Dynamic Systems and Applications 18 (2009) 375-392

OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR DAMPED DIFFERENTIAL EQUATIONS

YIN-LIAN FU AND QI-RU WANG

Department of Applied Mathematics, School of Sciences
South China Agricultural University, Guangzhou 510640, PR China
Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, PR China mcswqr@mail.sysu.edu.cn

ABSTRACT. By employing a class of kernel functions $\Phi(t, s, l)$ and a generalized Riccati technique, some new oscillation criteria are established for second-order nonlinear damped differential equations, which extend, improve and unify some related results known in the literature.

AMS (MOS) Subject Classification. 34C10

1. INTRODUCTION

Consider the second-order nonlinear damped differential equation

$$(1.1)\qquad (r(t)\psi(x(t))\varphi(x'(t)))' + p(t)\varphi(x'(t)) + q(t)f(x(t)) = 0,\ t \geq t_0,$$

where $r(t), p(t), q(t) \in C([t_0, \infty), \mathbb{R})$, and $\psi(x), \varphi(x), f(x) \in C(\mathbb{R}, \mathbb{R})$.

Throughout this paper we shall assume the following conditions hold.

(C1) $r(t) > 0$ and $xf(x) > 0$ for all $x \neq 0$;

(C2) $0 < c_1 \leq \psi(x) \leq c_2$ for all $x \in \mathbb{R}$;

(C3) $k > 0$ and $k\varphi^2(y) \leq y\varphi(y)$ for all $y \in \mathbb{R}$;

(C4) $f'(x)$ exists, $f'(x) \geq \mu > 0$ for $x \neq 0$;

or

(C4′) $q(t) \geq 0, \frac{f(x)}{x} \geq \lambda > 0$ for $x \neq 0$, where c_1, c_2, k, μ and λ are constants.

We say that a function $x : [t_0, t_1) \to \mathbb{R},\ t_1 > t_0$ is a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In the sequel, we always assume that solutions of Eq. (1.1) exist on some half-line $[T, \infty)(T \geq t_0)$. A solution $x(t)$ of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

This project is supported by the NNSF of China (Nos. 10571183, 10626032) and the NSF of Guangdong Province of China (No. 8151027501000053).

Received March 16, 2008 1056-2176 $15.00 ©Dynamic Publishers, Inc.

The oscillation problem for Eq. (1.2) and its various particular cases such as the nonlinear damped differential equation

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0 \tag{1.2}$$

has been studied extensively in recent years, e.g., see [1, 3–8, 10, 11] and the references therein. In 2004, by using a new kernel function of the form $\Phi(t,s,l)$ and a Riccati transformation

$$w(t) = \frac{r(t)x'(t)}{f(x(t))},$$

Sun [8] studied the oscillatory behavior of Eq. (1.2) and obtained the following results for the equation with $r(t) \equiv 1$.

Theorem A ([8, Theorem 2.3])**.** Eq. (1.2) with $r(t) \equiv 1$ is oscillatory provided that for each $l \geq t_0$, there exists a constant $\alpha > 1/2$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_l^t (t-s)^{2\alpha}(s-l)^2 \left[4\mu q(s) - p^2(s) + 4\frac{t-(1+\alpha)s+\alpha l}{(t-s)(s-l)}p(s)\right] ds > \frac{4\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Theorem B ([8, Theorem 2.4])**.** Eq. (1.2) with $r(t) \equiv 1$ is oscillatory provided that for each $l \geq t_0$, there exists a constant $\beta > 1/2$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{2\beta+1}} \int_l^t (t-s)^{2}(s-l)^{2\beta} \left[4\mu q(s) - p^2(s) + 4\frac{\beta t-(1+\beta)s+l}{(t-s)(s-l)}p(s)\right] ds > \frac{4\beta}{(2\beta-1)(2\beta+1)}.$$

However, in Theorems A and B of Sun [8], the author required that $r(t) \equiv 1$, $\psi(x(t)) \equiv 1$ and $\varphi(x) = x$ in Eq. (1.1), which restrict their applications.

In 2000, Ayanlar and Tiryaki [1] used the following generalized Riccati type substitution

$$w(t) = A(t)\left[\frac{r(t)\psi(x(t))\varphi(x'(t))}{x(t)} + r(t)B(t) + \frac{1}{2k}p(t)\right] \tag{1.3}$$

and obtained several oscillation theorems for Eq. (1.1) which required $q(t) \geq 0$ and $p(t)$ to be differentiable.

In 2004, by using the following generalized Riccati transformations

$$v(t) = A(t)r(t)\left[\frac{\psi(x(t))\varphi(x'(t))}{f(x(t))} + B(t)\right]$$

and

$$w(t) = A(t)r(t)\left[\frac{\psi(x(t))\varphi(x'(t))}{x(t)} + B(t)\right],$$

Wang [10] obtained more general results for Eq. (1.1) without the term $\frac{1}{2k}p(t)$ as in (1.3) and any restriction on the sign and differentiability of $p(t)$.

The results of Ayanlar and Tiryaki [1] and Wang [10] involve a class of functions $H(t,s)$ defined by Philos [6] which is used extensively and are given in the form that $\limsup_{t\to\infty}[\cdot] = +\infty$.

Recently, Fu [3] employed $\Phi(t,s,l)$ functions and Riccati transformations

$$w(t) = \frac{r(t)\psi(x(t))\varphi(x'(t))}{f(x(t))} + \frac{p(t)}{2\mu k}$$

and

$$w(t) = \frac{r(t)\psi(x(t))\varphi(x'(t))}{x(t)} + \frac{p(t)}{2k}$$

to extend the main results of Sun [8] to Eq. (1.1) and obtained the following results which required $p(t)$ to be differentiable.

Theorem C ([3, Theorem 2.4])**.** Assume that conditions (C1)–(C4) hold and $\lim_{t\to\infty} R(t) = \infty$, where $R(t) = \int_l^t ds/r(s)$ for $t \ge l \ge t_0$. If for every $l \ge t_0$, there exists a constant $\alpha > 1/2$ such that

$$\limsup_{t\to\infty} \frac{1}{R^{2\alpha+1}(t)} \int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 \frac{\mu k}{c_2} Q(s)\, ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

where

$$Q(t) = q(t) - \frac{p^2(t)}{4\mu k c_1 r(t)} - \frac{p'(t)}{2\mu k}, \tag{1.4}$$

then Eq. (1.1) is oscillatory.

Theorem D ([3, Theorem 2.5])**.** Assume that conditions (C1)–(C4) hold and $\lim_{t\to\infty} R(t) = \infty$, where $R(t) = \int_l^t ds/r(s)$ for $t \ge l \ge t_0$. If for every $l \ge t_0$, there exists a constant $\beta > 1/2$ such that

$$\limsup_{t\to\infty} \frac{1}{R^{2\beta+1}(t)} \int_l^t [R(t)-R(s)]^{2}[R(s)-R(l)]^{2\beta} \frac{\mu k}{c_2} Q(s)\, ds > \frac{\beta}{(2\beta-1)(2\beta+1)},$$

where $Q(t)$ is defined by (1.4), then Eq. (1.1) is oscillatory.

Motivated by the ideas of Wang [10], Sun [8], Sun and Meng [9], Dubé and Mingarelli [2], in the present paper, we shall establish several new oscillation criteria for Eq. (1.1) by introducing functions of the form $\Phi(t,s,l)$ and employing two more generalized Riccati transformations due to Wang [10]. The criteria extend, improve and unify the results of Sun [8] and Fu [3]. Our results are different from most known ones in the sense that they are given in the form that $\limsup_{t\to\infty}[\cdot]$ is greater than a constant, rather than in the form $\limsup_{t\to\infty}[\cdot] = +\infty$. Thus, our results can be applied to many cases, which are not covered by existing ones. Finally, several interesting examples are also included to show the applications of our results.

2. KAMENEV-TYPE OSCILLATION CRITERIA

Following Sun [8] and Sun and Meng [9], we shall define a class of functions Y. We say that a function $\Phi = \Phi(t,s,l)$ belongs to the function class Y, denoted by $\Phi \in Y$, if $\Phi \in C(E,\mathbb{R})$, where $E = \{(t,s,l) : t \geq s \geq l \geq t_0\}$, which satisfies $\Phi(t,t,l) = \Phi(t,l,l) = 0$ for $t \geq l \geq t_0$ and has the partial derivative $\Phi_s = \frac{\partial \Phi}{\partial s}$ on E such that $\Phi_s \in L^2_{loc}(E,\mathbb{R})$.

Now, we are in a position to give our first result.

Theorem 2.1. *Suppose that conditions* (C1)–(C4) *hold. If for each* $l \geq t_0$, *there exist functions* $\Phi \in Y, A \in C^1([t_0,\infty),\mathbb{R}^+)$ *and* $B \in C([t_0,\infty),\mathbb{R})$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \int_l^t \left[\Phi^2(t,s,l)Q_1(s) - \frac{c_2 A(s) r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(t,s,l) - \Phi_s(t,s,l)\right)^2\right] ds > 0, \tag{2.1}$$

where $\mathbb{R}^+ = (0,\infty)$,

$$Q_1(t) = A(t)\Bigg[q(t) - \frac{1}{4\mu k}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{p^2(t)}{r(t)} - \frac{1}{c_2}p(t)B(t) + \frac{\mu k}{c_2} r(t) B^2(t) - (r(t)B(t))'\Bigg] \tag{2.2}$$

and

$$G_1(t) = -\frac{A'(t)}{A(t)} - \frac{2\mu k}{c_2}B(t) + \frac{p(t)}{c_2 r(t)}, \tag{2.3}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $x(t) \neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0 \geq t_0$. Define

$$v(s) = A(s)r(s)\left[\frac{\psi(x(s))\varphi(x'(s))}{f(x(s))} + B(s)\right], \quad \text{for } s \geq T_0. \tag{2.4}$$

Then differentiating (2.4) and using (1.1) and (C1)-(C4), it follows that for $s \geq T_0$

$$\begin{aligned} v'(s) &= \frac{A'(s)}{A(s)}v(s) - A(s)p(s)\frac{\varphi(x'(s))}{f(x(s))} - A(s)q(s) \\ &\quad - \frac{A(s)r(s)\psi(x(s))\varphi(x'(s))x'(s)f'(x(s))}{f^2(x(s))} + A(s)(r(s)B(s))' \\ &\leq \frac{A'(s)}{A(s)}v(s) - A(s)q(s) + A(s)(r(s)B(s))' \\ &\quad - \frac{\mu k A(s) r(s)}{\psi(x(s))}\left[\left(\frac{\psi(x(s))\varphi(x'(s))}{f(x(s))}\right)^2 + \frac{p(s)}{\mu k r(s)}\frac{\psi(x(s))\varphi(x'(s))}{f(x(s))}\right] \end{aligned} \tag{2.5}$$

$$\begin{aligned}
&= \frac{A'(s)}{A(s)}v(s) - A(s)q(s) + A(s)(r(s)B(s))' + \frac{A(s)p^2(s)}{4\mu k r(s)\psi(x(s))} \\
&\quad -\frac{\mu k A(s)r(s)}{\psi(x(s))}\left[\frac{\psi(x(s))\varphi(x'(s))}{f(x(s))} + \frac{p(s)}{2\mu k r(s)}\right]^2 \\
&\leq \frac{A'(s)}{A(s)}v(s) - A(s)q(s) + A(s)(r(s)B(s))' + \frac{A(s)p^2(s)}{4\mu k c_1 r(s)} \\
&\quad -\frac{\mu k A(s)r(s)}{c_2}\left[\frac{\psi(x(s))\varphi(x'(s))}{f(x(s))} + \frac{p(s)}{2\mu k r(s)}\right]^2 \\
&= \frac{A'(s)}{A(s)}v(s) - A(s)q(s) + A(s)(r(s)B(s))' + \frac{A(s)p^2(s)}{4\mu k c_1 r(s)} \\
&\quad -\frac{\mu k A(s)r(s)}{c_2}\left[\frac{v(s)}{A(s)r(s))} - B(s) + \frac{p(s)}{2\mu k r(s)}\right]^2 \\
&= -Q_1(s) - G_1(s)v(s) - \frac{\mu k}{c_2 A(s)r(s)}v^2(s),
\end{aligned}$$

where $Q_1(s)$ and $G_1(s)$ are defined by (2.2) and (2.3), respectively.

Multiplying (2.5) by $\Phi^2(t,s,T_0)(t \geq T_0)$, and integrating it with respect to s from T_0 to t, we have

$$\begin{aligned}
\int_{T_0}^t \Phi^2(t,s,T_0)Q_1(s)\,ds &\leq \int_{T_0}^t \Phi^2(t,s,T_0)[-v'(s) - G_1(s)v(s)]\,ds \\
&\quad - \int_{T_0}^t \Phi^2(t,s,T_0)\frac{\mu k}{c_2 A(s)r(s)}v^2(s)\,ds.
\end{aligned}$$

Integrating by parts, we obtain for $t \geq T_0$

$$\begin{aligned}
\int_{T_0}^t \Phi^2(t,s,T_0)Q_1(s)\,ds \leq \int_{T_0}^t \bigg[&- [G_1(s)\Phi(t,s,T_0) - 2\Phi_s(t,s,T_0)]\Phi(t,s,T_0)v(s) \\
&- \frac{\mu k}{c_2 A(s)r(s)}\Phi^2(t,s,T_0)v^2(s)\bigg]\,ds \\
= -\int_{T_0}^t \bigg[&\sqrt{\frac{\mu k}{c_2 A(s)r(s)}}\Phi(t,s,T_0)v(s) \\
&+ \sqrt{\frac{c_2 A(s)r(s)}{\mu k}}\left(\frac{G_1(s)}{2}\Phi(t,s,T_0) - \Phi_s(t,s,T_0)\right)\bigg]^2 ds \\
&+ \int_{T_0}^t \frac{c_2 A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(t,s,T_0) - \Phi_s(t,s,T_0)\right)^2 ds \\
\leq \int_{T_0}^t &\frac{c_2 A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(t,s,T_0) - \Phi_s(t,s,T_0)\right)^2 ds
\end{aligned}$$

that is,

(2.6)
$$\int_{T_0}^{t}\left[\Phi^2(t,s,T_0)Q_1(s)-\frac{c_2A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(t,s,T_0)-\Phi_s(t,s,T_0)\right)^2\right]ds\le 0.$$

Taking the superior limit in (2.6), we have

$$\limsup_{t\to\infty}\int_{T_0}^{t}\left[\Phi^2(t,s,T_0)Q_1(s)-\frac{c_2A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(t,s,T_0)-\Phi_s(t,s,T_0)\right)^2\right]ds\le 0,$$

which contradicts the assumption (2.1). The proof is complete. □

From Theorem 2.1, we can obtain different sufficient conditions for oscillation of Eq. (1.1) by different choices of $\Phi(t,s,l)$. For instance, let

$$\Phi(t,s,l)=\sqrt{\rho(s)(t-s)^{\alpha}(s-l)^{\beta}},$$

where $\rho(s)\in C^1([t_0,\infty),\mathbb{R}^+)$, and $\alpha,\beta>1$ are constants, then we have

$$\Phi_s(t,s,l)=\frac{\Phi(t,s,l)}{2}\left(\frac{\rho'(s)}{\rho(s)}+\frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)}\right).$$

Thus, by Theorem 2.1, we have the following oscillation result.

Theorem 2.2. *Suppose that conditions* (C1)–(C4) *hold. If for each* $l\ge t_0$, *there exist functions* $A,\rho\in C^1([t_0,\infty),\mathbb{R}^+)$, $B\in C([t_0,\infty),\mathbb{R})$ *and two constants* $\alpha,\beta>1$ *such that* $(rB)\in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty}\int_{l}^{t}\rho(s)(t-s)^{\alpha}(s-l)^{\beta}\times\left[Q_1(s)-\frac{c_2A(s)r(s)}{4\mu k}\left(G_1(s)-\frac{\rho'(s)}{\rho(s)}-\frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)}\right)^2\right]ds>0,$$

where $Q_1(s)$ *and* $G_1(s)$ *are defined by* (2.2) *and* (2.3), *respectively, then* Eq. (1.1) *is oscillatory.*

Define

$$R(t)=\int_{t_0}^{t}\frac{1}{r(s)}\,ds,\qquad t\ge t_0,$$

and let

$$\Phi(t,s,l)=\sqrt{\rho(s)\,[R(t)-R(s)]^{\alpha}\,[R(s)-R(l)]^{\beta}},$$

where $\rho(s)\in C^1([t_0,\infty),\mathbb{R}^+)$, and $\alpha,\beta>1$ are constants, then we have

$$\Phi_s(t,s,l)=\frac{\Phi(t,s,l)}{2}\left(\frac{\rho'(s)}{\rho(s)}+\frac{\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)}{r(s)[R(t)-R(s)][R(s)-R(l)t]}\right).$$

By Theorem 2.1, we get the following oscillation criterion.

Theorem 2.3. *Suppose that conditions* (C1)–(C4) *hold. If for each* $l \geq t_0$*, there exist functions* $A, \rho \in C^1([t_0,\infty),\mathbb{R}^+)$*,* $B \in C([t_0,\infty),\mathbb{R})$ *and two constants* $\alpha, \beta > 1$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \int_l^t \rho(s)[R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \Bigg[Q_1(s) - \frac{c_2 A(s) r(s)}{4\mu k}\left(G_1(s) - \frac{\rho'(s)}{\rho(s)} - \frac{\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)}{r(s)[R(t)-R(s)][R(s)-R(l)]}\right)^2 \Bigg] ds > 0, \tag{2.7}$$

where $Q_1(s)$ *and* $G_1(s)$ *are defined by* (2.2) *and* (2.3)*, respectively, then* Eq. (1.1) *is oscillatory.*

Taking $\rho(t) \equiv 1$ and $A(t) \equiv 1$ in Theorem 2.3, we have the following interesting theorem.

Theorem 2.4. *Suppose that conditions* (C1)–(C4) *hold and* $\lim_{t\to\infty} R(t) = \infty$*. If for each* $l \geq t_0$*, there exist a function* $B \in C([t_0,\infty),\mathbb{R})$ *and two constants* $\alpha, \beta > 1$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{R^{\alpha+\beta-1}(t)} \int_l^t [R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \\ &\qquad \times \left[\frac{\mu k}{c_2} Q_2(s) - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)]}{2[R(t)-R(s)][R(s)-R(l)]} \right] ds \\ &\qquad > \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{4\Gamma(\alpha+\beta)}, \end{aligned} \tag{2.8}$$

where

$$Q_2(t) = q(t) - \frac{1}{4\mu k}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{p^2(t)}{r(t)} - \frac{1}{c_2}p(t)B(t) + \frac{\mu k}{c_2} r(t) B^2(t) - (r(t)B(t))' \tag{2.9}$$

and

$$G_2(t) = -\frac{2\mu k}{c_2}B(t) + \frac{p(t)}{c_2 r(t)}, \tag{2.10}$$

then Eq. (1.1) *is oscillatory.*

Proof. By setting $u = R(s) - R(l)$ and $w = R(t) - R(l)$, we have

(2.11)

$$\begin{aligned} &\int_l^t [R(t)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}[\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)]^2 \frac{1}{r(s)}\, ds \\ &= \int_l^t [R(t)-R(s)]^{\alpha-2}\,[R(s)-R(l)]^{\beta-2}\,[\beta[R(t)-R(s)] - \alpha[R(s)-R(l)]]^2\, dR(s) \\ &= \int_0^{R(t)-R(l)} u^{\beta-2}\,[R(t)-R(l)-u]^{\alpha-2}\,[\beta[R(t)-R(l)-u] - \alpha u]^2\, du \end{aligned}$$

$$= \int_0^w u^{\beta-2}(w-u)^{\alpha-2}[\beta(w-u)-\alpha u]^2\, du$$
$$= \beta^2 \int_0^w u^{\beta-2}(w-u)^{\alpha} du - 2\alpha\beta \int_0^w u^{\beta-1}(w-u)^{\alpha-1} du$$
$$+ \alpha^2 \int_0^w u^{\beta}(w-u)^{\alpha-2} du.$$

Using the following Euler's Beta function,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad Re(m,n) > 0.$$

we obtain (upon setting $x = \frac{u}{w}$)

$$\int_0^w u^{\beta-2}(w-u)^{\alpha} du = w^{\alpha+\beta-1}\frac{\Gamma(\beta-1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)},$$
$$\int_0^w u^{\beta-1}(w-u)^{\alpha-1} du = w^{\alpha+\beta-1}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)},$$
$$\int_0^w u^{\beta}(w-u)^{\alpha-2} du = w^{\alpha+\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)}.$$

Thus,

$$\begin{aligned}(2.12)\qquad &\int_0^w u^{\beta-2}(w-u)^{\alpha-2}[\beta(w-u)-\alpha u]^2\, du\\
&= \beta^2 w^{\alpha+\beta-1}\frac{\Gamma(\beta-1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)} - 2\alpha\beta w^{\alpha+\beta-1}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\\
&\quad + \alpha^2 w^{\alpha+\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)}\\
&= w^{\alpha+\beta-1}\frac{\Gamma(\beta-1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)}\Big[\beta^2\alpha(\alpha-1)\\
&\quad - 2\alpha\beta(\beta-1)(\alpha-1) + \alpha^2\beta(\beta-1)\Big]\\
&= \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta)}w^{\alpha+\beta-1}.\end{aligned}$$

Substituting back in for $w = R(t) - R(l)$, (2.11) and (2.12) give

$$\int_l^t [R(t)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}[\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)]^2 \frac{1}{r(s)}\, ds$$
$$= \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta)}[R(t)-R(l)]^{\alpha+\beta-1}.$$

So we have that

$$\begin{aligned}(2.13)\qquad &\frac{\mu k}{c_2}\limsup_{t\to\infty}\frac{1}{R^{\alpha+\beta-1}(t)}\int_l^t [R(t)-R(s)]^{\alpha}[R(s)-R(l)]^{\beta}\\
&\times\left[Q_2(s) - \frac{c_2 r(s)}{4\mu k}\left(G_2(s) - \frac{\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)}{r(s)[R(t)-R(s)][R(s)-R(l)]}\right)^2\right] ds\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \frac{1}{R^{\alpha+\beta-1}(t)} \int_l^t [R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \\
&\quad \times \left[\frac{\mu k}{c_2} Q_2(s) - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)]}{2[R(t)-R(s)][R(s)-R(l)]}\right] ds \\
&\quad - \frac{1}{4} \limsup_{t\to\infty} \frac{1}{R^{\alpha+\beta-1}(t)} \int_l^t [R(t)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}[\beta R(t) \\
&\quad - (\alpha+\beta)R(s)+\alpha R(l)]^2 \frac{1}{r(s)} ds \\
&= \limsup_{t\to\infty} \frac{1}{R^{\alpha+\beta-1}(t)} \int_l^t [R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \left[\frac{\mu k}{c_2} Q_2(s) \right.\\
&\quad \left. - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)]}{2[R(t)-R(s)][R(s)-R(l)]}\right] ds \\
&\quad - \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{4\Gamma(\alpha+\beta)}.
\end{aligned}
$$

From (2.8) and (2.13), we can easily obtain

$$
\begin{aligned}
&\limsup_{t\to\infty} \int_l^t [R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \\
&\times \left[Q_2(s) - \frac{c_2 r(s)}{4\mu k}\left(G_2(s) - \frac{\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)}{r(s)[R(t)-R(s)][R(s)-R(l)]}\right)^2\right] ds > 0,
\end{aligned}
$$

and hence, Eq. (1.1) is oscillatory by Theorem 2.3. The proof is complete. □

Remark 2.1. The values $\alpha = \beta = 1$ are prohibited as a simple evaluation of the integrals in (2.11) with these values shows. Hence, the restriction on α and β is greater than 1.

By

$$
\int_l^t \Phi^2(t,s,l) g'(s)\, ds = -2\int_l^t \Phi(t,s,l)\Phi_s(t,s,l) g(s)\, ds, \qquad g \in C^1([t_0,\infty),\mathbb{R})
$$

and Theorem 2.4, we have the following corollaries.

Corollary 2.1. *In Theorem* 2.4, *suppose that* $p(t) \in C^1([t_0,\infty),\mathbb{R})$ *and condition* (2.8) *is replaced by the condition*

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{\alpha+\beta-1}(t)} \int_l^t [R(t)-R(s)]^\alpha [R(s)-R(l)]^\beta \\
&\times \left[\frac{\mu k}{c_2} Q_2(s) - \frac{r(s)G_2^2(s)}{4} - \frac{(r(s)G_2(s))'}{2}\right] ds \\
&> \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{4\Gamma(\alpha+\beta)},
\end{aligned}
$$

then Eq. (1.1) *is oscillatory.*

Corollary 2.2. *Suppose that conditions* (C1)–(C4) *hold and* $\lim_{t\to\infty} R(t) = \infty$*. If for each* $l \geq t_0$*, there exist a function* $B \in C([t_0,\infty),\mathbb{R})$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\alpha > 1/2$ *or* $\beta > 1/2$ *such that*

$$(1)\ \limsup_{t\to\infty} \frac{1}{R^{2\alpha+1}(t)} \int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 \times \left\{\frac{\mu k}{c_2}Q_2(s) - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[R(t)-(\alpha+1)R(s)+\alpha R(l)]}{[R(t)-R(s)][R(s)-R(l)]}\right\} ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}$$

or

$$(2)\ \limsup_{t\to\infty} \frac{1}{R^{2\beta+1}(t)} \int_l^t [R(t)-R(s)]^{2}[R(s)-R(l)]^{2\beta} \times \left\{\frac{\mu k}{c_2}Q_2(s) - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[\beta R(t)-(\beta+1)R(s)+R(l)]}{[R(t)-R(s)][R(s)-R(l)]}\right\} ds > \frac{\beta}{(2\beta-1)(2\beta+1)},$$

where $Q_2(s)$ *and* $G_2(s)$ *are defined by* (2.9) *and* (2.10)*, respectively, then* Eq. (1.1) *is oscillatory.*

Proof. (1) In (2.8), replaced α and β by 2α and 2, respectively, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{2\alpha+1}(t)} \int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 \\
&\times \left[\frac{\mu k}{c_2}Q_2(s) - \frac{r(s)G_2^2(s)}{4} + \frac{G_2(s)[R(t)-(\alpha+1)R(s)+\alpha R(l)]}{[R(t)-R(s)][R(s)-R(l)]}\right] ds \\
&> (2\alpha)2(2\alpha+2-2)\frac{\Gamma(2\alpha-1)\Gamma(2-1)}{4\Gamma(2\alpha+2)} \\
&= 4\alpha(2\alpha)\frac{\Gamma(2\alpha-1)\Gamma(1)}{4(2\alpha+1)(2\alpha)(2\alpha-1)\Gamma(2\alpha-1)} \\
&= \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.
\end{aligned}$$

(2) In (2.8), replaced α and β by 2 and 2β, respectively, the rest of the proof is similar to that of (1) and hence omitted. □

Corollary 2.3. *Suppose that conditions* (C1)–(C4) *hold,* $p(t) \in C^1([t_0,\infty),\mathbb{R})$ *and* $\lim_{t\to\infty} R(t) = \infty$*. If for each* $l \geq t_0$*, there exist a function* $B \in C([t_0,\infty),\mathbb{R})$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\alpha > 1/2$ *or* $\beta > 1/2$ *such that*

$$\limsup_{t\to\infty} \frac{1}{R^{2\alpha+1}(t)} \int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 \times \left[\frac{\mu k}{c_2}Q_2(s) - \frac{r(s)G_2^2(s)}{4} - \frac{(r(s)G_2(s))'}{2}\right] ds$$

$$> \frac{\alpha}{(2\alpha-1)(2\alpha+1)}$$

or

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{2\beta+1}(t)} \int_l^t [R(t)-R(s)]^2[R(s)-R(l)]^{2\beta} \\
&\times \left[\frac{\mu k}{c_2} Q_2(s) - \frac{r(s)G_2^2(s)}{4} - \frac{(r(s)G_2(s))'}{2}\right] ds \qquad (2.14)\\
&> \frac{\beta}{(2\beta-1)(2\beta+1)},
\end{aligned}$$

where $Q_2(s)$ *and* $G_2(s)$ *are defined by* (2.9) *and* (2.10), *respectively, then* Eq. (1.1) *is oscillatory.*

Remark 2.2. Applying Corollary 2.3 with $r(t) \equiv 1$ and $B(t) \equiv 0$ to the following Euler equation

$$x''(t) + \frac{\gamma}{t^2} x(t) = 0, \qquad t \geq t_0 > 0,$$

or as the results in [2, 8] show, we can obtain that the above Euler equation is oscillatory when $\gamma > 1/4$, and nonoscillatory when $\gamma \leq 1/4$ (If $\gamma \leq 1/4$, evidently, the above Euler equation has a nonoscillatory solution $x(t) = t^{\frac{1+\sqrt{1-4\gamma}}{2}}$). Therefore, the oscillation constants in the right hand sides of inequalities in Corollaries 2.1, 2.2, 2.3, etc. are sharp.

Remark 2.3. It is easy to see that Sun's two main theorems in [8] (Theorems 2.3 and 2.4, see also Theorems A and B in Section 1) are the special cases of Corollary 2.2 when $r(t) \equiv 1$, $\psi(x) \equiv 1$, $\varphi(x) = x$ and $B(t) \equiv 0$. In addition, Theorem 2.4 above can be applied to the case when $r(t) \not\equiv 1$ and $\lim_{t\to\infty} \int_{t_0}^t \frac{1}{r(s)} ds = \infty$.

Remark 2.4. Fu's Theorems 2.4 and 2.5 in [3] (see Theorems C and D in Section 1) are the special cases of Corollary 2.2 when $B(t) = \frac{p(t)}{2\mu k r(t)}$.

If $f(x)$ is of no monotonicity and satisfies condition (C4′), we have the following oscillation criterion.

Theorem 2.5. *Suppose that conditions* (C1)–(C3) *and* (C4′) *hold. If for each* $l \geq t_0$, *there exist functions* $\Phi \in Y, A \in C^1([t_0,\infty),\mathbb{R}^+)$ *and* $B \in C([t_0,\infty),\mathbb{R})$ *such that* $(rB) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \int_l^t \left[\Phi^2(t,s,l)Q_3(s) - \frac{c_2 A(s) r(s)}{k}\left(\frac{G_3(s)}{2}\Phi(t,s,l) - \Phi_s(t,s,l)\right)^2\right] ds > 0,$$

where

$$Q_3(t) = A(t)\left[\lambda q(t) - \frac{1}{4k}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{p^2(t)}{r(t)} \right. \qquad (2.15)$$

$$- \frac{1}{c_2}p(t)B(t) + \frac{k}{c_2}r(t)B^2(t) - (r(t)B(t))'\Big]$$

and

$$G_3(t) = -\frac{A'(t)}{A(t)} - \frac{2k}{c_2}B(t) + \frac{p(t)}{c_2 r(t)}, \tag{2.16}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficiently large $T_0 \geq t_0$. Define

$$w(s) = A(s)r(s)\left[\frac{\psi(x(s))\varphi(x'(s))}{x(s)} + B(s)\right], \quad \text{for } s \geq T_0. \tag{2.17}$$

Then differentiating (2.17) and using (1.1), (C1)–(C3) and (C4′), we obtain that for $s \geq T_0$

$$\begin{aligned}
w'(s) = {} & \frac{A'(s)}{A(s)}w(s) - A(s)p(s)\frac{\varphi(x'(s))}{x(s)} - A(s)q(s)\frac{f(x(s))}{x(s)} \\
& - \frac{A(s)r(s)\psi(x(s))\varphi(x'(s))x'(s)}{x^2(s)} + A(s)(r(s)B(s))' \\
\leq {} & \frac{A'(s)}{A(s)}w(s) - \lambda A(s)q(s) + A(s)(r(s)B(s))' \\
& - \frac{kA(s)r(s)}{\psi(x(s))}\left[\frac{p(s)}{kr(s)}\frac{\psi(x(s))\varphi(x'(s))}{x(s)} + \left(\frac{\psi(x(s))\varphi(x'(s))}{x(s)}\right)^2\right] \\
= {} & \frac{A'(s)}{A(s)}w(s) - \lambda A(s)q(s) + A(s)(r(s)B(s))' + \frac{A(s)p^2(s)}{4kr(s)\psi(x(s))} \\
& - \frac{kA(s)r(s)}{\psi(x(s))}\left[\frac{\psi(x(s))\varphi(x'(s))}{x(s)} + \frac{p(s)}{2kr(s)}\right]^2 \\
\leq {} & \frac{A'(s)}{A(s)}w(s) - \lambda A(s)q(s) + A(s)(r(t)B(s))' + \frac{A(s)p^2(s)}{4kc_1 r(s)} \\
& - \frac{kA(s)r(s)}{c_2}\left[\frac{\psi(x(s))\varphi(x'(s))}{x(s)} + \frac{p(s)}{2kr(s)}\right]^2 \\
= {} & \frac{A'(s)}{A(s)}w(s) - \lambda A(s)q(s) + A(s)(r(s)B(s))' + \frac{A(s)p^2(s)}{4kc_1 r(s)} \\
& - \frac{kA(s)r(s)}{c_2}\left[\frac{w(s)}{A(s)r(s)} - B(s) + \frac{p(s)}{2kr(s)}\right]^2 \\
= {} & -Q_3(s) - G_3(s)w(s) - \frac{k}{c_2 A(s)r(s)}w^2(s),
\end{aligned}$$

where $Q_3(s)$ and $G_3(s)$ are defined by (2.15) and (2.16), respectively. The remainder of the proof is similar to that of Theorem 2.1, so we omit the details. The proof is complete. □

Remark 2.5. Similar to Theorem 2.5, we can obtain some corresponding oscillation theorems and corollaries for Eq. (1.1) if we replace condition (C4) with (C4′) in Theorems 2.2–2.4 and Corollaries 2.1–2.3.

3. INTERVAL OSCILLATION CRITERIA

We can see that Theorems 2.1–2.5 and other results in [1, 3–6, 8, 10, 11] involve the integral of the coefficients r, p and q, and hence require the information of the coefficients on the entire half-line $[t_0, \infty)$. It is difficult to apply them to the cases when Eq. (1.1) is "bad" on a big part of $[t_0, \infty)$, e.g., when $\int_{t_0}^{\infty} q(t)dt = -\infty$. This should motivate further study of the interval property for Eq. (1.1). In the following, we will establish several new interval oscillation criteria for Eq. (1.1), that is, criteria given by the behavior of Eq. (1.1) (or r, p and q) only on a sequence of subintervals of $[t_0, \infty)$. The results may be applied to the extreme cases such as $\int_{t_0}^{\infty} q(t)dt = -\infty$.

Theorem 3.1. *Assume that conditions* (C1)–(C4) *hold. If for each* $l \geq t_0$, *there exist functions* $\Phi \in Y, A \in C^1([t_0, \infty), \mathbb{R}^+), B \in C([t_0, \infty), \mathbb{R})$ *and two constants* $b > a \geq l$ *such that* $(rB) \in C^1([t_0, \infty), \mathbb{R})$ *and*

$$\int_a^b \left[\Phi^2(b, s, a)Q_1(s) - \frac{c_2 A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(b, s, a) - \Phi_s(b, s, a)\right)^2\right] ds > 0, \tag{3.1}$$

where $Q_1(s)$ *and* $G_1(s)$ *are defined by* (2.2) *and* (2.3), *respectively, then* Eq. (1.1) *is oscillatory.*

Proof. With the proof of Theorem 2.1, where t and l are replaced by b and a, respectively, we can easily see that every solution of Eq. (1.1) has at least one zero in (a, b), i.e., every solution of Eq. (1.1) has arbitrarily large zero on $[t_0, \infty)$. This completes the proof of Theorem 3.1. □

Similar to the discussion in Section 2, we have the following corollaries and theorem.

Corollary 3.1. *Assume that conditions* (C1)–(C4) *hold. If for each* $l \geq t_0$, *there exist functions* $A, \rho \in C^1([t_0, \infty), \mathbb{R}^+)$, $B \in C([t_0, \infty), \mathbb{R})$, *two constants* $\alpha, \beta > 1$, *and two constants* $b > a \geq l$ *such that* $(rB) \in C^1([t_0, \infty), \mathbb{R})$ *and*

$$\int_a^b \rho(s)(b-s)^{\alpha}(s-a)^{\beta}\left[Q_1(s) - \frac{c_2 A(s)r(s)}{4\mu k}\left(G_1(s) - \frac{\rho'(s)}{\rho(s)} - \frac{\beta b - (\alpha+\beta)s + \alpha a}{(b-s)(s-a)}\right)^2\right] ds > 0, \tag{3.2}$$

where $Q_1(s)$ *and* $G_1(s)$ *are defined by* (2.2) *and* (2.3), *respectively, then* Eq. (1.1) *is oscillatory.*

Corollary 3.2. *Assume that conditions* (C1)–(C4) *hold. If for each* $l \geq t_0$, *there exist functions* $A, \rho \in C^1([t_0, \infty), \mathbb{R}^+)$, $B \in C([t_0, \infty), \mathbb{R})$, *two constants* $\alpha, \beta > 1$, *and two constants* $b > a \geq l$ *such that* $(rB) \in C^1([t_0, \infty), \mathbb{R})$ *and*

$$\int_a^b \rho(s)[R(b) - R(s)]^\alpha [R(s) - R(a)]^\beta \Bigg[Q_1(s) - \frac{c_2 A(s) r(s)}{4\mu k} \Bigg(G_1(s) - \frac{\rho'(s)}{\rho(s)} - \frac{\beta R(b) - (\alpha + \beta) R(s) + \alpha R(a)}{r(s)[R(b) - R(s)][R(s) - R(a)]} \Bigg)^2 \Bigg] ds > 0,$$

where $Q_1(s)$ *and* $G_1(s)$ *are defined by* (2.2) *and* (2.3), *respectively, then* Eq. (1.1) *is oscillatory.*

Theorem 3.2. *Assume that conditions* (C1)–(C3) *and* (C4′) *hold. If for each* $l \geq t_0$, *there exist functions* $\Phi \in Y$, $A \in C^1([t_0, \infty), \mathbb{R}^+)$, $B \in C([t_0, \infty), \mathbb{R})$ *and two constants* $b > a \geq l$ *such that* $(rB) \in C^1([t_0, \infty), \mathbb{R})$ *and*

$$\int_a^b \left[\Phi^2(b, s, a) Q_3(s) - \frac{c_2 A(s) r(s)}{k} \left(\frac{G_3(s)}{2} \Phi(b, s, a) - \Phi_s(b, s, a) \right)^2 \right] ds > 0,$$

where $Q_3(s)$ *and* $G_3(s)$ *are defined by* (2.15) *and* (2.16), *respectively, then* Eq. (1.1) *is oscillatory.*

Remark 3.1. Similar to Theorem 3.2, we can obtain some corresponding oscillation results for Eq. (1.1) if we replace condition (C4) with (C4′) in Corollaries 3.1 and 3.2.

Remark 3.2. The results in this paper are presented in the form of a high degree of generality: they give many possibilities for oscillation criteria with appropriate choices of the functions $\Phi \in Y$, $A \in C^1([t_0, \infty), \mathbb{R}^+)$ and $B \in C([t_0, \infty), \mathbb{R})$.

4. EXAMPLES

In this section, we will show the applications of our oscillation criteria by three examples. The first example illustrates Corollary 2.3.

Example 4.1. Consider the nonlinear damped differential equation

$$\left(\frac{1}{t^2} \frac{1}{1 + e^{-|x(t)|}} \frac{x'(t)}{1 + \sigma x'^2(t)} \right)' + \frac{2}{t^3 - 1} \frac{x'(t)}{1 + \sigma x'^2(t)} + \frac{\gamma}{t^4} x(t) \left(1 + x^4(t) \right) = 0, \ t \geq t_0 > 1, \tag{4.1}$$

where $\sigma \geq 0$ and $\gamma > 5/4$ are constants.

Clearly, the conditions (C1)–(C4) hold for $c_1 = 1/2$, $c_2 = \mu = k = 1$, and $R(t) = \int_{t_0}^t \frac{1}{r(s)} ds = \frac{1}{3}(t^3 - t_0^3)$, $\lim_{t \to \infty} R(t) = \infty$. Let us apply Corollary 2.3 with $B(t) = \frac{t^2}{t^3 - 1}$, then

$$Q_2(t) = \frac{\gamma}{t^4} - \frac{1}{4} t^2 \frac{4}{(t^3 - 1)^2} - \frac{2t^2}{(t^3 - 1)^2} + \frac{1}{t^2} \frac{t^4}{(t^3 - 1)^2} - \left(\frac{1}{t^2} \frac{t^2}{t^3 - 1} \right)'$$

$$= \frac{\gamma}{t^4} + \frac{t^2}{(t^3-1)^2},$$

$$G_2(t) = -2\frac{t^2}{t^3-1} + t^2\frac{2}{t^3-1} = 0.$$

For any $l \geq t_0$, a straightforward computation yields

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{2\beta+1}(t)} \int_l^t [R(t)-R(s)]^2[R(s)-R(l)]^{2\beta} \\
&\times \left[\frac{\mu k}{c_2}Q_2(s) - \frac{r(s)G_2^2(s)}{4} - \frac{(r(s)G_2(s))'}{2}\right] ds \\
&= \lim_{t\to\infty} \frac{1}{R^{2\beta+1}(t)} \int_l^t [R(t)-R(s)]^2[R(s)-R(l)]^{2\beta} \left(\frac{\gamma}{s^4} + \frac{s^2}{(s^3-1)^2}\right) ds \\
&= \lim_{t\to\infty} \frac{\int_l^t [R(s)-R(l)]^{2\beta} \left(\frac{\gamma}{s^4} + \frac{s^2}{(s^3-1)^2}\right) ds}{R^{2\beta-1}(t)} \\
&\quad - \lim_{t\to\infty} \frac{2\int_l^t R(s)[R(s)-R(l)]^{2\beta} \left(\frac{\gamma}{s^4} + \frac{s^2}{(s^3-1)^2}\right) ds}{R^{2\beta}(t)} \\
&\quad + \lim_{t\to\infty} \frac{\int_l^t R^2(s)[R(s)-R(l)]^{2\beta} \left(\frac{\gamma}{s^4} + \frac{s^2}{(s^3-1)^2}\right) ds}{R^{2\beta+1}(t)} \\
&= \lim_{t\to\infty} \frac{[R(t)-R(l)]^{2\beta} \left(\frac{\gamma}{t^4} + \frac{t^2}{(t^3-1)^2}\right)}{(2\beta-1)R^{2\beta-2}(t)t^2} \\
&\quad - \lim_{t\to\infty} \frac{R(t)[R(t)-R(l)]^{2\beta} \left(\frac{\gamma}{t^4} + \frac{t^2}{(t^3-1)^2}\right)}{\beta R^{2\beta-1}(t)t^2} \\
&\quad + \lim_{t\to\infty} \frac{R^2(t)[R(t)-R(l)]^{2\beta} \left(\frac{\gamma}{t^4} + \frac{t^2}{(t^3-1)^2}\right)}{(2\beta+1)R^{2\beta}(t)t^2} \\
&= \frac{\gamma+1}{9} \frac{1}{\beta(2\beta-1)(2\beta+1)}.
\end{aligned}$$

Since $\gamma > 5/4$, i.e., $\gamma+1 > 9/4$, we can choose an appropriate constant $\beta > 1/2$ such that $\gamma+1 > 9\beta^2$, and hence

$$\frac{\gamma+1}{9}\frac{1}{\beta(2\beta-1)(2\beta+1)} > \frac{\beta}{(2\beta-1)(2\beta+1)}.$$

Thus, the condition (2.14) holds. From Corollary 2.3, we have that Eq. (4.1) is oscillatory for $\gamma > 5/4$.

The second example illustrates Theorem 3.1.

Example 4.2. Consider the nonlinear damped differential equation

$$\left((1+\cos^2 t)\frac{1+e^{-|x(t)|}}{2}x'(t)\left(1-e^{-x'^2(t)}\right)\right)' + \sin 2t x'(t)\left(1-e^{-x'^2(t)}\right) \tag{4.2}$$

$$+\frac{c}{1+\cos^2 t}x(t)\left(1+x^2(t)\right)=0,\ t\geq 1,$$

where $c > 9/4$ is a constant.

Obviously, the conditions (C1)–(C4) hold for $c_1 = 1/2,\ c_2 = \mu = k = 1$. For any $l \geq 1$, there exists $n \in N_0 = \{0, 1, 2, \dots\}$ such that $2n\pi \geq l$. Let $a = 2n\pi$, and $b = (2n+1)\pi$. Choose $\Phi(b, s, a) = \sqrt{\sin(b-s)\sin(s-a)} = \sin s$ for $a \leq s \leq b$, then $\Phi_s(b, s, a) = \cos s$. Let us apply Theorem 3.1 with $A(t) = 1 + \cos^2 t, B(t) = \frac{\sin 2t}{1+\cos^2 t}$ for all $t \geq 1$, then

$$Q_1(t) = (1+\cos^2 t)\left(\frac{c}{1+\cos^2 t} - \frac{1}{4}\frac{\sin^2 2t}{1+\cos^2 t} - \sin 2t\frac{\sin 2t}{1+\cos^2 t} + \frac{\sin^2 2t}{1+\cos^2 t} - 2\cos 2t\right)$$
$$= c - \frac{1}{4}\sin^2 2t - 2\cos 2t(1+\cos^2 t),$$
$$G_1(t) = \frac{2\cos t\sin t}{1+\cos^2 t} - 2\frac{\sin 2t}{1+\cos^2 t} + \frac{\sin 2t}{1+\cos^2 t} = 0.$$

A direct computation yields

$$\int_a^b \left[\Phi^2(b,s,a)Q_1(s) - \frac{c_2 A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(b,s,a) - \Phi_s(b,s,a)\right)^2\right]ds$$
$$= \int_0^\pi \left\{\sin^2 s\left[c - \frac{1}{4}\sin^2 2s - 2\cos 2s(1+\cos^2 s)\right] - (1+\cos^2 s)^2\cos^2 s\right\}ds$$
$$= \int_0^\pi \left(\frac{4c-9}{8} - \frac{16c+95}{32}\cos 2s + \frac{1}{8}\cos 4s + \frac{3}{32}\cos 6s - \frac{1}{8}\cos^3 2s\right)ds$$
$$= \frac{4c-9}{8}\pi,$$

thus, by $c > 9/4$, we have

$$\int_a^b \left[\Phi^2(b,s,a)Q_1(s) - \frac{c_2 A(s)r(s)}{\mu k}\left(\frac{G_1(s)}{2}\Phi(b,s,a) - \Phi_s(b,s,a)\right)^2\right]ds > 0.$$

This means that (3.1) holds. From Theorem 3.1, we find that Eq. (4.2) is oscillatory.

The third example illustrates Corollary 3.1.

Example 4.3. Consider the nonlinear damped differential equation

$$\left(t^2\frac{1+x^2(t)}{2+x^2(t)}x'(t)\left(1-e^{-x'^2(t)}\right)\right)' + p(t)x'(t)\left(1-e^{-x'^2(t)}\right) + q(t)x(t)\left(5+x^4(t)\right) = 0,\ t \geq 1, \tag{4.3}$$

where

$$\delta p(t) = q(t) = \begin{cases} \delta(t-2n)(2n+1-t), & 2n \leq t < 2n+1, \\ n(2n+1-t)(2n+2-t), & 2n+1 \leq t < 2n+2, \end{cases}$$

for $\delta > 51/20$ is a constant, $n \in N_0 = \{0, 1, 2, \cdots\}$.

Clearly, the conditions (C1)–(C4) hold for $c_1 = 1/2,\ c_2 = k = 1,\ \mu = 5$. For any $l \geq 1$, there exists $n \in N_0$ such that $2n \geq l$. Let $a = 2n,\ b = 2n+1,\ \alpha = \beta = 2$, and $\rho(t) \equiv 1$. Choose $A(t) = 1/t,\ B(t) \equiv 0$, then the left-hand side of (3.2) becomes

$$\begin{aligned}
&\int_0^1 (1-s)^2 s^2 \left[\delta(1-s) - \frac{1}{20s}(1-s)^2 - \frac{s}{20}\left(\frac{1+s}{1-s}\right)^2\right] ds \\
&= \delta \int_0^1 (1-s)^3 s^2 ds - \frac{1}{20}\int_0^1 (1-s)^4 s ds - \frac{1}{20}\int_0^1 (1+s)^2 s^3 ds \\
&= \frac{\delta}{60} - \frac{17}{400} > 0,
\end{aligned}$$

since $\delta > 51/20$.

Therefore, (3.2) holds and we conclude by Corollary 3.1 that Eq. (4.3) is oscillatory. Note that in this equation, we have $\int_1^\infty p(t)dt = \int_1^\infty q(t)dt = -\infty$ and $p(t)$ is not differentiable when $\delta \neq n$. Also, the criteria in [1–11] fail to apply to Eq. (4.3).

REFERENCES

[1] B. Ayanlar and A. Tiryaki, Oscillation theorems for nonlinear second order differential equations with damping, *Acta Math. Hungar.*, 89(2000), 1–13.

[2] S.G.A. Dubé and A.B. Mingarelli, Nonlinear functionals and a theorem of Sun, *J. Math. Anal. Appl.*, 308(2005), 208–220.

[3] Y.L. Fu, Oscillation criteria for second-order nonlinear differential equations with damping, *Mathematics in Practice and Theory*, 37(13)(2007), 180–188.

[4] I.V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, *Mat. Zametki*, 23(1978), 249–251.

[5] H.J. Li, Oscillation criteria for second order linear differential equations, *J. Math. Anal. Appl.*, 194(1995), 217–234.

[6] Ch.G. Philos, Oscillation theorems for linear differential equations of second order, *Arch. Math. (Basel)*, 53(1989), 483–492.

[7] Y.V. Rogovchenko and Fatos Tuncay, Interval oscillation criteria for second order nonlinear differential equations with damping, *Dynam. Systems Appl.*, 16(2)(2007), 337–343.

[8] Y.G. Sun, New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping, *J. Math. Anal. Appl.*, 291(2004), 341–351.

[9] Y.G. Sun and F.W. Meng, Oscillatory behavior of linear matrix Hamiltonian systems, *Math. Nachr.*, 280(2007), 1310–1316.

[10] Q.R. Wang, Oscillation criteria for nonlinear second order damped differential equations, *Acta Math. Hungar.*, 102(1-2)(2004), 117–139.

[11] J. Yan, Oscillation theorems for second order linear differential equations with damping, *Proc. Amer. Math. Soc.*, 98(1986), 276–282.