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ON THE STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION II

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ABSTRACT. In this paper, we prove the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation

$$2f(x+y, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$$

in the spirit of P. Găvruta.

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1. INTRODUCTION

In 1940, S. M. Ulam [11] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [10] provided a generalization of Hyers' Theorem by allowing the Cauchy difference to be controlled by a sum of powers of norms like

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p).$$

P. Găvruta [1] provided a further generalization of Th. M. Rassias' Theorem in the following way.

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Theorem 1.1. *Let X be a vector space, let Y a Banach space and let $\varphi : X \times X \to [0, \infty)$ be a function satisfying*

$$\psi_(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies the functional inequality $\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$ for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ which satisfies

$$\|f(x) - T(x)\| \leq \psi(x, x)$$

for all $x \in X$.

Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians [4,5,7,8].

Throughout this paper, let X be a real vector space and Y a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x + y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$). For a given mapping $f : X \times X \to Y$, we define

$$Cf(x, y, z, w) := 2f(x + y, \frac{z + w}{2}) - f(x, z) - f(x, w) - f(y, z) - f(y, w)$$

for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping if f satisfies the functional equation

$$Cf(x, y, z, w) = 0$$

for all $x, y, z, w \in X$ and the functional equation $Cf = 0$ is called a Cauchy-Jensen functional equation. In 2006, Park and Bae [9] obtained the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation.

The authors [3] investigated the stability in the spirit of Th. M. Rassias for a Cauchy-Jensen functional equation in the following theorem.

Theorem 1.2. *Let $p \neq 1, 2$, $\theta > 0$ and let X be a normed space. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|Cf(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X \backslash \{0\}$ if $p < 1$ ($x, y, z, w \in X$ if $p > 1$), then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \frac{\theta}{2 - 2^p}\|x\|^p + \theta\|y\|^p$$

for all $x, y \in X \backslash \{0\}$ if $p < 1$,

$$\|f(x, y) - F(x, y)\| \leq (\frac{\theta}{|2^p - 2|} + \frac{4\theta}{|4 - 2^p|})\|x\|^p + \frac{5 \cdot 2^p\theta}{|4 - 2^p|}\|y\|^p$$

for all $x, y \in X$ if $p > 1$. In particular, f is a Cauchy-Jensen mapping if $p < 0$.

In this paper, we investigate the generalized Hyers-Ulam stability in the spirit of P. Găvruta [1] for a Cauchy-Jensen functional equation and obtain Theorem 1.2 as an application. We also have better stability results than those of Park and Bae [9].

2. STABILITY OF A CAUCHY-JENSEN MAPPING

We need the following lemma [3] to prove the main theorem.

Lemma 2.1. *Let $f : X \times X \to Y$ be a Cauchy-Jensen mapping. Then the following equalities hold;*

$$\begin{aligned} f(x,y) &= 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) - (4^n - 2^n) f(\frac{x}{2^n}, 0), \\ f(x,y) &= \frac{f(2^n x, 2^n y)}{4^n} + (\frac{1}{2^n} - \frac{1}{4^n}) f(2^n x, 0) \\ &= \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{2}(\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, 2^n y) + f(2^n x, -2^n y)) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we have the stability result in the spirit of P. Găvruta for a Cauchy-Jensen mapping.

Theorem 2.2. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Cf(x,y,z,w)\| \le \varphi(x,y,z,w), \tag{2.1}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned} \|f(x,y) - F(x,y)\| \le & \sum_{j=0}^{\infty} [\frac{2(\varphi(2^j x, 2^j x, 2^{j+1} y, 2^{j+1} y)}{4^{j+2}} + \frac{\varphi(2^j x, 2^j x, 0, 0)}{2^{j+2}} \\ & + \frac{2\varphi(2^j x, 2^j x, 0, 0) + 8\varphi(2^j x, 0, 2^{j+1} y, 0)}{4^{j+2}} \\ & + \frac{\varphi(0, 0, 2^{j+1} y, 2^{j+1} y)}{4^{j+1}}] + \frac{\varphi(0,0,0,0)}{3} \end{aligned} \tag{2.2}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{n\to\infty} \frac{f(2^n x, 0)}{2^n} + \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n}$$

for all $x, y \in X$.

Proof. Since

$$\left\|\frac{f(2^nx,0)}{2^n}-\frac{f(2^{n+1}x,0)}{2^{n+1}}\right\|\le\frac{\varphi(2^nx,2^nx,0,0)}{2^{n+2}}$$

and

$$\begin{aligned}&\left\|\frac{f(2^nx,2^ny)-f(2^nx,0)}{4^n}-\frac{f(2^{n+1}x,2^{n+1}y)-f(2^{n+1}x,0)}{4^{n+1}}\right\|\\ &=\frac{2}{4^{n+2}}\|Cf(2^nx,2^nx,2^{n+1}y,2^{n+1}y)+2Cf(0,0,2^{n+1}y,2^{n+1}y)\\ &\quad -Cf(2^nx,2^nx,0,0)-4Cf(2^nx,0,2^{n+1}y,0)+2Cf(0,0,0,0)\|\\ &\le\frac{2}{4^{n+2}}\big(\varphi(2^nx,2^nx,2^{n+1}y,2^{n+1}y)+2\varphi(0,0,2^{n+1}y,2^{n+1}y)\\ &\quad+\varphi(2^nx,2^nx,0,0)+4\varphi(2^nx,0,2^{n+1}y,0)+2\varphi(0,0,0,0)\big)\end{aligned}$$

for all $x,y\in X$ and $n\in\mathbb{N}$, we get

$$\left\|\frac{f(2^lx,0)}{2^l}-\frac{f(2^mx,0)}{2^m}\right\|\le\sum_{j=l}^{m-1}\frac{\varphi(2^jx,2^jx,0,0)}{2^{j+2}}\tag{2.3}$$

and

$$\begin{aligned}&\left\|\frac{(f(2^lx,2^ly)-f(2^lx,0))}{4^l}-\frac{(f(2^mx,2^my)-f(2^mx,0))}{4^m}\right\|\\ &\le\sum_{j=l}^{m-1}\frac{2}{4^{j+2}}\big(\varphi(2^jx,2^jx,2^{j+1}y,2^{j+1}y)+2\varphi(0,0,2^{j+1}y,2^{j+1}y)\\ &\quad+\varphi(2^jx,2^jx,0,0)+4\varphi(2^jx,0,2^{j+1}y,0)+2\varphi(0,0,0,0)\big)\end{aligned}\tag{2.4}$$

for all $x,y\in X$ and given integers l,m $(0\le l<m)$. Hence the sequences $\{\frac{f(2^nx,0)}{2^n}\}$ and $\{\frac{(f(2^nx,2^ny)-f(2^nx,0))}{4^n}\}$ are Cauchy sequences for all $x,y\in X$. Since Y is complete, the sequences $\{\frac{f(2^nx,0)}{2^n}\}$ and $\{\frac{(f(2^nx,2^ny)-f(2^nx,0))}{4^n}\}$ converge for all $x,y\in X$. Define the maps $F_1,F_2:X\times X\to Y$ by

$$\begin{aligned}F_1(x,y)&=\lim_{n\to\infty}\frac{f(2^nx,0)}{2^n},\\ F_2(x,y)&=\lim_{n\to\infty}\frac{f(2^nx,2^ny)-f(2^nx,0)}{4^n}=\lim_{n\to\infty}\frac{f(2^nx,2^ny)}{4^n}\end{aligned}$$

for all $x,y\in X$. Since

$$\begin{aligned}CF_1(x,y,z,w)&=\lim_{n\to\infty}\frac{Cf(2^nx,2^ny,0,0)}{2^n}=0\quad\text{and}\\ CF_2(x,y,z,w)&=\lim_{n\to\infty}\frac{Cf(2^nx,2^ny,2^nz,2^nw)}{4^n}=0\end{aligned}$$

for all $x,y\in X$, F is a Cauchy-Jensen mapping, where F is defined by

$$F(x,y)=F_1(x,y)+F_2(x,y).$$

Putting $l=0$, taking $m\to\infty$ in (2.3) and (2.4) and using the inequality

$$\|f(x,y)-F(x,y)\|\le\|f(x,y)-f(x,0)-F_2(x,y)\|+\|f(x,0)-F_1(x,y)\|$$

we obtain the inequality (2.2). Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.2). By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\leq \|\frac{(F-F')(2^n x, 2^n y)}{4^n} + (\frac{1}{2^n} - \frac{1}{4^n})(F-F')(2^n x, 0)\| \\
&\leq \frac{1}{4^n}\|(F-f)(2^n x, 2^n y)\| + \frac{1}{4^n}\|(f-F')(2^n x, 2^n y)\| \\
&\quad + \frac{1}{2^n}(\|(F-f)(2^n x, 0)\| + \frac{1}{2^n}\|(f-F')(2^n x, 0)\|) \\
&\leq \sum_{j=n}^{\infty}[\frac{\varphi(2^j x, 2^j x, 2^{j+1}y, 2^{j+1}y) + 2\varphi(0,0,2^{j+1}y, 2^{j+1}y)}{4^{j+1}} \\
&\quad + \frac{\varphi(2^j x, 2^j x, 0, 0) + 4\varphi(2^j x, 0, 2^{j+1}y, 0)}{4^{j+1}} + \frac{2\varphi(2^j x, 2^j x, 0, 0)}{2^j} \\
&\quad + \frac{\varphi(2^j x, 0, 0, 0)}{2^j}] + \frac{2\varphi(0,0,0,0)}{3 \cdot 4^n} + \frac{4\varphi(0,0,0,0)}{2^n}
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $F(x,y) = F'(x,y)$ as desired. □

Now we have another stability result of a Cauchy-Jensen mapping for the several cases in the applications.

Theorem 2.3. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{j=0}^{\infty} 4^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}) < \infty$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Cf(x,y,z,w)\| \leq \varphi(x,y,z,w)$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
&\|f(x,y) - F(x,y)\| \\
&\leq \sum_{j=0}^{\infty}[2 \cdot 4^{j-1}\big(\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^j}, \frac{y}{2^j}) + \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, 0) \\
&\quad + 2\varphi(0, 0, \frac{y}{2^j}, \frac{y}{2^j}) + 4\varphi(\frac{x}{2^{j+1}}, 0, \frac{y}{2^j}, 0)\big) + 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, 0)] \qquad (2.5)
\end{aligned}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n}, 0) + \lim_{n\to\infty} 4^n (f(\frac{x}{2^n}, \frac{y}{2^n}) - f(\frac{x}{2^n}, 0))$$

for all $x, y \in X$.

Proof. Apply (2.1) with $x = y = z = w = 0$ to get $f(0,0) = 0$. Since

$$\|2^n f(\frac{x}{2^n}, 0) - 2^{n+1} f(\frac{x}{2^{n+1}}, 0)\| \leq 2^{n-1}\varphi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0, 0)$$

and

$$\begin{aligned}
&\|4^n(f(\frac{x}{2^n},\frac{y}{2^n})-f(\frac{x}{2^n},0))-4^{n+1}f(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}})-f(\frac{x}{2^{n+1}},0)\| \\
&= 2\cdot 4^{n-1}\|Cf(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},\frac{y}{2^n},\frac{y}{2^n})-Cf(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0,0) \\
&\quad +2Cf(0,0,\frac{y}{2^n},\frac{y}{2^n})-4Cf(\frac{x}{2^{n+1}},0,\frac{y}{2^n},0)\| \\
&\le 2\cdot 4^{n-1}\big(\varphi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},\frac{y}{2^n},\frac{y}{2^n})+\varphi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0,0) \\
&\quad +2\varphi(0,0,\frac{y}{2^n},\frac{y}{2^n})+4\varphi(\frac{x}{2^{n+1}},0,\frac{y}{2^n},0)\big)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$, we can use the similar method in Theorem 2.2 to get the sequences $\{2^n f(\frac{x}{2^n},0)\}$ and $\{4^n(f(\frac{x}{2^n},\frac{y}{2^n})-f(\frac{x}{2^n},0))\}$ converge for all $x, y \in X$. Define the maps $F_1, F_2 : X \times X \to Y$ by

$$\begin{aligned}
F_1(x,y) &= \lim_{n\to\infty} 2^n f(\frac{x}{2^n},0), \\
F_2(x,y) &= \lim_{n\to\infty} 4^n(f(\frac{x}{2^n},\frac{y}{2^n})-f(\frac{x}{2^n},0))
\end{aligned}$$

for all $x, y \in X$. Using the similar method in Theorem 2.2, one can obtain the inequalities

$$\begin{aligned}
&\|f(x,0)-F_1(x,0)\| \le \sum_{j=0}^{\infty} 2^{j-1}\varphi(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0,0), \\
&\|f(x,y)-f(x,0)-F_2(x,y)\| \\
&\le \sum_{j=0}^{\infty} 2\cdot 4^{j-1}\big(\varphi(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{y}{2^j},\frac{y}{2^j})+\varphi(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0,0) \\
&\quad +2\varphi(0,0,\frac{y}{2^j},\frac{y}{2^j})+4\varphi(\frac{x}{2^{j+1}},0,\frac{y}{2^j},0)\big)
\end{aligned}$$

for all $x, y \in X$. Since

$$\begin{aligned}
CF_1(x,y,z,w) &= \lim_{n\to\infty} 2^n Cf(\frac{x}{2^n},\frac{y}{2^n},0,0)=0 \\
CF_2(x,y,z,w) &= \lim_{n\to\infty} 4^n\big(Cf(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n})-Cf(\frac{x}{2^n},\frac{y}{2^n},0,0)\big)=0,
\end{aligned}$$

for all $x, y \in X$, F is a Cauchy-Jensen mapping satisfying (2.5), where F is defined by

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.5). By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\leq \|4^n(F-F')(\frac{x}{2^n}, \frac{y}{2^n}) - (4^n - 2^n)(F-F')(\frac{x}{2^n}, 0)\| \\
&\leq 4^n\|(F-f)(\frac{x}{2^n}, \frac{y}{2^n})\| + 4^n\|(f-F')(\frac{x}{2^n}, \frac{y}{2^n})\| \\
&\quad +4^n(\|(F-f)(\frac{x}{2^n}, 0)\| + \|(f-F')(\frac{x}{2^n}, 0)\|) \\
&\leq \sum_{j=n}^{\infty} 4^j \big(\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^j}, \frac{y}{2^j}) + 5\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, 0) \\
&\quad +2\varphi(0, 0, \frac{y}{2^j}, \frac{y}{2^j}) + 4\varphi(\frac{x}{2^{j+1}}, 0, \frac{y}{2^j}, 0)\big) + 4\varphi(\frac{x}{2^{j+1}}, 0, 0, 0)\big)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $F(x,y) = F'(x,y)$ as desired. □

Theorem 2.4. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty$$

and

$$\sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}) < \infty$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Cf(x,y,z,w)\| \leq \varphi(x,y,z,w),$$

for all $x, y, z, w \in X$. Then there exists a Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
\|f(x,y) - F(x,y)\| \leq & \sum_{j=0}^{\infty}[\frac{2}{4^{j+2}}(\varphi(2^j x, 2^j x, 0, 0) + 4\varphi(2^j x, 0, 2^{j+1}y, 0) \\
& +2\varphi(0, 0, 2^{j+1}y, 2^{j+1}y) + \varphi(2^j x, 2^j x, 2^{j+1}y, 2^{j+1}y)) \\
& +2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, 0)] \qquad (2.6)
\end{aligned}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n}, 0) + \lim_{n\to\infty} \frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n}$$

for all $x, y \in X$.

Proof. As in the proof of Theorem 2.2 and Theorem 2.3, we obtain two Cauchy-Jensen mappings F_1 and F_2 satisfying the inequalities

$$\|f(x,0) - F_1(x,0)\| \leq \sum_{j=0}^{\infty} 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, 0),$$

$$\begin{aligned}&\|f(x,y)-f(x,0)-F_2(x,y)\|\\&\leq \sum_{j=0}^{\infty}\frac{2}{4^{j+2}}(\varphi(2^jx,2^jx,0,0)+4\varphi(2^jx,0,2^{j+1}y,0)\\&\quad+2\varphi(0,0,2^{j+1}y,2^{j+1}y)+\varphi(2^jx,2^jx,2^{j+1}y,2^{j+1}y)\end{aligned}$$

and which are defined by

$$\begin{aligned}F_1(x,y) &= \lim_{n\to\infty} 2^n f(\frac{x}{2^n},0),\\ F_2(x,y) &= \lim_{n\to\infty}\frac{f(2^nx,2^ny)-f(2^nx,0)}{4^n}\end{aligned}$$

for all $x, y \in X$. Hence, F is a Cauchy-Jensen mapping satisfying (2.6) for all $x, y \in X$, where F is defined by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

□

As an application of Theorems 2.2, 2.3 and 2.4, we have corollary which was obtained by authors [3].

Corollary 2.5. *Let $0 \leq p(\neq 1,2)$ and let θ, f be as in Theorem 1.2. Then there exists a Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y)-F(x,y)\| \leq (\frac{\theta}{|2^p-2|}+\frac{4\theta}{|4-2^p|})\|x\|^p+\frac{5\cdot 2^p\theta}{|4-2^p|}\|y\|^p$$

for all $x, y \in X$.

Proof. Applying Theorem 2.2 (Theorem 2.3 and Theorem 2.4, respectively) for the case $0 \leq p < 1$ ($2 < p$ and $1 < p < 2$, respectively), we obtain the desired result. □

3. STABILITY OF A CAUCHY-JENSEN MAPPING ON THE PUNCTURED DOMAIN

The following lemma is easily verified by the same method in the proof of Lemma 3.3 in [6].

Lemma 3.1. *Let a set $A(\subset X)$ satisfy the following condition: for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all $|n| \geq n_x$ and $nx \in A$ for all $|n| < n_x$. If $F : X \times X \to Y$ satisfies the equality*

$$CF(x,y,z,w) = 0$$

for all $x, y, z, w \in X \backslash A$, then the map $F : X \times X \to Y$ is a Cauchy-Jensen mapping.

Proof. Let $A_x = \{n \in \mathbb{N} \mid nx \notin A\}$ for each $x \neq 0$. Choose $k \in A_x \cap A_y \cap A_{x+y} \cap A_z \cap A_w \cap A_{z+w}$ for the case $x, y, z, w \neq 0$ with $x + y \neq 0$ and $z + w \neq 0$. Then

$$\begin{aligned}
&CF(x,y,z,w)\\
&= \ CF((k+1)(x+y), -k(x+y), \frac{(k+2)(z+w)}{2}, \frac{-k(z+w)}{2})\\
&-\frac{1}{2}[CF((k+1)x, -kx, (k+2)z, -kz) + CF((k+1)x, -kx, (k+2)w, -kw)\\
&+CF((k+1)y, -ky, (k+2)z, -kz) + CF((k+1)y, -ky, (k+2)w, -kw)\\
&-CF((k+1)x, (k+1)y, (k+2)z, (k+2)w)\\
&-CF((k+1)x, (k+1)y, -kz, -kw) - CF(-kx, -ky, (k+2)z, (k+2)w)\\
&-CF(-kx, -ky, -kz, -kw)] = 0.
\end{aligned}$$

Choose $k \in A_x \cap A_z \cap A_w \cap A_{z+w}$ for the case $x, z, w \neq 0$ with $z + w \neq 0$, then

$$\begin{aligned}
&CF(x,-x,z,w)\\
&= \ CF((2k+1)x, -(2k+1)x, \frac{(k+2)(z+w)}{2}, \frac{-k(z+w)}{2})\\
&-\frac{1}{2}[CF((k+1)x, -kx, (k+2)z, -kz) + CF((k+1)x, -kx, (k+2)w, -kw)\\
&+CF(-(k+1)x, kx, (k+2)z, -kz) + CF(-(k+1)x, kx, (k+2)w, -kw)\\
&-CF((k+1)x, kx, (k+2)z, (k+2)w) - CF((k+1)x, kx, -kz, -kw)\\
&-CF(-(k+1)x, -kx, (k+2)z, (k+2)w)\\
&-CF(-(k+1)x, -kx, -kz, -kw)] = 0,\\
&CF(x,0,z,w)\\
&= \ CF((2k+1)x, -2kx, \frac{(k+2)(z+w)}{2}, \frac{-k(z+w)}{2})\\
&-\frac{1}{2}[CF((k+1)x, -kx, (k+2)z, -kz) + CF((k+1)x, -kx, (k+2)w, -kw)\\
&+CF(kx, -kx, (k+2)z, -kz) + CF(kx, -kx, (k+2)w, -kw)\\
&-CF((k+1)x, kx, (k+2)z, (k+2)w) - CF((k+1)x, kx, -kz, -kw)\\
&-CF(-kx, -kx, (k+2)z, (k+2)w) - CF(-kx, -kx, -kz, -kw)] = 0.
\end{aligned}$$

Similarly we can show that $CF(x, y, z, w) = 0$ for the other cases. □

Lemma 3.2. *Let A, F be as in Lemma 3.1. Then there exists a unique Cauchy-Jensen map $F' : X \times X \to Y$ satisfying the equality*

$$F'(x,y) = F(x,y)$$

for all $x, y \in X \backslash A$.

Proof. Let $F' : X \times X \to Y$ be another Cauchy-Jensen map satisfying the equality

$$F'(x, y) = F(x, y)$$

for all $x, y \in X \backslash A$. Choose $n \in A_x \cap A_y$ for the case $x, y \neq 0$, then

$$\begin{aligned}
F(x, y) - F'(x, y) &= \frac{1}{2}[CF((n+1)x, -nx, (n+2)y, -ny) \\
&\quad -CF'((n+1)x, -nx, (n+2)y, -ny)] = 0, \\
F(0, y) - F'(0, y) &= \frac{1}{2}[CF(ny, -ny, (n+2)y, -ny) \\
&\quad -CF'(ny, -ny, (n+2)y, -ny)] = 0, \\
F(x, 0) - F'(x, 0) &= \frac{1}{2}[CF((n+1)x, -nx, nx, -nx) \\
&\quad -CF'((n+1)x, -nx, nx, -nx)] = 0,
\end{aligned}$$

for all $x, y \in X \backslash \{0\}$ as we desired. □

Lemma 3.3. *Let $f : X \times X \to Y$ be a mapping and let $f' : X \times X \to Y$ be the map defined by*

$$f'(x, y) = f(x, y) - f(x, 0) + \frac{2}{3} f(0, 0)$$

for all $x, y \in X$. Then

$$f(x, 0) - f(0, 0) - \frac{f(2x, 0) - f(0, 0)}{2} = B_1(x, y),$$

$$f'(x, y) - \frac{f'(2x, 2y)}{4} = B_2(x, y)$$

$$f(0, y) - f(0, 0) - \frac{f(0, 2y) - f(0, 0)}{2} = B_3(x, y)$$

for all $x, y, z, w \in X \backslash \{0\}$, where

$$\begin{aligned}
B_1(x, y) &= \frac{1}{4}[-Cf(\frac{3}{2}x, \frac{1}{2}x, y, -y) + Cf(\frac{3}{2}x, -\frac{1}{2}x, y - y) \\
&\quad +Cf(\frac{1}{2}x, \frac{1}{2}x, y, -y) - Cf(\frac{1}{2}x, -\frac{1}{2}x, y, -y)], \\
B_2(x, y) &= \frac{1}{8}[-Cf(x, x, 3y, y) + Cf(x, x, 3y, -y) - Cf(x, x, y, y) \\
&\quad -Cf(x, x, y, -y)] - B_1(x, y), \\
B_3(x, y) &= \frac{1}{4}[Cf(x, -x, y, y) - Cf(x, -x, y, -y) \\
&\quad -Cf(x, -x, 3y, y) + Cf(x, -x, 3y, -y)],
\end{aligned}$$

From Lemmas 3.1, 3.2 and 3.3, we have the stability result of a Cauchy-Jensen mapping on the punctured domain.

Theorem 3.4. *Let A be as in Lemma 3.1. Let $\varphi : X \times X \times X \times X \to [0,\infty)$ be a function satisfying*

$$\sum_{i=0}^{\infty} \frac{1}{2^i}\varphi(2^i x, 2^i y, 2^i z, 2^i w) < \infty \tag{3.1}$$

for all $x, y, z, w \in X$. If $f : X \times X \to Y$ satisfies

$$\|Cf(x,y,z,w)\| \le \varphi(x,y,z,w) \tag{3.2}$$

for all $x, y, z, w \in X\backslash A$, then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - \frac{1}{3}f(0,0) - F(x,y)\| \le \bar{\varphi}(x,y) \tag{3.3}$$

for all $x, y \in X\backslash A$, where

$$\begin{aligned}
\bar{\varphi}(x,y) &= \sum_{j=0}^{\infty}[\frac{\varphi_1(2^j x, 2^j y)}{2^j} + \frac{\varphi_2(2^j x, 2^j y)}{4^j}],\\
\varphi_1(x,y) &= \frac{1}{4}[\varphi(\frac{3}{2}x, \frac{1}{2}x, y, -y) + \varphi(\frac{3}{2}x, -\frac{1}{2}x, y - y)\\
&\quad +\varphi(\frac{1}{2}x, \frac{1}{2}x, y, -y) + \varphi(\frac{1}{2}x, -\frac{1}{2}x, y, -y)],\\
\varphi_2(x,y) &= \frac{1}{8}[\varphi(x,x,3y,y) + \varphi(x,x,3y,-y) + \varphi(x,x,y,y)\\
&\quad +\varphi(x,x,y,-y)] + \varphi_1(x,y).
\end{aligned}$$

The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{4^j}f(2^j x, 2^j y) + \lim_{j\to\infty}\frac{1}{2^j}f(2^j x, 0) + \frac{1}{3}f(0,0)$$

for all $x, y \in X$.

Proof. Let f', B_1, B_2 be as in Lemma 3.3. Using Lemma 3.3 and (3.2), we get

$$\begin{aligned}
\|\frac{f(2^n x, 0) - f(0,0)}{2^n} - \frac{f(2^{n+1}x, 0) - f(0,0)}{2^{n+1}}\| &= \|\frac{B_1(2^n x, 2^n y)}{2^n}\|\\
&\le \frac{\varphi_1(2^n x, 2^n y)}{2^n},\\
\|\frac{f'(2^n x, 2^n y)}{4^n} - \frac{f'(2^{n+1}x, 2^{n+1}y)}{4^{n+1}}\| &= \|\frac{B_2(2^n x, 2^n y)}{4^n}\|\\
&\le \frac{\varphi_2(2^n x, 2^n y)}{4^n},\\
\|\frac{f(0, 2^n y) - f(0,0)}{2^n} - \frac{f(0, 2^{n+1}y) - f(0,0)}{2^{n+1}}\| &= \|\frac{B_3(2^n x, 2^n y)}{2^n}\|\\
&\le \frac{\varphi_3(2^n x, 2^n y)}{2^n}
\end{aligned}$$

for all $x, y \in X\backslash A$, where

$$\varphi_3(x,y) = \frac{1}{4}[\varphi(x,-x,y,y)+\varphi(x,-x,y,-y) +\varphi(x,-x,3y,y)+\varphi(x,-x,3y,-y)].$$

For given integers l, m $(0 \le l < m)$, the inequalities

$$\|\frac{f(2^l x,0)-f(0,0)}{2^l} - \frac{f(2^m x,0)-f(0,0)}{2^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_1(2^j x, 2^j y)}{2^j}, \tag{3.4}$$

$$\|\frac{f'(2^l x,2^l y)}{4^l} - \frac{f'(2^m x,2^m y)}{4^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_2(2^j x, 2^j y)}{4^j}, \tag{3.5}$$

$$\|\frac{f(0,2^l y)-f(0,0)}{2^l} - \frac{f(0,2^m y)-f(0,0)}{2^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_3(2^j x, 2^j y)}{2^j}$$

hold for all $x, y \in X\backslash A$. By the above inequalities and (3.1), the sequences $\{\frac{f(2^n x,0)-f(0,0)}{2^n}\}$, $\{\frac{f(0,2^n y)-f(0,0)}{2^n}\}$, $\{\frac{f'(2^n x,2^n y)}{4^n}\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{f(2^n x,0)-f(0,0)}{2^n}\}$, $\{\frac{f(0,2^n y)-f(0,0)}{2^n}\}$, $\{\frac{f'(2^n x,2^n y)}{4^n}\}$ converge for all $x, y \in X\backslash A$. Since

$$\lim_{j\to\infty} \frac{f(2^j x,0)}{4^j} = 0, \quad \lim_{j\to\infty} \frac{f(0,2^j y)}{4^j} = 0,$$

for all $x, y \in X$, we can define $F_1, F_2 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \tfrac{f(2^j x,0)}{2^j}, \quad F_2(x,y) := \lim_{j\to\infty} \tfrac{f(2^j x,2^j y)}{4^j}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.4) and (3.5), one can obtain the inequalities

$$\|f(x,0)-f(0,0)-F_1(x,y)\| \le \textstyle\sum_{j=0}^{\infty} \frac{\varphi_1(2^j x,2^j y)}{2^j},$$
$$\|f(x,y)-f(x,0)+\tfrac{2}{3}f(0,0)-F_2(x,y)\| \le \textstyle\sum_{j=0}^{\infty} \frac{\varphi_2(2^j x,2^j y)}{4^j}$$

for all $x, y \in X\backslash A$. Using (3.2) and the definitions of F_1, F_2, we have

$$\begin{aligned} CF_1(x,y,z,w) &= \lim_{n\to\infty} \frac{1}{2^{n+1}}(2Cf(2^n x,2^n y,2^n z,-2^n z) \\ &\quad -Cf(2^n x,2^n x,2^n z,-2^n z) - Cf(2^n y,2^n y,2^n z,-2^n z)) = 0, \\ CF_2(x,y,z,w) &= \lim_{n\to\infty} \frac{Cf(2^n x,2^n y,2^n z,2^n w)}{4^n} = 0 \end{aligned}$$

for all $x, y, z, w \in X\backslash A$. Since

$$\|f(x,y) - \frac{1}{3}f(0,0) - F(x,y)\| \le \|f'(x,y) - F_2(x,y)\| + \|f(x,0)-f(0,0)-F_1(x,y)\|,$$

F is a Cauchy-Jensen mapping satisfying (3.3) by Lemma 3.1, where

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (3.3). By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&= \ \|\frac{(F-F')(2^n x, 2^n y)}{4^n} \\
&+\frac{1}{2}(\frac{1}{2^n} - \frac{1}{4^n})[(F-F')(2^n x, -2^n y) + (F-F')(2^n x, 2^n y)]\| \\
&\le \ \frac{\|(F-f)(2^n x, 2^n y) + \frac{1}{3}f(0,0)\| + \|(f-F')(2^n x, 2^n y) - \frac{1}{3}f(0,0)\|}{4^n} \\
&+\frac{\|(F-f)(2^n x, -2^n y) + \frac{1}{3}f(0,0)\| + \|(f-F')(2^n x, -2^n y) - \frac{1}{3}f(0,0)\|}{2^{n+1}} \\
&+\frac{\|(F-f)(-2^n x, 2^n y) + \frac{1}{3}f(0,0)\| + \|(f-F')(-2^n x, 2^n y) - \frac{1}{3}f(0,0)\|}{2^{n+1}} \\
&\le \ \frac{2}{4^n}\bar{\varphi}(2^n x, 2^n y) + \frac{1}{2^n}[\bar{\varphi}(2^n x, -2^n y) + \bar{\varphi}(2^n x, 2^n y)]
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X \backslash A$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X \backslash A$. By Lemma 3.2, $F(x,y) = F'(x,y)$ for all $x, y \in X$ as we desired. □

Corollary 3.5. *Let X be a normed space and $B = \{x \in X | \|x\| \le 1\}$. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|Cf(x,y,z,w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X \backslash B$ with fixed real numbers $p < 1$ and $\theta > 0$, then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
\|f(x,y) - F(x,y)\| \ \le \ &(\frac{3+3^p}{2^p(2-2^p)} + \frac{2(3 + 2\cdot 2^p + 3^p)}{2^p(4-2^p)})\theta\|x\|^p + (\frac{4}{2-2^p} \\
&+\frac{8}{4-2^p})\theta\|y\|^p
\end{aligned}$$

for all $x, y \in X \backslash B$.

Theorem 3.6. *Let A be as in Theorem 3.4 and let $\varphi : X \times X \times X \times X \to [0, \infty)$ a function such that*

$$\lim_{(x,y,z,w)\to\infty} \varphi(x,y,z,w) = 0$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (3.2) for all $x, y, z, w \in X \backslash A$. Then f is a Cauchy-Jensen mapping.

Proof. Let $\bar{\varphi}, F$ be as in Theorem 3.4. Using (3.2), (3.3) and the equality

$$\begin{aligned}
\frac{f(0,0)}{3} - F(0,0) \ &= \tfrac{1}{2}[Cf(kx, -kx, ky, -ky) + (f-F)(-kx, -ky) - \tfrac{f(0,0)}{3} \\
&+(f-F)(kx, -ky) - \tfrac{f(0,0)}{3} + (f-F)(kx, ky) - \tfrac{f(0,0)}{3} \\
&+(f-F)(-kx, ky) - \tfrac{f(0,0)}{3} - CF(kx, -kx, ky, -ky)]
\end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$, we get

$$\begin{aligned}\|\frac{f(0,0)}{3} - F(0,0)\| &\leq \tfrac{1}{2}[\varphi(kx,-kx,ky,-ky) + \bar{\varphi}(-kx,-ky) \\ &\quad +\bar{\varphi}(kx,-ky) + \bar{\varphi}(kx,ky) + \bar{\varphi}(-kx,ky)]\end{aligned}$$

for all $x, y \neq 0$ with $kx, ky \notin A$. As $n \to \infty$, we have

$$f(0,0) = 3F(0,0) = 0$$

and

$$\|f(x,y) - F(x,y)\| \leq \ \bar{\varphi}(x,y)$$

for all $x, y \in X \backslash A$. Similarly, using (3.5) and the inequalities

$$\begin{aligned}
f(x,y) - F(x,y) &= \frac{1}{2}[Cf((k+1)x,-kx,(k+2)y,-ky) \\
&\quad +(f-F)(-kx,-ky) + (f-F)((k+1)x,(k+2)y) \\
&\quad +(f-F)((k+1)x,-ky) + (f-F)(-kx,(k+2)y) \\
&\quad -CF((k+1)x,-kx,(k+2)y,-ky)], \\
f(x,0) - F(x,0) &= \frac{1}{2}[Cf((k+1)x,-kx,ky,-ky) + (f-F)(-kx,-ky) \\
&\quad +(f-F)((k+1)x,-ky) + (f-F)((k+1)x,ky) \\
&\quad +(f-F)(-kx,ky) - CF((k+1)x,-kx,ky,-ky)], \\
f(0,y) - F(0,y) &= \frac{1}{2}[Cf(kx,-kx,(k+2)y,-ky) + (f-F)(-kx,-ky) \\
&\quad +(f-F)(kx,-ky) - CF(kx,-kx,(k+2)y,-ky) \\
&\quad +(f-F)(kx,(k+2)y) + (f-F)(-kx,(k+2)y)]
\end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$, we easily get

$$f(x,y) = F(x,y), \quad f(x,0) = F(x,0), \quad f(0,y) = F(0,y)$$

for all $x, y \neq 0$ as we desired. □

Corollary 3.7. *Let $p < 0$ and let $f : X \times X \to Y$ be as in Corollary* 3.5. *Then f is a Cauchy-Jensen mapping.*

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