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ON THE DIAMOND-ALPHA RIEMANN INTEGRAL AND MEAN VALUE THEOREMS ON TIME SCALES

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ABSTRACT. We study diamond-alpha integrals on time scales. A diamond-alpha version of Fermat's theorem for stationary points is also proved, as well as Rolle's, Lagrange's, and Cauchy's mean value theorems on time scales.

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1. INTRODUCTION

The calculus on time scales has been initiated by Aulbach and Hilger in order to create a theory that can unify and extend discrete and continuous analysis [1]. Two versions of the calculus on time scales, the delta and nabla calculus, are now standard in the theory of time scales [4, 5]. In 2006, a combined diamond-alpha dynamic derivative was introduced by Sheng, Fadag, Henderson and Davis [9], as a linear combination of the delta and nabla dynamic derivatives on time scales. The diamond-alpha derivative reduces to the standard delta-derivative for $\alpha = 1$ and to the standard nabla derivative for $\alpha = 0$. On the other hand, it represents a weighted dynamic derivative formula on any uniformly discrete time scale when $\alpha = \frac{1}{2}$. We refer the reader to [6–10] for a complete account of the recent diamond-alpha calculus on time scales. In Section 2 we briefly review the necessary definitions and calculus on time scales; our results are given in Sections 3, 4, and 5.

The diamond-alpha integral on time scales is defined in [7–10] by means of a linear combination of the delta and nabla integrals. In the present paper we use a Darboux approach to define the Riemann diamond-alpha integral on time scales and to prove the corresponding main theorems of the diamond-alpha integral calculus (Section 3). In addition, we briefly investigate diamond-alpha improper integrals (Section 4), and prove some new versions of mean value theorems on time scales via diamond-α derivatives and integrals (Section 5). A new notion of local extremum on time scales is also proposed, which leads to a diamond-alpha Fermat's theorem for

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stationary points (Theorem 5.4) more similar in aspect to the classical condition than those of delta or nabla derivatives.

2. PRELIMINARIES ON TIME SCALES

In this section we introduce basic definitions and results from the theory of delta, nabla, and diamond-alpha time scales [4, 7, 9].

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and is denoted by $\mathbb{T}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M), and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively.

Throughout the paper we let $\mathbb{T} = [a, b] \cap \mathbb{T}_0$ with $a < b$ and $\mathbb{T}_0$ a time scale containing a and b.

The *delta graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T};$$

the *nabla graininess function* is defined by $\eta(t) := t - \rho(t)$.

We introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, and $\mathbb{T}^k_k$, which are derived from the time scale $\mathbb{T}$, as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, we define $\mathbb{T}^k_k = \mathbb{T}^k \cap \mathbb{T}_k$.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^k$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t and say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

We define $f^\nabla(t)$ to be the number value, if one exists, such that for all $\epsilon > 0$, there is a neighborhood V of t such that for all $s \in V$,

$$\left|f(\rho(t)) - f(s) - f^\nabla(t)\,(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|.$$

We say that f is nabla differentiable on $\mathbb{T}_k$, provided that $f^\nabla(t)$ exists for all $t \in \mathbb{T}_k$.

For delta differentiable functions f and g, the next formula holds:

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t)\\ &= f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t),\end{aligned}$$

where we abbreviate here and throughout the text $f \circ \sigma$ by f^{σ}. Similarly property holds for nabla derivatives (and we then use the notation $f^{\rho} = f \circ \rho$).

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be a *regulated* function if its left-sided limits exist at left-dense points, and its right-sided limits exist at right-dense points.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by $\mathrm{C_{rd}}$ and the set of all delta differentiable functions with rd-continuous derivative by $\mathrm{C^1_{rd}}$.

Analogously, a function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist finite at all right-dense points in $\mathbb{T}$.

It is known that rd-continuous functions possess a *delta antiderivative*, i.e., there exists a function F with $F^{\Delta} = f$, and in this case the *delta integral* is defined by $\int_c^d f(t)\Delta t = F(c) - F(d)$ for all $c, d \in \mathbb{T}$. The delta integral has the following property:

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a *nabla antiderivative* of $g : \mathbb{T} \to \mathbb{R}$, provided $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then, the nabla integral of g is defined by $\int_a^b g(t)\nabla t = G(b) - G(a)$.

Let $\mathbb{T}$ be a time scale, and t, $s \in \mathbb{T}$. Following [7], we define $\mu_{ts} = \sigma(t) - s$, $\eta_{ts} = \rho(t) - s$, and $f^{\diamond_\alpha}(t)$ to be the value, if one exists, such that for all $\epsilon > 0$ there is a neighborhood U of t such that for all $s \in U$

$$\left|\alpha\left[f^{\sigma}(t) - f(s)\right]\eta_{ts} + (1-\alpha)\left[f^{\rho}(t) - f(s)\right]\mu_{ts} - f^{\diamond_\alpha}(t)\mu_{ts}\eta_{ts}\right| < \epsilon\left|\mu_{ts}\eta_{ts}\right| .$$

A function f is said diamond-α differentiable on $\mathbb{T}^k_k$ provided $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}^k_k$. Let $0 \le \alpha \le 1$. If $f(t)$ is differentiable on $t \in \mathbb{T}^k_k$ both in the delta and nabla senses, then f is diamond-α differentiable at t and the dynamic derivative $f^{\diamond_\alpha}(t)$ is given by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t) \tag{2.1}$$

(see [7, Theorem 3.2]). Equality (2.1) is given as definition of $f^{\diamond_\alpha}(t)$ in [9]. The diamond-α derivative reduces to the standard Δ derivative for $\alpha = 1$, or the standard ∇ derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" for $\alpha \in (0, 1)$. Furthermore, the combined dynamic derivative offers a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $\alpha = \frac{1}{2}$.

Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}^k_k$. Then (cf. [9, Theorem 2.3]),

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}^k_k$ with

$$(f+g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + g^{\diamond_\alpha}(t);$$

(ii) For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}^k_k$ with

$$(cf)^{\diamond_\alpha}(t) = cf^{\diamond_\alpha}(t);$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}^k_k$ with

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

Let $a, b \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. The diamond-$\alpha$ integral of h from a to b is defined in [7, 9] by

$$\int_a^b h(\tau) \diamond_\alpha \tau = \alpha \int_a^b h(\tau)\Delta\tau + (1-\alpha)\int_a^b h(\tau)\nabla\tau, \quad 0 \le \alpha \le 1, \tag{2.2}$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$. In Section 3 we introduce a more basic notion of diamond-α integral. We use a Darboux approach without the need to define previously delta and nabla integrals. Improper integrals are introduced in Section 4. We end with Section 5, proving some generalizations of the mean value theorems on time scales via diamond-α derivatives and integrals. Moreover, a new notion of local extremum on time scales is proposed, which leads to a diamond-alpha first order optimality condition more similar to the classical condition ($\mathbb{T} = \mathbb{R}$) than those of delta or nabla derivatives (cf. [4]).

3. THE RIEMANN DIAMOND-α INTEGRAL

Let $\mathbb{T}$ be a one-dimensional time scale, $a, b \in \mathbb{T}$, $a < b$ and $[a, b]$ a closed, bounded interval in $\mathbb{T}$. A partition of $[a, b]$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a, b],$$

where $a = t_0 < t_1 < \cdots < t_n = b$. The number n depends on the particular partition, so we have $n = n(P)$. We denote by $\mathcal{P} = \mathcal{P}(a, b)$ the set of all partitions of $[a, b]$. Let f be a real-valued bounded function on $[a, b]$. We set:

$$\overline{M} = \sup\{f(t) : t \in [a, b)\}, \quad \overline{m} = \inf\{f(t) : t \in [a, b)\},$$

$$\underline{M} = \sup\{f(t) : t \in (a, b]\}, \quad \underline{m} = \inf\{f(t) : t \in (a, b]\},$$

and for $1 \le i \le n$

$$\overline{M_i} = \sup\{f(t) : t \in [t_{i-1}, t_i)\}, \quad \overline{m_i} = \inf\{f(t) : t \in [t_{i-1}, t_i)\},$$

$$\underline{M_i} = \sup\{f(t) : t \in (t_{i-1}, t_i]\}, \quad \underline{m_i} = \inf\{f(t) : t \in (t_{i-1}, t_i]\}.$$

Let $\alpha \in [0,1] \subset \mathbb{R}$. The upper Darboux $\diamond_\alpha$-sum $U(f,P)$ and the lower Darboux $\diamond_\alpha$-sum $L(f,P)$ of f with respect to P are defined respectively by

$$U(f,P) = \sum_{i=1}^{n} (\alpha \overline{M_i} + (1-\alpha)\underline{M_i})(t_i - t_{i-1}) \,,$$

$$L(f,P) = \sum_{i=1}^{n} (\alpha \overline{m_i} + (1-\alpha)\underline{m_i})(t_i - t_{i-1}) \,.$$

Note that

$$U(f,P) \leq \sum_{i=1}^{n} (\alpha \overline{M} + (1-\alpha)\underline{M})(t_i - t_{i-1}) = (\alpha \overline{M} + (1-\alpha)\underline{M})(b-a)$$

and

$$L(f,P) \geq \sum_{i=1}^{n} (\alpha \overline{m} + (1-\alpha)\underline{m})(t_i - t_{i-1}) = (\alpha \overline{m} + (1-\alpha)\underline{m})(b-a) \,.$$

Thus, we have:

$$(\alpha \overline{m} + (1-\alpha)\underline{m})(b-a) \leq L(f,P) \leq U(f,P) \leq (\alpha \overline{M} + (1-\alpha)\underline{M})(b-a). \tag{3.1}$$

The upper Darboux $\diamond_\alpha$-integral $U(f)$ of f from a to b is defined by

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}(a,b)\}$$

and the lower Darboux $\diamond_\alpha$-integral $L(f)$ of f from a to b is defined by

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}(a,b)\} \,.$$

In view of (3.1), $U(f)$ and $L(f)$ are finite real numbers.

Definition 3.1. We say that f is $\diamond_\alpha$-integrable from a to b (or on $[a,b]$) provided $L(f) = U(f)$. In this case, we write $\int_a^b f(t) \diamond_\alpha t$ for this common value. We call this integral the Darboux $\diamond_\alpha$-integral.

Let $\overline{U}(f)$ and $\overline{L}(f)$ denote the upper and the lower Darboux Δ-integral of f from a to b, respectively; $\underline{U}(f)$ and $\underline{L}(f)$ denote the upper and the lower Darboux ∇-integral of f from a to b, respectively. Given the construction of $U(f)$ and $L(f)$, the equality (2.2) follows from the properties of supremum and infimum.

Corollary 3.2. *If f is Δ-integrable from a to b and ∇-integrable from a to b, then it is $\diamond_\alpha$-integrable from a to b and*

$$\int_a^b f(t) \diamond_\alpha t = \alpha \int_a^b f(t)\Delta t + (1-\alpha) \int_a^b f(t) \nabla t.$$

Now, suppose that f is $\diamond_\alpha$-integrable from a to b. Then $L(f) = U(f)$ and

$$\alpha\overline{U}(f) + (1-\alpha)\underline{U}(f) = \alpha\overline{L}(f) + (1-\alpha)\underline{L}(f),$$
$$\alpha(\overline{U}(f) - \overline{L}(f)) = (1-\alpha)(\underline{L}(f) - \underline{U}(f)).$$

Since $\overline{U}(f) \geq \overline{L}(f)$ and $\underline{U}(f) \geq \underline{L}(f)$, we get the following result.

Corollary 3.3. *Let f be $\diamond_\alpha$-integrable from a to b.*

(i) If $\alpha = 1$, then f is Δ-integrable from a to b.

(ii) If $\alpha = 0$, then f is ∇-integrable from a to b.

(iii) If $0 < \alpha < 1$, then f is Δ-integrable and ∇-integrable from a to b.

Example 3.4. Note that the strict inequalities in (iii) of the above corollary are necessary. Consider the function $f(t) = \frac{1}{t}$ and the time scale $\mathbb{T} = \mathbb{Z}$. We have $\int_{-1}^{0} f(t)\Delta t = -1$ and $\int_{0}^{1} f(t)\nabla t = 1$. However, both $\int_{0}^{1} f(t)\Delta t$ and $\int_{-1}^{0} f(t)\nabla t$ do not exist.

The following theorems may be showed in the same way as Theorem 5.5 and Theorem 5.6 in [2].

Theorem 3.5. *If $U(f,P) = L(f,P)$ for some $P \in \mathcal{P}(a,b)$, then the function f is $\diamond_\alpha$-integrable from a to b and $\int_a^b f(t) \diamond_\alpha t = U(f,P) = L(f,P)$.*

Theorem 3.6. *A bounded function f on $[a,b]$ is $\diamond_\alpha$-integrable if and only if for each $\varepsilon > 0$ there exists $P \in \mathcal{P}(a,b)$ such that $U(f,P) - L(f,P) < \varepsilon$.*

Lemma 3.7 ([2]). *For every $\delta > 0$ there exists some partition $P \in \mathcal{P}(a,b)$ given by $a = t_0 < t_1 < \cdots < t_n = b$ such that for each $i \in \{1,2,\ldots,n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$.*

We denote by $\mathcal{P}_\delta = \mathcal{P}_\delta(a,b)$ the set of all $P \in \mathcal{P}(a,b)$ that possess the property indicated in Lemma 3.7.

Theorem 3.8. *A bounded function f on $[a,b]$ is $\diamond_\alpha$-integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P \in \mathcal{P}_\delta(a,b) \Rightarrow U(f,P) - L(f,P) < \varepsilon\,. \tag{3.2}$$

Proof. By Theorem 3.6 condition (3.2) implies integrability. Conversely, suppose that f is $\diamond_\alpha$-integrable from a to b. If $\alpha = 1$ or $\alpha = 0$, then f is Δ-integrable from a to b or ∇-integrable from a to b. Therefore condition (3.2) holds (see [2]). Now, let $0 < \alpha < 1$. By Corollary 3.3, f is Δ-integrable and ∇-integrable from a to b. According to [2, Theorem 5.9], for each $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $P_1 \in \mathcal{P}_{\delta_1}(a,b)$ implies $\overline{U}(f,P_1) - \overline{L}(f,P_1) < \varepsilon$ and $P_2 \in \mathcal{P}_{\delta_2}(a,b)$ implies $\underline{U}(f,P_2) - \underline{L}(f,P_2) < \varepsilon$. If $P \in \mathcal{P}_\delta(a,b)$ where $\delta = min\{\delta_1,\delta_2\}$, then we have $U(f,P) - L(f,P) = \alpha\overline{U}(f,P) + (1-\alpha)\underline{U}(f,P) - \alpha\overline{L}(f,P) - (1-\alpha)\underline{L}(f,P) < \varepsilon$. □

Let f be a bounded function on $[a,b]$ and let $P \in \mathcal{P}(a,b)$ be given by $a = t_0 < t_1 < \cdots < t_n = b$. For $1 \leq i \leq n$, choose arbitrary points $\overline{\xi}_i$ in $[t_{i-1}, t_i)$, $\underline{\xi}_i$ in $(t_{i-1}, t_i]$, and form the sum

$$S = \Sigma_{i=1}^{n}(\alpha f(\overline{\xi}_i) + (1-\alpha) f(\underline{\xi}_i))(t_i - t_{i-1})\,.$$

We call S a Riemann $\diamond_\alpha$-sum of f corresponding to $P \in \mathcal{P}(a,b)$. We say that f is Riemann $\diamond_\alpha$-integrable from a to b if there exists a real number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $P \in \mathcal{P}_\delta(a,b)$ implies $|S - I| < \varepsilon$ independent of the choice of $\overline{\xi}_i$, $\underline{\xi}_i$ for $1 \leq i \leq n$. The number I is called the Riemann $\diamond_\alpha$-integral of f from a to b.

Theorem 3.9. *If f is Riemann Δ-integrable and Riemann ∇-integrable from a to b, then it is Riemann $\diamond_\alpha$-integrable from a to b and $I = \alpha \int_a^b f(t)\Delta t + (1-\alpha)\int_a^b f(t)\nabla t$.*

Proof. Assume that f is Riemann Δ-integrable and Riemann ∇-integrable from a to b. Then, for each $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $P_1 \in \mathcal{P}_{\delta_1}(a,b)$ implies $|\overline{S} - \int_a^b f(t)\Delta t| < \varepsilon$, and $P_2 \in \mathcal{P}_{\delta_2}(a,b)$ implies $|\underline{S} - \int_a^b f(t)\nabla t| < \varepsilon$, where $\overline{S}$ is the Riemann Δ-sum of f corresponding to P_1, and $\underline{S}$ is the Riemann ∇-sum of f corresponding to P_2. Now, if $P \in \mathcal{P}_\delta(a,b)$ with $\delta = min\{\delta_1, \delta_2\}$, then we have

$$\begin{aligned}
&\left|S - \alpha\int_a^b f(t)\Delta t - (1-\alpha)\int_a^b f(t)\nabla t\right| \\
&\qquad = \left|\alpha\overline{S} + (1-\alpha)\underline{S} - \alpha\int_a^b f(t)\Delta t - (1-\alpha)\int_a^b f(t)\nabla t\right| \\
&\qquad \leq \left|\alpha\overline{S} - \alpha\int_a^b f(t)\Delta t\right| + \left|(1-\alpha)\underline{S} - (1-\alpha)\int_a^b f(t)\nabla t\right| \\
&\qquad \leq \varepsilon\,.
\end{aligned}$$

Thus, f is Riemann $\diamond_\alpha$-integral from a to b and $I = \alpha\int_a^b f(t)\Delta t + (1-\alpha)\int_a^b f(t)\nabla t$. □

By construction of the $\diamond_\alpha$-Riemann sum S, the following theorem may be proved in much the same way as [2, Theorem 5.11].

Theorem 3.10. *A bounded function f on $[a,b]$ is Riemann $\diamond_\alpha$-integrable if and only if it is Darboux $\diamond_\alpha$-integrable, in which case the values of the integrals are equal.*

We define

$$\int_a^a f(t) \diamond_\alpha t = 0$$

and

$$\int_a^b f(t) \diamond_\alpha t = -\int_b^a f(t) \diamond_\alpha t,\ \ a > b.$$

Corollary 3.11. *Let f be Riemann $\diamond_\alpha$-integrable from a to b.*

(i) If $\alpha = 1$, then f is Riemann Δ-integrable from a to b.

(ii) If $\alpha = 0$, then f is Riemann ∇-integrable from a to b.

(iii) If $0 < \alpha < 1$, then f is is Riemann Δ-integrable and Riemann ∇-integrable from a to b.

Theorem 3.12. *Let $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$. Then,*

(i) f is integrable from t to $\sigma(t)$ and

$$\int_t^{\sigma(t)} f(s) \diamond_\alpha s = \mu(t)(\alpha f(t) + (1-\alpha) f^\sigma(t)) ; \tag{3.3}$$

(ii) f is integrable from $\rho(t)$ to t and

$$\int_{\rho(t)}^{t} f(s) \diamond_\alpha s = \eta(t)(\alpha f^\rho(t) + (1-\alpha) f(t)).$$

Proof. (i) If $\sigma(t) = t$, then $\mu(t) = 0$ and equality (3.3) is obvious. If $\sigma(t) > t$, then $\mathcal{P}(t, \sigma(t))$ contains only one element given by $t = s_0 < s_1 = \sigma(t)$. Since $[s_0, s_1) = t$ and $(s_0, s_1] = \sigma(t)$, we have $U(f, P) = \alpha f(t)(\sigma(t) - t) + (1-\alpha) f^\sigma(t)(\sigma(t) - t) = L(f, P)$. By Theorem 3.5, f is $\diamond_\alpha$-integrable from t to $\sigma(t)$ and (3.3) holds. Proof of (ii) is done in a similar way. □

Corollary 3.13. *Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:*

(i) If $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[a, b]$ is $\diamond_\alpha$-integrable from a to b if and only if is Riemann integrable on $[a, b]$ in the classical sense, and in this case $\int_a^b f(t) \diamond_\alpha t = \int_a^b f(t)dt$.

(ii) If $\mathbb{T} = \mathbb{Z}$, then every function f defined on $\mathbb{Z}$ is $\diamond_\alpha$-integrable from a to b, and $\int_a^b f(t) \diamond_\alpha t = \sum_{t=a+1}^{b-1} f(t) + \alpha f(a) + (1-\alpha) f(b)$.

(iii) If $\mathbb{T} = h\mathbb{Z}$, then then every function f defined on $h\mathbb{Z}$ is $\diamond_\alpha$-integrable from a to b, and $\int_a^b f(t) \diamond_\alpha t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} f(kh)h + \alpha f(a)h + (1-\alpha) f(b)h$.

The following results are straightforward consequences of Theorems 3.9 and 3.10 and properties of the Riemann delta (nabla) integral:

1. Let $a, b \in \mathbb{T}$ and $a < b$. Every constant function $f : \mathbb{T} \to \mathbb{R}$ is $\diamond_\alpha$-integrable from a to b and $\int_a^b f(t) \diamond_\alpha t = c(b - a)$.
2. Every monotone function $f : \mathbb{T} \to \mathbb{R}$ on $[a, b]$ is $\diamond_\alpha$-integrable from a to b.
3. Every continuous function $f : \mathbb{T} \to \mathbb{R}$ on $[a, b]$ is $\diamond_\alpha$-integrable from a to b.
4. Every bounded function $f : \mathbb{T} \to \mathbb{R}$ on $[a, b]$ with only finitely many discontinuity points is $\diamond_\alpha$-integrable from a to b.
5. Every regulated function $f : \mathbb{T} \to \mathbb{R}$ on $[a, b]$ is $\diamond_\alpha$-integrable from a to b.
6. Let $f : \mathbb{T} \to \mathbb{R}$ be a bounded function on $[a, b]$ that is $\diamond_\alpha$-integrable from a to b. Then, f is $\diamond_\alpha$-integrable on every subinterval $[c, d]$ of $[a, b]$.

7. Let f, g be $\diamond_\alpha$-integrable from a to b and $c \in \mathbb{R}$, $d \in \mathbb{T}$, $a < d < b$. Then:
   (i) cf is $\diamond_\alpha$-integrable from a to b and $\int_a^b (cf)(t) \diamond_\alpha t = c \int_a^b f(t) \diamond_\alpha t$,
   (ii) $f+g$ is $\diamond_\alpha$-integrable from a to b and $\int_a^b (f+g)(t) \diamond_\alpha t = \int_a^b f(t) \diamond_\alpha t + \int_a^b g(t) \diamond_\alpha t$,
   (iii) fg is $\diamond_\alpha$-integrable from a to b,
   (iv) $\int_a^b f(t) \diamond_\alpha t = \int_a^d f(t) \diamond_\alpha t + \int_d^b f(t) \diamond_\alpha t$.
8. If f, g are $\diamond_\alpha$-integrable from a to b and if $f(t) \leq g(t)$ for all $t \in [a, b]$, then $\int_a^b f(t) \diamond_\alpha t \leq \int_a^b f(t) \diamond_\alpha t$.
9. If f is $\diamond_\alpha$-integrable from a to b, then so is $|f|$. Moreover, $|\int_a^b f(t) \diamond_\alpha t| \leq \int_a^b |f(t)| \diamond_\alpha t$.

4. IMPROPER INTEGRALS

Let $\mathbb{T}$ be a time scale, $a \in \mathbb{T}$. Throughout this section we assume that there exists a subset

$$\{t_k : k \in \mathbb{N}_0\} \subset \mathbb{T}, \quad a = t < t_1 < t_2 < \cdots, \quad \lim_{k\to\infty} t_k = \infty.$$

Let us suppose that the real-valued function f is defined on $[a, \infty)$ and is Riemann $\diamond_\alpha$-integrable from a to any point $A \in \mathbb{T}$ with $A \geq a$. If the integral

$$F(A) = \int_a^A f(t) \diamond_\alpha t$$

approaches a finite limit as $A \to \infty$, we call that limit the improper diamond-α integral of first kind of f from a to ∞ and write

$$\int_a^\infty f(t) \diamond_\alpha t = \lim_{A\to\infty} \left(\int_a^A f(t) \diamond_\alpha t \right). \tag{4.1}$$

In such case we say that the improper integral $\int_a^\infty f(t) \diamond_\alpha t$ exists or that it is convergent. If the limit (4.1) does not exist, we say that the improper integral does not exist or is divergent. Note that

$$\int_a^\infty f(t) \diamond_\alpha t = \lim_{A\to\infty} \left(\alpha \int_a^A f(t) \triangle t + (1-\alpha) \int_a^A f(t) \nabla t \right).$$

Corollary 4.1. *If improper integrals $\int_a^\infty f(t) \triangle t$ and $\int_a^\infty f(t) \nabla t$ exist, then*

$$\int_a^\infty f(t) \diamond_\alpha t = \alpha \int_a^\infty f(t) \triangle t + (1-\alpha) \int_a^\infty f(t) \nabla t.$$

On the other hand, the improper diamond-α integral may exist even if improper delta and nabla integrals do not exist.

Example 4.2. Consider the function

$$f(t) = \begin{cases} 1 & \text{if } t = 2l \\ -1 & \text{if } t = 2l+1, \end{cases} \qquad l \in \mathbb{T},$$

on the time scale $\mathbb{T} = \mathbb{N} \cup \{0\}$. We have

$$\int_0^{\infty} f(t)\triangle t = \sum_{k=0}^{\infty} f(k)\,, \quad \int_0^{\infty} f(t)\nabla t = \sum_{k=0}^{\infty} f(k+1).$$

Hence, the improper integrals $\int_0^\infty f(t)\triangle t$ and $\int_0^\infty f(t)\nabla t$ do not exist. On the other hand, for $\alpha = 1/2$ we have

$$\int_a^A f(t)\diamond_\alpha t = \frac{1}{2}\left(\sum_{k=0}^{A-1} f(k) + \sum_{k=0}^{A-1} f(k+1)\right) = \frac{1}{2}\sum_{k=0}^{A-1}(f(k)+f(k+1)) = 0$$

and

$$\int_a^{\infty} f(t)\diamond_\alpha t = \lim_{A\to\infty}\left(\int_a^A f(t)\diamond_\alpha t\right) = 0.$$

5. FERMAT'S AND MEAN VALUE THEOREMS

Theorem 5.1 and Theorem 5.2 are exact analogies of mean value theorems for delta (nabla) integrals and there are no differences in proofs of these theorems. However, the formulation of the mean value theorems 5.6 and 5.7 below, for the diamond-α derivative, provide a generalization more similar to the classical results than the ones previously proved for the delta or nabla derivatives (cf. [3]).

Theorem 5.1. *Let f and g be bounded and $\diamond_\alpha$-integrable functions from a to b, and let g be nonnegative (or nonpositive) on $[a,b]$. Let m and M be the infimum and supremum, respectively, of the function f on $[a,b]$. Then, there exists a real number K satisfying the inequalities $m \le K \le M$ such that*

$$\int_a^b f(t)g(t)\diamond_\alpha t = K\int_a^b g(t)\diamond_\alpha t.$$

Theorem 5.2. *Let f be bounded and $\diamond_\alpha$-integrable function on $[a,b]$. Let m and M be the infimum and supremum, respectively, of the function $F(s) = \int_a^t f(s)\diamond_\alpha s$ on $[a,b]$. We have:*

(i) If a function g is non-increasing with $g(t) \ge 0$ on $[a,b]$, then there is a number K such that $m \le K \le M$ and

$$\int_a^b f(t)g(t)\diamond_\alpha t = Kg(a).$$

(ii) If g is any monotonic function on $[a,b]$, then there is a number K such that $m \le K \le M$ and

$$\int_a^b f(t)g(t)\diamond_\alpha t = [g(a)-g(b)]K + g(b)\int_a^b f(t)\diamond_\alpha t.$$

We now define a local maximum of a real function defined on a time scale $\mathbb{T}$. The definition of a local minimum is done in a similar way.

Definition 5.3. We say that a function $f : \mathbb{T} \to \mathbb{R}$ assumes its local maximum at $t_0 \in \mathbb{T}_k^k$ provided
(i) if t_0 is scattered, then $f(\sigma(t_0)) \leq f(t_0)$ and $f(\rho(t_0)) \leq f(t_0)$;
(ii) if t_0 is dense, then there is a neighborhood U of t_0 such that $f(t) \leq f(t_0)$ for all $t \in U$;
(iii) if t_0 is left-scattered and right-dense, then $f(\rho(t_0)) \leq f(t_0)$ and there is a neighborhood U of t_0 such that $f(t) \leq f(t_0)$ for all $t \in U$ with $t > t_0$;
(iv) if t_0 is right-scattered and left-dense, then $f(\sigma(t_0)) \leq f(t_0)$ and there is a neighborhood U of t_0 such that $f(t) \leq f(t_0)$ for all $t \in U$ with $t < t_0$.

Theorem 5.4 permits to introduce the notion of critical point in a similar way as done in classical calculus. We remark that equality $f^{\diamond_\alpha}(t_0) = 0$ in Theorem 5.4 does not always hold for the delta ($\alpha = 1$) or nabla ($\alpha = 0$) cases (cf. [3]).

Theorem 5.4 (diamond-alpha Fermat's theorem for stationary points)**.** *Suppose f assumes its local extremum at $t_0 \in \mathbb{T}_k^k$ and f is delta and nabla differentiable at t_0. Then, there exists $\alpha \in [0,1]$ such that $f^{\diamond_\alpha}(t_0) = 0$.*

Proof. Suppose that f assumes its local maximum at $t_0 \in \mathbb{T}_k^k$. Then, we have $f^{\triangle}(t_0) \leq 0$ and $f^{\nabla}(t_0) \geq 0$. If $f^{\triangle}(t_0) = 0$ ($f^{\nabla}(t_0) = 0$), then we put $\alpha = 1$ ($\alpha = 0$). Therefore, we can assume that $f^{\triangle}(t_0) < 0$ and $f^{\nabla}(t_0) > 0$. Setting

$$\alpha = \frac{f^{\nabla}(t_0)}{f^{\nabla}(t_0) - f^{\triangle}(t_0)},$$

it is easy to see that $0 < \alpha < 1$, and we obtain the intended result. □

Example 5.5. Let $\mathbb{T} = \{-1, 0, 1, 2, 3, 4\}$, and f be defined by $f(-1) = f(0) = 5$, $f(1) = 0$, $f(2) = 1$, and $f(3) = f(4) = 3$. The point 1 is a local minimizer, points 0 and 3 are local maximizers, and point 2 is neither a minimizer neither a maximizer (as well as -1 and 4, by definition). The delta-derivative is only zero at point 3 while the nabla-derivative is zero only at zero. According with Theorem 5.4, at all extremizers there exists an alpha such that the diamond-alpha derivative vanishes. Indeed, $f^{\diamond_0}(0) = 0$, $f^{\diamond_{5/6}}(1) = 0$, and $f^{\diamond_1}(3) = 0$. Observe that $f^{\diamond_\alpha}(2) \neq 0$ for all $\alpha \in [0,1]$.

Theorem 5.6 (diamond-alpha Rolle's mean value theorem)**.** *Let f be a continuous function on $[a,b]$ that is delta and nabla differentiable on (a,b) with $f(a) = f(b)$. Then, there exists $\alpha \in [0,1]$ and $c \in (a,b)$ such that $f^{\diamond_\alpha}(c) = 0$.*

Proof. If $f = const$, then $f^{\diamond_\alpha}(c) = 0$ for all $\alpha \in [0,1]$ and $c \in (a,b)$. Hence, assume that f is not the constant function and $f(t) \geq f(a)$ for all $t \in [a,b]$. Since function f is continuous on the compact set $[a,b]$, f assumes its maximum $M > f(a)$. Therefore, there exists $c \in [a,b]$ such that $M = f(c)$. As $f(a) = f(b)$, $a < c < b$, clearly f assumes its local maximum at c and there exists $\alpha \in [0,1]$ such that $f^{\diamond_\alpha}(c) = 0$. □

The following mean value theorem is a generalization of Theorem 5.6.

Theorem 5.7 (diamond-alpha Lagrange's mean value theorem)**.** *Let f be a continuous function on $[a,b]$ that is delta and nabla differentiable on (a,b). Then, there exists $\alpha \in [0,1]$ and $c \in (a,b)$ such that*

$$f^{\diamond_\alpha}(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Consider the function g defined on $[a,b]$ by

$$g(t) = f(a) - f(t) + (t-a)\frac{f(b)-f(a)}{b-a}.$$

Clearly g is continuous on $[a,b]$ and $\triangle$ and ∇ differentiable on (a,b). Also $g(a) = g(b) = 0$. Hence, there exists $\alpha \in [0,1]$ and $c \in (a,b)$ such that $g^{\diamond_\alpha}(c) = 0$. Since

$$\begin{aligned} g^{\diamond_\alpha}(t) &= \alpha g^{\triangle}(t) + (1-\alpha)g^{\nabla}(t) \\ &= \alpha\left(-f^{\triangle}(t) + \frac{f(b)-f(a)}{b-a}\right) + (1-\alpha)\left(-f^{\nabla}(t) + \frac{f(b)-f(a)}{b-a}\right), \end{aligned}$$

we conclude that

$$0 = -\alpha f^{\triangle}(c) - (1-\alpha)f^{\nabla}(c) + \frac{f(b)-f(a)}{b-a} \Leftrightarrow f^{\diamond_\alpha}(c) = \frac{f(b)-f(a)}{b-a}.$$

□

We end by proving a diamond-alpha Cauchy mean value theorem, which is the more general form of the diamond-alpha mean value theorem.

Theorem 5.8 (diamond-alpha Cauchy's mean value theorem)**.** *Let f and g be continuous functions on $[a,b]$ that are delta and nabla differentiable on (a,b). Suppose that $g^{\diamond_\alpha}(t) \neq 0$ for all $t \in (a,b)$ and all $\alpha \in [0,1]$. Then, there exists $\bar{\alpha} \in [0,1]$ and $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{\diamond_{\bar{\alpha}}}(c)}{g^{\diamond_{\bar{\alpha}}}(c)}.$$

Proof. Let us first observe that from the condition $g^{\diamond_\alpha}(t) \neq 0$ for all $t \in (a,b)$ and all $\alpha \in [0,1]$ it follows from Theorem 5.6 that $g(b) \neq g(a)$. Hence, we may consider the function F defined on $[a,b]$ by

$$F(t) = f(t) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(t)-g(a)].$$

Clearly, F is continuous on $[a,b]$ and delta and nabla differentiable on (a,b). Also, $F(a) = F(b)$. Applying Theorem 5.6 to the function F and taking into account that

$$\begin{aligned} F^{\diamond_\alpha}(t) &= \alpha F^{\triangle}(t) + (1-\alpha)F^{\nabla}(t) \\ &= \alpha\left(f^{\triangle}(t) - \frac{f(b)-f(a)}{g(b)-g(a)}g^{\triangle}(t)\right) + (1-\alpha)\left(f^{\nabla}(t) - \frac{f(b)-f(a)}{g(b)-g(a)}g^{\nabla}(t)\right), \end{aligned}$$

we conclude that there exists $\bar{\alpha} \in [0,1]$ and $c \in (a,b)$ such that

$$0 = f^{\diamond_{\bar{\alpha}}}(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g^{\diamond_{\bar{\alpha}}}(c).$$

Hence,

$$f^{\diamond_{\bar{\alpha}}}(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g^{\diamond_{\bar{\alpha}}}(c)\,,$$

and dividing by $g^{\diamond_{\bar{\alpha}}}(c)$ we complete the proof. □

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