phi_n(2^{-k}x, y) = 0$$

for all $x, y \in \mathcal{A}$. In the proof of Theorem 2.1, we assume that $\mathcal{R} = \mathcal{A}$. If we replace (2.8) and (2.9) by

$$d_n(x) = \lim_{k\to\infty} 2^k g_n\Big(\frac{1}{2^k}x\Big), \quad h_n(x) = \lim_{k\to\infty} 2^k f_n\Big(\frac{1}{2^k}x\Big),$$

respectively, and replace (2.11) and (2.16) by

$$\lim_{k\to\infty} 2^k\Delta_n\Big(x, \frac{1}{2^k}y\Big) = 0, \quad \lim_{k\to\infty} 2^k\Delta_n\Big(\frac{1}{2^k}x, y\Big) = 0,$$

respectively, then Theorem 2.1 is still true.

Corollary 2.3. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{B}$ a Banach algebra. Let $\theta_n, \vartheta_n \in (0,\infty)$ for each $n = 0, 1, \dots$ and p, q, r real numbers such that either $p < 1, q < 1$ or $p > 1, q > 1$. Suppose that $G = \{g_0, g_1, \dots, g_n, \dots\}$ is a sequence of maps from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \dots,$*

$$\|g_n(x+y) - g_n(x) - g_n(y)\| \le \theta_n(\|x\|^p + \|y\|^p) \tag{2.18}$$

for all $x, y \in \mathcal{A}$. *If* $F = \{f_0, f_1, \dots, f_n, \dots\}$ *is a sequence of maps from* $\mathcal{A}$ *into* $\mathcal{B}$ *such that for each* $n = 0, 1, \dots,$

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le \theta_n(\|x\|^p + \|y\|^p),$$

and

$$\left\| f_n(xy) - \sum_{i=0}^{n} f_{n-i}(x) g_i(y) \right\| \le \vartheta_n(\|x\|^q \|y\|^r)$$

for all $x, y \in \mathcal{A}$, *then there exists a unique higher generalized ring derivation* $H = \{h_0, h_1, \dots, h_n, \dots\}$ *of any rank from* $\mathcal{A}$ *into* $\mathcal{B}$ *associated with a sequence* $D = \{d_0, d_1, \dots, d_n, \dots\}$ *of additive maps from* $\mathcal{A}$ *into* $\mathcal{B}$ *such that for each* $n = 0, 1, \dots$ *and all* $x \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \le \frac{2\theta_n}{2 - 2^p}\|x\|^p$$

and

$$\|g_n(x) - d_n(x)\| \le \frac{2\theta_n}{2 - 2^p}\|x\|^p.$$

Moreover,

$$\sum_{i=0}^{n}\{f_{n-i}(y) - h_{n-i}(y)\}d_i(x) = 0 \quad \text{and} \quad \sum_{i=0}^{n} h_{n-i}(x)\{g_i(y) - d_i(y)\} = 0$$

for each $n = 0, 1, \dots$ *and all* $x, y \in \mathcal{A}$.

Proof. Letting $\varphi_n(x, y) = \theta_n(\|x\|^p + \|y\|^p)$ and $\phi_n(x, y) = \vartheta_n(\|x\|^q\|y\|^r)$ for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{A}$, and applying Theorem 2.1, we obtain the conclusion. □

By setting $\varphi_n(x, y) = \varepsilon_n$ and $\phi_n(x, y) = \delta_n$ for each $n = 0, 1, \dots$ and all $x, y \in \mathcal{A}$, Theorem 2.1 also gives us the following corollary.

Corollary 2.4. *Let* $\mathcal{A}$ *be a normed algebra and* $\mathcal{B}$ *a Banach algebra. Suppose that* $G = \{g_0, g_1, \dots, g_n, \dots\}$ *is a sequence of maps from* $\mathcal{A}$ *into* $\mathcal{B}$ *such that for each* $n = 0, 1, \dots,$ *there exist* $\varepsilon_n \ge 0$ *such that*

$$\|g_n(x+y) - g_n(x) - g_n(y)\| \le \varepsilon_n \tag{2.19}$$

for all $x, y \in \mathcal{A}$. *If* $F = \{f_0, f_1, \dots, f_n, \dots\}$ *is a sequence of maps from* $\mathcal{A}$ *into* $\mathcal{B}$ *such that for each* $n = 0, 1, \dots,$ *there exist* $\varepsilon_n \ge 0$ *and* $\delta_n \ge 0$ *such that*

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le \varepsilon_n, \tag{2.20}$$

and

$$\left\| f_n(xy) - \sum_{i=0}^{n} f_{n-i}(x) g_i(y) \right\| \le \delta_n \tag{2.21}$$

for all $x, y \in \mathcal{A}$, *then there exists a unique higher generalized ring derivation* $H = \{h_0, h_1, \dots, h_n, \dots\}$ *of any rank from* $\mathcal{A}$ *into* $\mathcal{B}$ *associated with a sequence* $D =$

$\{d_0, d_1, \ldots, d_n, \ldots\}$ of additive maps from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \ldots$ and all $x \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \le \varepsilon_n$$

and

$$\|g_n(x) - d_n(x)\| \le \varepsilon_n.$$

Moreover,

$$\sum_{i=0}^{n} \{f_{n-i}(y) - h_{n-i}(y)\} d_i(x) = 0 \quad \text{and} \quad \sum_{i=0}^{n} h_{n-i}(x)\{g_i(y) - d_i(y)\} = 0 \tag{2.22}$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$.

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with unit. Suppose that g is a map from $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$\|g(x+y) - g(x) - g(y)\| \le \varepsilon,$$
$$\|g(xy) - g(x)g(y)\| \le \delta.$$

for some $\delta \ge 0$, $\varepsilon \ge 0$ and all $x, y \in \mathcal{A}$. If f is a map from $\mathcal{A}$ into $\mathcal{B}$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon,$$

and

$$\|f(xy) - f(x)g(y)\| \le \delta$$

for some $\delta \ge 0$, $\varepsilon \ge 0$ and all $x, y \in \mathcal{A}$, then g is a ring homomorphism and f is a generalized ring homomorphism associated with g.

Proof. Let e be the unit in $\mathcal{A}$. In order to apply Corollary 2.4 to the case $n = 0$, set $f = f_0$, $g = g_0$, $h = h_0$ and $d = d_0$. From Bourgin's result [3], (2.8) and Corollary 2.4, we see that g is a ring homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which gives

$$d(x) = \lim_{k \to \infty} \frac{1}{2^k} g(2^k x) = g(x)$$

for all $x \in \mathcal{A}$ and that h is a generalized ring homomorphism associated with d. From the case $n = 0$ of (2.22), we get

$$\{f(y) - h(y)\} d(x) = 0 \tag{2.23}$$

for all $x, y \in \mathcal{A}$. Since $d(e) = g(e)$ is the unit in $\mathcal{B}$, the relation (2.23) implies that $f(y) = h(y)$ for all $y \in \mathcal{A}$. Therefore it follows that f is a generalized ring homomorphism associated with g. □

Using Theorem 2.1, we get the following Bourgin-type superstability results.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra with unit e and $\mathcal{B}$ a Banach algebra with unit e^*. Suppose that $G = \{g_0, g_1, \dots, g_n, \dots\}$ is a sequence of maps from $\mathcal{A}$ into $\mathcal{B}$ satisfying* (2.19)*, where g_0 is onto such that*

$$\|g_0(xy) - g_0(x)g_0(y)\| \leq \delta_0$$

for all $x, y \in \mathcal{A}$. If $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a sequence of maps from $\mathcal{A}$ into $\mathcal{B}$ satisfying (2.20) *and* (2.21)*, where $f_0(e) = e^*$, then $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a higher generalized ring derivation of any rank from $\mathcal{A}$ into $\mathcal{B}$ associated with G.*

Proof. By induction, we lead the conclusion. Utilizing Lemma 2.5, we deduce from (2.8) and (2.9) that g_0 is a ring homomorphism which yields

$$d_0(x) = \lim_{k\to\infty} \frac{1}{2^k} g_0(2^k x) = g_0(x)$$

for all $x \in \mathcal{A}$ and that f_0 is a generalized ring homomorphism associated with g_0 and so we have

$$h_0(x) = \lim_{k\to\infty} \frac{1}{2^k} f_0(2^k x) = f_0(x)$$

for all $x \in \mathcal{A}$. If $n = 1$, then it follows from the left of (2.22) that $f_1(y) = h_1(y)$ holds for all $y \in \mathcal{A}$ since $d_0(e) = g_0(e)$ is the unit in $\mathcal{B}$. Let us assume that $f_m(y) = h_m(y)$ is valid for all $y \in \mathcal{A}$ and all $m < n$. Then (2.22) implies that $\{f_n(y) - h_n(y)\}d_0(x) = 0$ for all $x, y \in \mathcal{A}$. Since $d_0(e) = g_0(e)$ is the unit in $\mathcal{B}$, we have $f_n(y) = h_n(y)$ for all $y \in \mathcal{A}$. Hence we conclude that $f_n(x) = h_n(x)$ holds for all $n = 0, 1, \dots$ and all $x \in \mathcal{A}$. On the other hand, from the right of (2.22) and the similar argument as above, we arrive at $g_n(x) = d_n(x)$ for all $n = 0, 1, \dots$ and all $x \in \mathcal{A}$ since $h_0(e) = f_0(e) = e^*$. Now, Corollary 2.4 tells us that $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a higher generalized ring derivation of any rank from $\mathcal{A}$ into $\mathcal{B}$ associated with G. The proof of the theorem is complete. □

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra with unit. Suppose that $G = \{g_0, g_1, \dots, g_n, \dots\}$ is a sequence of maps on $\mathcal{A}$ satisfying* (2.19)*, where g_0 is an identity map on $\mathcal{A}$. If $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a sequence of maps on $\mathcal{A}$ satisfying* (2.20) *and* (2.21)*, where f_0 is an identity map on $\mathcal{A}$, then $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a strongly higher generalized ring derivation of any rank on $\mathcal{A}$ associated with G.*

Proof. For all $x \in \mathcal{A}$, we have, by (2.8) and (2.9),

$$d_0(x) = \lim_{k\to\infty} \frac{1}{k} g_0(kx) = x \quad \text{and} \quad h_0(x) = \lim_{k\to\infty} \frac{1}{k} f_0(kx) = x,$$

i.e., $d_0(= g_0) = h_0(= f_0)$ is an identity operator on $\mathcal{A}$. Following the same method as in the proof of Theorem 2.6 using the induction and the relation (2.22), we get

$$\{f_n(y) - h_n(y)\}x = 0 \quad \text{and} \quad x\{g_n(y) - d_n(y)\} = 0$$

for all $n \in \mathbb{N}$ and all $x, y \in \mathcal{A}$. Since $\mathcal{A}$ contains the unit, it follows that $f_n(y) = h_n(y)$ for all $n \in \mathbb{N}$ and all $y \in \mathcal{A}$ and that $g_n(y) = d_n(y)$ for all $n \in \mathbb{N}$ and all $y \in \mathcal{A}$. So, by Corollary 2.4, we see that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ is a strongly higher generalized ring derivation of any rank on $\mathcal{A}$ associated with G. This completes the proof. □

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