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TRIGONOMETRIC AND HYPERBOLIC SYSTEMS ON TIME SCALES

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ABSTRACT. In this paper we discuss trigonometric and hyperbolic systems on time scales. These systems generalize and unify their corresponding continuous-time and discrete-time analogies, namely the systems known in the literature as trigonometric and hyperbolic linear Hamiltonian systems and discrete symplectic systems. We provide time scale matrix definitions of the usual trigonometric and hyperbolic functions and show that many identities known from the basic calculus extend to this general setting, including the time scale differentiation of these functions.

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1. INTRODUCTION

In this paper we study trigonometric and hyperbolic systems on time scales and properties of their solutions, the time scale matrix trigonometric functions Sin, Cos, Tan, Cotan, and time scale matrix hyperbolic functions Sinh, Cosh, Tanh, Cotanh, which are all properly defined in this work. The trigonometric and hyperbolic systems from this paper generalize and unify their corresponding continuous-time and discrete-time analogies, namely the systems known in the literature as trigonometric and hyperbolic linear Hamiltonian systems and discrete symplectic systems. More precisely, the system of the form

$$\text{(CTS)} \qquad X' = Q(t)\, U, \quad U' = -Q(t)\, X,$$

where $t \in [a,b]$, $X(t)$, $U(t)$, and $Q(t)$ are $n \times n$ matrices and additionally the matrix $Q(t)$ is symmetric for all $t \in [a,b]$, is called a *continuous trigonometric system* (CTS). Basic properties of this system can be found in [3, 20, 30].

The discrete analog of (CTS) has the form

$$\text{(DTS)} \qquad X_{k+1} = P_k X_k + Q_k U_k, \quad U_{k+1} = -Q_k X_k + P_k U_k,$$

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where we use the notation $X_k := X(k)$, $k \in [a,b] \cap \mathbb{Z}$, X_k, U_k, P_k, Q_k are $n \times n$ matrices and, additionally, for all $k \in [a,b] \cap \mathbb{Z}$ the following holds

$$P_k^T P_k + Q_k^T Q_k = I = P_k P_k^T + Q_k Q_k^T, \tag{1.1}$$

$$P_k^T Q_k \quad \text{and} \quad P_k Q_k^T \quad \text{are symmetric}, \tag{1.2}$$

and it is called a *discrete trigonometric system* (DTS). Basic properties of this system can be found in [2,6,32].

In a similar way we can define the *continuous hyperbolic system* (CHS) as

$$X' = Q(t)\, U, \quad U' = Q(t)\, X, \tag{CHS}$$

where $t \in [a,b]$, $X(t)$, $U(t)$ and $Q(t)$ are $n \times n$ matrices and, additionally, the matrix $Q(t)$ is symmetric for all $t \in [a,b]$. The system of this form was studied in [22].

The *discrete hyperbolic system* (DHS) is defined as

$$X_{k+1} = P_k X_k + Q_k U_k, \quad U_{k+1} = Q_k X_k + P_k U_k, \tag{DHS}$$

where $k \in [a,b] \cap \mathbb{Z}$, X_k, U_k, P_k, Q_k are $n \times n$ matrices and, in addition to (1.2),

$$P_k^T P_k - Q_k^T Q_k = I = P_k P_k^T - Q_k Q_k^T$$

holds for $k \in [a,b] \cap \mathbb{Z}$. The reader can get acquainted with these systems in [19,32].

The conditions for the coefficient matrices are set in such a way so that the considered system is Hamiltonian, resp. symplectic. That is, for the relevant matrices

$$\begin{pmatrix} 0 & Q(t) \\ -Q(t) & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & Q(t) \\ Q(t) & 0 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} P_k & Q_k \\ -Q_k & P_k \end{pmatrix}, \quad \begin{pmatrix} P_k & Q_k \\ Q_k & P_k \end{pmatrix},$$

we have the identities

$$\mathcal{S}^T(t)\, \mathcal{J} + \mathcal{J} \mathcal{S}(t) = 0, \quad \text{resp.} \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \text{where} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The aim of this paper is to unify and generalize the theories of continuous and discrete trigonometric systems, as well as the theories of continuous and discrete hyperbolic systems. This will be done within the theory of symplectic (or Hamiltonian) systems on time scales $\mathbb{T}$. We derive for general time scales $\mathbb{T}$ the same identities which are known for the special cases of the continuous time $\mathbb{T} = \mathbb{R}$ or the discrete time $\mathbb{T} = \mathbb{Z}$.

In the continuous time case the study of elementary properties of scalar and matrix trigonometric functions goes back to the paper [5] of Bohl and to the work of Barrett, Etgen, Došlý, and Reid, see [3, 12–15, 20, 21, 30]. Discrete time scalar and matrix trigonometric functions were studied by Anderson, Bohner, and Došlý in [2,6–8], and more recently by Došlá, Došlý, Pechancová, and Škrabáková in [11,18]. Parallel considerations but for the hyperbolic systems, both continuous and discrete, can be found in the work [19, 22, 32] by Došlý, Filakovský, Pospíšil, and the second author. As for the general time scale setting, scalar trigonometric and hyperbolic functions were defined in [9, Chapter 3] by Bohner and Peterson and in [27] by Pospíšil. Some properties of the matrix analogs of the time scale trigonometric and hyperbolic functions were established in the papers [16, 28, 29] by Došlý and Pospíšil.

By the same technique as in [14], namely considering two different systems with the same initial conditions, we establish additive and difference formulas for trigonometric and hyperbolic systems on time scales. In particular, utilizing these identities in the continuous time we derive n-dimensional analogies of many classical formulas which are known for trigonometric and hyperbolic systems in the scalar case. The second purpose of this paper is to provide a concise but complete treatment of properties of time scale matrix trigonometric and hyperbolic functions, as well as to point out to the analogies between them.

The setup of this paper is the following. In the next section we collect preliminary properties of time scales and time scale symplectic systems which will be needed throughout the paper. In Section 3 we present the theory and new formulas for the trigonometric systems on time scales. Similar results are then derived in Section 4 for the hyperbolic systems on time scales. Finally, in Section 5 we discuss the difficulties which arise in extending some of the scalar addition formulas to time scales.

We remark, that we present all results in the real case (i.e., for the real-valued coefficients and solutions), but they hold in the complex domain as well. For this we only need to replace the transpose of a matrix by the conjugate transpose, the term "symmetric" by "Hermitian", and "orthogonal" by "unitary". Note also that all our results hold on arbitrary infinite (continuous, discrete, or time scale) intervals. This fact is important for the future possible development of the oscillation theory of the trigonometric and hyperbolic systems.

2. PRELIMINARIES ON TIME SCALES AND SYMPLECTIC SYSTEMS

In this section we introduce basic concepts, notations, and fundamental properties of time scales and time scale symplectic systems. The founder of the time scale theory is Stefan Hilger who established in [23] the calculus of time scales or measure chains. The monographs [9] and [10] offer extended knowledge of this theory.

By definition, a *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. In this paper we will consider a bounded time scale $\mathbb{T}$ which can therefore be identified with the time scale interval $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$, where $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$, and where $[a, b]$ is the usual interval of real numbers. Similarly, we use the notation $[a, b]_{\mathbb{Z}}$ for a discrete interval with $a, b \in \mathbb{Z}$, i.e., $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$. Open and half-open time scale intervals are defined accordingly.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$ (and simultaneously we put $\inf \emptyset := \sup \mathbb{T}$). The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$ (simultaneously we put $\sup \emptyset := \inf \mathbb{T}$). A point $t \in (a, b]_{\mathbb{T}}$ is said to be *left-dense* if $\rho(t) = t$, while a point $t \in [a, b)_{\mathbb{T}}$ *right-dense* if $\sigma(t) = t$. Also a point $t \in [a, b]_{\mathbb{T}}$ is *left-scattered*, resp. *right-scattered* if $\rho(t) < t$, resp. $\sigma(t) > t$. The *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

For a function f on $[a, \rho(b)]_{\mathbb{T}}$ the time scale *derivative* $f^{\Delta}(t)$ is defined in such a way that it reduces to the usual derivative $f'(t)$ at every right-dense point t (in particular, in the continuous time at all t), and it reduces to the forward difference $\Delta f(t) = f(t+1) - f(t)$ at every right-scattered point t (in particular, in the discrete time at all t). In addition, if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t = \rho(t)$ and $\mu(t) \equiv 0$, while if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) \equiv 1$. For brevity, we use the notation $f^{\sigma}(t) := f(\sigma(t))$ and $f^{\rho}(t) := f(\rho(t))$.

A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is said to be *rd-continuous* ($f \in \mathrm{C}_{\mathrm{rd}}$) if it is continuous at each right-dense point and there exists a finite left-hand limit at each left-dense point. A function f is said to be *rd-continuously Δ-differentiable* ($f \in \mathrm{C}^1_{\mathrm{rd}}$) if f^{Δ} exists on $[a, \rho(b)]_{\mathbb{T}}$ and $f^{\Delta} \in \mathrm{C}_{\mathrm{rd}}$. Note that if $f^{\Delta}(t)$ exists, then

$$f^{\sigma}(t) = f(t) + \mu(t)\, f^{\Delta}(t). \tag{2.1}$$

If b is a left-scattered point, then $f^{\Delta}(b)$ is not well-defined. The usual differential rules take in this case the form $(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t)$ and $(fg)^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f^{\sigma}(t)\, g^{\Delta}(t)$. The time scale integral is defined accordingly but it will not be used in this paper.

A matrix function $A : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^{n \times n}$ is said to be *regressive* on an interval $J \subseteq [a, \rho(b)]_{\mathbb{T}}$ if the matrix $I + \mu(t)\, A(t)$ is invertible for all $t \in J$. It is known, see e.g. [23, Theorem 5.7], that if $A \in \mathrm{C}_{\mathrm{rd}}$ and A is regressive on $[a, t_0)_{\mathbb{T}}$, then the initial value problem

$$z^{\Delta} = A(t)\, z, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad z(t_0) = z_0,$$

has a unique solution $z(t)$ on $[a, b]_{\mathbb{T}}$ for any $t_0 \in [a, b]_{\mathbb{T}}$ and any z_0. Observe that the regressivity assumption is void if $t_0 = a$.

Let A be a differentiable $n \times n$ matrix-valued function such that AA^{σ} is invertible. Then the differentiation of the identity $AA^{-1} = I$ yields

$$(A^{-1})^{\Delta} = -(A^{\sigma})^{-1} A^{\Delta}\, A^{-1} = -A^{-1} A^{\Delta} (A^{\sigma})^{-1}. \tag{2.2}$$

The time scale *symplectic* (or Hamiltonian) *system* is the first order linear system

$$X^{\Delta} = \mathcal{A}(t)\, X + \mathcal{B}(t)\, U, \quad U^{\Delta} = \mathcal{C}(t)\, X + \mathcal{D}(t)\, U, \tag{$\mathcal{S}$}$$

where $X, U : [a,b]_{\mathbb{T}} \to \mathbb{R}^{n\times n}$, the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{R}^{n\times n})$ on $[a,\rho(b)]_{\mathbb{T}}$, and the matrix $\mathcal{S}(t) := \left(\begin{smallmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{smallmatrix}\right)$ satisfies

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0 \quad \text{on } [a,\rho(b)]_{\mathbb{T}}. \tag{2.3}$$

This identity implies that the matrix $I + \mu(t)\,\mathcal{S}(t)$ is symplectic. Recall that a $2n\times 2n$ matrix M is symplectic if $M^T\mathcal{J}M = \mathcal{J}$. Basic references for time scale symplectic systems are [17, 25].

Since every symplectic matrix is invertible, it follows that the matrix function $\mathcal{S}$ is regressive on $[a,\rho(b)]_{\mathbb{T}}$. Therefore, the initial value problems associated with $(\mathcal{S})$ have unique solutions for any initial point $t_0 \in [a,b]_{\mathbb{T}}$ and any (vector or matrix) initial values, see also [9, Corollary 7.12].

If $\mathbb{T} = \mathbb{R}$, then with $A(t) := \mathcal{A}(t)$, $B(t) := \mathcal{B}(t)$, and $C(t) := \mathcal{C}(t)$ the system $(\mathcal{S})$ is the linear *Hamiltonian system* (see e.g. [26] or [31])

$$X' = A(t)\,X + B(t)\,U, \quad U' = C(t)\,X - A^T(t)\,U,$$

where the matrix $S(t) := \left(\begin{smallmatrix} A(t) & B(t) \\ C(t) & -A^T(t) \end{smallmatrix}\right)$ satisfies $\mathcal{J}S(t) + S^T(t)\,\mathcal{J} = 0$ for all $t \in [a,b]$, i.e., the matrix $S(t)$ is *Hamiltonian*. If $\mathbb{T} = \mathbb{Z}$, then with $\mathcal{A}_k := I + \mathcal{A}(k)$, $\mathcal{B}_k := \mathcal{B}(k)$, $\mathcal{C}_k := \mathcal{C}(k)$, and $\mathcal{D}_k := I + \mathcal{D}(k)$ the system $(\mathcal{S})$ is the *discrete symplectic system* (see e.g. [1, 32])

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k,$$

where the matrix $\mathcal{S}_k := \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{smallmatrix}\right)$ is symplectic.

In the block notation identity (2.3) is equivalent to (we omit the argument t)

$$\begin{aligned} \mathcal{B}^T - \mathcal{B} + \mu\,(\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B}) &= 0, \\ \mathcal{C}^T - \mathcal{C} + \mu\,(\mathcal{C}^T\mathcal{A} - \mathcal{A}^T\mathcal{C}) &= 0, \\ \mathcal{A}^T + \mathcal{D} + \mu\,(\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B}) &= 0. \end{aligned}$$

This implies that the matrices $\mathcal{B}^T(I + \mu\mathcal{D})$ and $\mathcal{C}^T(I + \mu\mathcal{A})$ are symmetric. Note that other equivalent identities are derived in [10, Remark 10.1] by using the fact that $I + \mu(t)\,\mathcal{S}^T(t)$ is symplectic. But these identities are not used in this paper.

If $Z = \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$ and $\tilde{Z} = \left(\begin{smallmatrix} \tilde{X} \\ \tilde{U} \end{smallmatrix}\right)$ are any solutions of $(\mathcal{S})$, then their *Wronskian matrix* is defined on $[a,b]_{\mathbb{T}}$ as $W(Z,\tilde{Z})(t) := X^T(t)\,\tilde{U}(t) - U^T(t)\,\tilde{X}(t)$. The following is a simple consequence of $W^{\Delta}(Z,\tilde{Z})(t) \equiv 0$.

Proposition 2.1. *Let $Z = \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$ and $\tilde{Z} = \left(\begin{smallmatrix} \tilde{X} \\ \tilde{U} \end{smallmatrix}\right)$ be any solutions of $(\mathcal{S})$. Then the Wronskian $W(Z,\tilde{Z})(t) \equiv W$ is constant on $[a,b]_{\mathbb{T}}$.*

A solution $Z = \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$ of $(\mathcal{S})$ is said to be a *conjoined solution* if $W(Z,Z) \equiv 0$, i.e., X^TU is symmetric at one and hence at any $t \in [a,b]_{\mathbb{T}}$. Two solutions Z and $\tilde{Z}$ are *normalized* if $W(Z,\tilde{Z}) \equiv I$. A solution Z and is said to be a *basis* if $\operatorname{rank} Z(t) \equiv n$

on $[a,b]_{\mathbb{T}}$. It is known that for any conjoined basis Z there always exists another conjoined basis $\tilde{Z}$ such that Z and $\tilde{Z}$ are normalized (see [9, Lemma 7.29]).

Proposition 2.2. *Let Z be any solution of $(\mathcal{S})$. Then* $\operatorname{rank} Z(t) \equiv r$ *is constant on* $[a,b]_{\mathbb{T}}$.

Proof. Let $\Phi(t)$ be a fundamental matrix of system $(\mathcal{S})$, i.e., $\Phi = (Z, \tilde{Z})$, where Z and $\tilde{Z}$ are normalized. Then every solution of $(\mathcal{S})$ is a constant multiple of $\Phi(t)$, that is, $Z(t) = \Phi(t)\, M$ on $[a,b]_{\mathbb{T}}$ for some $M \in \mathbb{R}^{2n\times n}$. If $\operatorname{rank} Z(t_0) = r$ at some $t_0 \in [a,b]_{\mathbb{T}}$, then $\operatorname{rank} M = r$. Consequently, $\operatorname{rank} Z(t) = r$ for all $t \in [a,b]_{\mathbb{T}}$. $\square$

From Propositions 2.1 and 2.2 we can see that the defining properties of conjoined bases of $(\mathcal{S})$ can be prescribed just at one point $t_0 \in [a,b]_{\mathbb{T}}$, for example by the initial condition $Z(t_0) = Z_0$ with $Z_0^T \mathcal{J} Z_0 = 0$ and $\operatorname{rank} Z_0 = n$.

Proposition 2.3. *Two solutions Z and $\widetilde{Z}$ of system $(\mathcal{S})$ are normalized conjoined bases if and only if the $2n \times 2n$ matrix $\Phi(t) := (Z(t), \widetilde{Z}(t))$ is symplectic for all $t \in [a,b]_{\mathbb{T}}$.*

The proof is straightforward and can be found e.g. in [9, Lemma 7.27]. It follows that $Z = \binom{X}{U}$ and $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ are normalized conjoined bases if and only if

$$\begin{cases} X^T\widetilde{U} - U^T\widetilde{X} = I = X\widetilde{U}^T - U\widetilde{X}^T, \\ X^T U = U^T X, \quad \widetilde{X}^T\widetilde{U} = \widetilde{U}^T\widetilde{X}, \quad X\widetilde{X}^T = \widetilde{X}X^T, \quad U\widetilde{U}^T = \widetilde{U}U^T. \end{cases} \tag{2.4}$$

The fact that the matrix Φ is symplectic for all $t \in [a,b]_{\mathbb{T}}$ implies that $\Phi^{-1} = \mathcal{J}^T\Phi^T\mathcal{J}$, and thus from $\Phi^\sigma = (I + \mu\mathcal{S})\,\Phi$ we get $\Phi^\sigma\mathcal{J}^T\Phi^T\mathcal{J} = I + \mu\mathcal{S}$. That is,

$$\begin{cases} X^\sigma\widetilde{U}^T - \widetilde{X}^\sigma U^T = I + \mu\mathcal{A}, \qquad \widetilde{X}^\sigma X^T - X^\sigma\widetilde{X}^T = \mu\mathcal{B}, \\ \widetilde{U}^\sigma X^T - U^\sigma\widetilde{X}^T = I + \mu\mathcal{D}, \qquad U^\sigma\widetilde{U}^T - \widetilde{U}^\sigma U^T = \mu\mathcal{C}. \end{cases} \tag{2.5}$$

For a given point $t_0 \in [a,b]_{\mathbb{T}}$, the conjoined basis $\binom{\hat{X}}{\hat{U}}$ of $(\mathcal{S})$ determined by the initial conditions $\hat{X}(t_0) = 0$ and $\hat{U}(t_0) = I$ is called the *principal solution at t_0*.

3. TIME SCALE TRIGONOMETRIC SYSTEMS

In this section we consider the system $(\mathcal{S})$, where the matrix $\mathcal{S}(t)$ takes the form $\mathcal{S}(t) = \begin{pmatrix} \mathcal{P}(t) & \mathcal{Q}(t) \\ -\mathcal{Q}(t) & \mathcal{P}(t) \end{pmatrix}$, where $\mathcal{P}, \mathcal{Q} \in \mathrm{C_{rd}}(\mathbb{R}^{n\times n})$ on $[a,\rho(b)]_{\mathbb{T}}$. Therefore, from (2.3) we get that the matrices $\mathcal{P}$ and $\mathcal{Q}$ satisfy the identities (we omit the argument t)

$$\mathcal{Q}^T - \mathcal{Q} + \mu\,(\mathcal{Q}^T\mathcal{P} - \mathcal{P}^T\mathcal{Q}) = 0, \tag{3.1}$$

$$\mathcal{P}^T + \mathcal{P} + \mu\,(\mathcal{Q}^T\mathcal{Q} + \mathcal{P}^T\mathcal{P}) = 0 \tag{3.2}$$

for all $t \in [a,\rho(b)]_{\mathbb{T}}$, see also [9, pg. 312] and [24, Theorem 7].

Definition 3.1 (Time scale trigonometric system)**.** The system

$$X^{\Delta} = \mathcal{P}(t)\, X + \mathcal{Q}(t)\, U, \quad U^{\Delta} = -\mathcal{Q}(t)\, X + \mathcal{P}(t)\, U, \tag{TTS}$$

where the coefficient matrices satisfy identities (3.1) and (3.2) for all $t \in [a, \rho(b)]_{\mathbb{T}}$, is called a *time scale trigonometric system* (TTS).

Remark 3.2. System ($\mathcal{S}$) is trigonometric if its coefficients satisfy, in addition to (2.3) the identity $\mathcal{J}^T \mathcal{S}(t)\, \mathcal{J} = \mathcal{S}(t)$ for all $t \in [a, \rho(b)]_{\mathbb{T}}$. Therefore, trigonometric systems are also called *self-reciprocal*, see [9, Definition 7.50].

Remark 3.3. Now, we compare the continuous time trigonometric system arising from Definition 3.1, with the system (CTS) introduced in Section 1. For $[a,b]_{\mathbb{T}} = [a,b]$, the system (TTS) from Definition 3.1 takes the form

$$X' = \mathcal{P}(t)\, X + \mathcal{Q}(t)\, U, \quad U' = -\mathcal{Q}(t)\, X + \mathcal{P}(t)\, U, \tag{3.3}$$

where $\mathcal{Q}(t)$ is symmetric and $\mathcal{P}(t)$ is antisymmetric, see (3.1) and (3.2) with $\mu = 0$. Now we use the special transformation to reduce the system (3.3) into (CTS), see [8, 30].

More precisely, let $H(t)$ be a solution of the system $H' = \mathcal{P}(t)\, H$ with the initial condition $H^T(a)\, H(a) = I$, i.e., the matrix $H(a)$ is orthogonal. Now, we consider the transformation $\bar{X} := H^{-1}(t)\, X$ and $\bar{U} := H^T(t)\, U$, which yields

$$\bar{X}' = H^{-1}(t)\, \mathcal{Q}(t)\, H^{T-1}(t)\, \bar{U}, \quad \bar{U}' = -H^T(t)\, \mathcal{Q}(t)\, H(t)\, \bar{X}.$$

Hence, this resulting system will be of the form (CTS) once we show that $H^T(t) = H^{-1}(t)$ for all $t \in [a,b]$. But this follows from the calculation $(H^T H)' = 0$ and from the initial condition on $H(a)$. Now, we put $\tilde{Q}(t) := H^T(t)\, \mathcal{Q}(t)\, H(t)$ which is symmetric, so that

$$\bar{X}' = \tilde{Q}(t)\, \bar{U}, \quad \bar{U}' = -\tilde{Q}(t)\, \bar{X}.$$

Remark 3.4. Now we consider the discrete case and show that the time scale trigonometric system (TTS) reduces for $[a,b]_{\mathbb{T}} = [a,b]_{\mathbb{Z}}$ to the system (DTS) introduced in Section 1. Upon setting $P_k := I + \mathcal{P}(k)$ and $Q_k := \mathcal{Q}(k)$ one can easily see that identities (3.1) and (3.2) are in this case equivalent to the properties of P_k and Q_k in (1.1)–(1.2).

Now we turn our attention to solutions of the general time scale trigonometric system.

Lemma 3.5. *The pair* $\binom{X}{U}$ *solves the trigonometric system* (TTS) *if and only if the pair* $\binom{U}{-X}$ *solves the same system. Equivalently* $\binom{U}{X}$ *solves* (TTS) *if and only if* $\binom{-X}{U}$ *does so.*

The following definition extend to time scales the matrix sine and cosine functions known in the continuous time in [3, pg. 511] and in the discrete case in [2, pg. 39].

Definition 3.6. Let $s \in [a,b]_{\mathbb{T}}$ be fixed. We define the $n \times n$ matrix-valued functions *sine* (denoted by Sin_s) and *cosine* (denoted by Cos_s) by

$$\mathrm{Sin}_s(t) := X(t), \quad \mathrm{Cos}_s(t) := U(t),$$

where the pair $\binom{X}{U}$ is the principal solution of system (TTS) at s, that is, it is given by the initial conditions $X(s) = 0$ and $U(s) = I$. We suppress the index s when $s = a$, i.e., we denote $\mathrm{Sin} := \mathrm{Sin}_a$ and $\mathrm{Cos} := \mathrm{Cos}_a$.

Remark 3.7. (i) The matrix functions Sin_s and Cos_s are n-dimensional analogs of the scalar trigonometric functions $\sin(t-s)$ and $\cos(t-s)$.

(ii) When $n = 1$ and $\mathcal{P} = 0$ and $\mathcal{Q} = p$ with $p \in \mathrm{C_{rd}}$, the matrix functions $\mathrm{Sin}_s(t)$ and $\mathrm{Cos}_s(t)$ reduce exactly to the scalar time scale trigonometric functions $\sin_p(t,s)$ and $\cos_p(t,s)$ from [9, Definition 3.25].

(iii) In the continuous time scalar case and when $\mathcal{P} = 0$, i.e., system (TTS) is the same as (CTS), the solutions $\mathrm{Sin}(t) = \sin \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$ and $\mathrm{Cos}(t) = \cos \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$. Similar formulas hold for the discrete scalar case, see [2, pg. 40].

Remark 3.8. By using Lemma 3.5, the above matrix sine and cosine functions can be alternatively defined as $\mathrm{Cos}_s(t) := \tilde{X}(t)$ and $\mathrm{Sin}_s(t) := -\tilde{U}(t)$, where $\binom{\tilde{X}}{\tilde{U}}$ is the solution of system (TTS) with the initial conditions $\tilde{X}(s) = I$ and $\tilde{U}(s) = 0$.

By definition, the Wronskian of the two solutions $\binom{\mathrm{Cos}}{\mathrm{Sin}}$ and $\binom{-\mathrm{Sin}}{\mathrm{Cos}}$ is $W(t) \equiv W(a) = I$. Hence, $\binom{\mathrm{Cos}}{\mathrm{Sin}}$ and $\binom{-\mathrm{Sin}}{\mathrm{Cos}}$ form normalized conjoined bases of the system (TTS) and

$$\hat{\Phi}(t) := \begin{pmatrix} \mathrm{Cos}(t) & -\mathrm{Sin}(t) \\ \mathrm{Sin}(t) & \mathrm{Cos}(t) \end{pmatrix} \tag{3.4}$$

is a fundamental matrix of (TTS). Therefore, every solution $\binom{X}{U}$ of (TTS) has the form

$$X(t) = \mathrm{Cos}(t)\,X(a) - \mathrm{Sin}(t)\,U(a) \quad \text{and} \quad U(t) = \mathrm{Sin}(t)\,X(a) + \mathrm{Cos}(t)\,U(a)$$

for all $t \in [a,b]_{\mathbb{T}}$. As a consequence of formulas (2.4) and (2.5) we get the following.

Corollary 3.9. *For all $t \in [a,b]_{\mathbb{T}}$ the identities*

$$\mathrm{Cos}^T \mathrm{Cos} + \mathrm{Sin}^T \mathrm{Sin} = I = \mathrm{Cos}\,\mathrm{Cos}^T + \mathrm{Sin}\,\mathrm{Sin}^T, \tag{3.5}$$

$$\mathrm{Cos}^T \mathrm{Sin} = \mathrm{Sin}^T \mathrm{Cos}, \quad \mathrm{Cos}\,\mathrm{Sin}^T = \mathrm{Sin}\,\mathrm{Cos}^T \tag{3.6}$$

hold, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have the identities

$$\mathrm{Cos}^\sigma \mathrm{Cos}^T + \mathrm{Sin}^\sigma \mathrm{Sin}^T = I + \mu\mathcal{P}, \quad \mathrm{Cos}^\sigma \mathrm{Sin}^T - \mathrm{Sin}^\sigma \mathrm{Cos}^T = \mu\mathcal{Q}.$$

The following result is a matrix analog of the fundamental formula $\cos^2(t) + \sin^2(t) = 1$ for scalar continuous time trigonometric functions, see also [9, Exercise 3.30]. Here $\|\cdot\|_F$ is the usual Frobenius norm, i.e., $\|V\|_F = \big(\sum_{i,j=1}^n v_{ij}^2\big)^{\frac{1}{2}}$, see [4, pg. 346].

Corollary 3.10. *For all $t \in [a,b]_{\mathbb{T}}$ we have the identity*

$$\|\mathrm{Cos}\|_F^2 + \|\mathrm{Sin}\|_F^2 = n. \tag{3.7}$$

Proof. Since for arbitrary matrix $V \in \mathbb{R}^{n\times n}$ the identity $\mathrm{tr}(V^T V) = \|V\|_F^2$ holds, equation (3.7) follows directly from (3.5). □

Corollary 3.11. *For all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have*

$$\mathrm{Cos}^\Delta \mathrm{Cos}^T + \mathrm{Sin}^\Delta \mathrm{Sin}^T = \mathcal{P}, \tag{3.8}$$

$$\mathrm{Sin}^\Delta \mathrm{Cos}^T - \mathrm{Cos}^\Delta \mathrm{Sin}^T = \mathcal{Q}. \tag{3.9}$$

Proof. Since $\binom{\mathrm{Sin}}{\mathrm{Cos}}$ is the solution of system (TTS), we have $\mathrm{Sin}^\Delta = \mathcal{P}\,\mathrm{Sin} + \mathcal{Q}\,\mathrm{Cos}$ and $\mathrm{Cos}^\Delta = -\mathcal{Q}\,\mathrm{Sin} + \mathcal{P}\,\mathrm{Cos}$. If we now multiply the first of these two identities by the matrix Sin^T from the right and the second one by Cos^T from the right, and if we add the two obtained equations, then formula (3.8) follows. In these computations we also used the second identities from (3.5) and (3.6). Similar calculations lead to formula (3.9). □

Remark 3.12. If the matrix Cos and/or Sin is invertible at some point $t \in [a,b]_{\mathbb{T}}$, then, by (3.5) and (3.6), we can write

$$\mathrm{Cos}^{-1} = \mathrm{Cos}^T + \mathrm{Sin}^T \mathrm{Cos}^{T-1} \mathrm{Sin}^T, \quad \mathrm{Sin}^{-1} = \mathrm{Sin}^T + \mathrm{Cos}^T \mathrm{Sin}^{T-1} \mathrm{Cos}^T. \tag{3.10}$$

Next we present additive formulae for matrix trigonometric functions on time scales. This result generalizes its continuous time counterpart in [21, Theorem 1.1] to time scales.

Theorem 3.13. *For $t, s \in [a,b]_{\mathbb{T}}$ we have*

$$\mathrm{Sin}_s(t) = \mathrm{Sin}(t)\,\mathrm{Cos}^T(s) - \mathrm{Cos}(t)\,\mathrm{Sin}^T(s), \tag{3.11}$$

$$\mathrm{Cos}_s(t) = \mathrm{Cos}(t)\,\mathrm{Cos}^T(s) + \mathrm{Sin}(t)\,\mathrm{Sin}^T(s), \tag{3.12}$$

$$\mathrm{Sin}(t) = \mathrm{Sin}_s(t)\,\mathrm{Cos}(s) + \mathrm{Cos}_s(t)\,\mathrm{Sin}(s), \tag{3.13}$$

$$\mathrm{Cos}(t) = \mathrm{Cos}_s(t)\,\mathrm{Cos}(s) - \mathrm{Sin}_s(t)\,\mathrm{Sin}(s). \tag{3.14}$$

Proof. We set

$$V(t) := \mathrm{Sin}(t)\,\mathrm{Cos}^T(s) - \mathrm{Cos}(t)\,\mathrm{Sin}^T(s), \quad Y(t) := \mathrm{Cos}(t)\,\mathrm{Cos}^T(s) + \mathrm{Sin}(t)\,\mathrm{Sin}^T(s).$$

Then we calculate

$$V^{\Delta}(t) = \mathrm{Sin}^{\Delta}(t)\,\mathrm{Cos}^{T}(s) - \mathrm{Cos}^{\Delta}(t)\,\mathrm{Sin}^{T}(s) = \mathcal{P}(t)V(t) + \mathcal{Q}(t)Y(t),$$
$$Y^{\Delta}(t) = \mathrm{Cos}^{\Delta}(t)\,\mathrm{Cos}^{T}(s) + \mathrm{Sin}^{\Delta}(t)\,\mathrm{Sin}^{T}(s) = -\mathcal{Q}(t)V(t) + \mathcal{P}(t)Y(t),$$

where we used the second identities from (3.5) and (3.6) at t. The initial values are $V(s) = 0$ and $Y(s) = I$, where we used the second identities from (3.5) and (3.6) at s. Hence, equations (3.11) and (3.12) follow from the uniqueness of solutions of time scale symplectic systems. That is $V(t) = \mathrm{Sin}_s(t)$ and $Y(t) = \mathrm{Cos}_s(t)$. Note that equations (3.11) and (3.12) can be written as

$$\begin{pmatrix} \mathrm{Sin}_s(t) & \mathrm{Cos}_s(t) \end{pmatrix} = \begin{pmatrix} \mathrm{Sin}(t) & \mathrm{Cos}(t) \end{pmatrix} \begin{pmatrix} \mathrm{Cos}^{T}(s) & \mathrm{Sin}^{T}(s) \\ -\mathrm{Sin}^{T}(s) & \mathrm{Cos}^{T}(s) \end{pmatrix}, \tag{3.15}$$

where the $2n \times 2n$ matrix on the right-hand side equals to $\hat{\Phi}^{-1}(s)$ and the matrix $\hat{\Phi}(s)$ is defined in (3.4). Multiplying equality (3.15) by $\hat{\Phi}(s)$ from the right, identities (3.13) and (3.14) follow. □

With respect to Remark 3.7 for the scalar continuous time case, identities (3.11)–(3.14) are matrix analogies of elementary trigonometric identities.

Interchanging the parameters t and s in (3.11) and (3.12) yields expected properties of the matrix trigonometric functions.

Corollary 3.14. *Let $t, s \in [a, b]_{\mathbb{T}}$. Then*

$$\mathrm{Sin}_s(t) = -\mathrm{Sin}_t^{T}(s) \quad \text{and} \quad \mathrm{Cos}_s(t) = \mathrm{Cos}_t^{T}(s). \tag{3.16}$$

Remark 3.15. In the scalar continuous time case and when $\mathcal{Q}(t) \equiv 1$ the formulas in (3.16) have the form $\sin(t-s) = -\sin(s-t)$ and $\cos(t-s) = \cos(s-t)$. Consequently, if we let $s = 0$, we obtain $\sin(t) = -\sin(-t)$ and $\cos(t) = \cos(-t)$, so that Corollary 3.14 is the matrix analogue of the statement about the parity for the scalar functions sine and cosine.

Next we wish to generalize the sum and difference formulae for solutions of two time scale symplectic systems. This can be done via the approach from [14]. This leads to a generalization of several formulas known in the scalar continuous case. Observe that, comparing to Theorem 3.13 in which we consider *one system* and solutions with *different initial conditions*, we shall now deal with *two systems* and solutions with the *same initial conditions*. Consider the following two time scale trigonometric systems

$$X^{\Delta} = \mathcal{P}_{(i)}(t)\,X + \mathcal{Q}_{(i)}(t)\,U, \quad U^{\Delta} = -\mathcal{Q}_{(i)}(t)\,X + \mathcal{P}_{(i)}(t)\,U \tag{3.17}$$

with initial conditions $X_{(i)}(a) = 0$ and $U_{(i)}(a) = I$, where $i = 1, 2$. Denote by $\mathrm{Sin}_{(i)}(t)$ and $\mathrm{Cos}_{(i)}(t)$ the corresponding matrix sine and cosine functions from Definition 3.6.

Put

$$\mathrm{Sin}^{\pm}(t) := \mathrm{Sin}_{(1)}(t)\,\mathrm{Cos}_{(2)}^{T}(t) \pm \mathrm{Cos}_{(1)}(t)\,\mathrm{Sin}_{(2)}^{T}(t), \tag{3.18}$$

$$\mathrm{Cos}^{\pm}(t) := \mathrm{Cos}_{(1)}(t)\,\mathrm{Cos}_{(2)}^{T}(t) \mp \mathrm{Sin}_{(1)}(t)\,\mathrm{Sin}_{(2)}^{T}(t). \tag{3.19}$$

Theorem 3.16. *Assume that $\mathcal{P}_{(i)}$ and $\mathcal{Q}_{(i)}$ satisfy* (3.1) *and* (3.2). *The pair* $\mathrm{Sin}^{\pm}$ *and* $\mathrm{Cos}^{\pm}$ *solves the system*

$$\begin{cases} X^{\Delta} = \mathcal{P}_{(1)}X + \mathcal{Q}_{(1)}U + X\mathcal{P}_{(2)}^{T} \pm U\mathcal{Q}_{(2)}^{T} \\ \qquad\qquad + \mu\,[\mathcal{P}_{(1)}\,(X\mathcal{P}_{(2)}^{T} \pm U\mathcal{Q}_{(2)}^{T}) + \mathcal{Q}_{(1)}\,(\mp X\mathcal{Q}_{(2)}^{T} + U\mathcal{P}_{(2)}^{T})], \\ U^{\Delta} = -\mathcal{Q}_{(1)}X + \mathcal{P}_{(1)}U \mp X\mathcal{Q}_{(2)}^{T} + U\mathcal{P}_{(2)}^{T} \\ \qquad\qquad + \mu\,[-\mathcal{Q}_{(1)}\,(X\mathcal{P}_{(2)}^{T} \pm U\mathcal{Q}_{(2)}^{T}) + \mathcal{P}_{(1)}\,(\mp X\mathcal{Q}_{(2)}^{T} + U\mathcal{P}_{(2)}^{T})] \end{cases} \tag{3.20}$$

with the initial conditions $X(a) = 0$ and $U(a) = I$. Moreover, for all $t \in [a, b]_{\mathbb{T}}$ we have

$$\mathrm{Sin}^{\pm}\,(\mathrm{Sin}^{\pm})^{T} + \mathrm{Cos}^{\pm}\,(\mathrm{Cos}^{\pm})^{T} = I = (\mathrm{Sin}^{\pm})^{T}\,\mathrm{Sin}^{\pm} + (\mathrm{Cos}^{\pm})^{T}\,\mathrm{Cos}^{\pm}, \tag{3.21}$$

$$\mathrm{Sin}^{\pm}\,(\mathrm{Cos}^{\pm})^{T} = \mathrm{Cos}^{\pm}\,(\mathrm{Sin}^{\pm})^{T}, \quad (\mathrm{Sin}^{\pm})^{T}\,\mathrm{Cos}^{\pm} = (\mathrm{Cos}^{\pm})^{T}\,\mathrm{Sin}^{\pm}. \tag{3.22}$$

Proof. All the statements in this theorem are proven by straightforward calculations. In these we use the identities, see (2.1),

$$\mathrm{Sin}_{(1)}^{\sigma} = \mathrm{Sin}_{(1)} + \mu\,\mathrm{Sin}_{(1)}^{\Delta} = \mathrm{Sin}_{(1)} + \mu\,(\mathcal{P}_{(1)}\,\mathrm{Sin}_{(1)} + \mathcal{Q}_{(1)}\,\mathrm{Cos}_{(1)}),$$
$$\mathrm{Cos}_{(1)}^{\sigma} = \mathrm{Cos}_{(1)} + \mu\,\mathrm{Cos}_{(1)}^{\Delta} = \mathrm{Cos}_{(1)} + \mu\,(-\mathcal{Q}_{(1)}\,\mathrm{Sin}_{(1)} + \mathcal{P}_{(1)}\,\mathrm{Cos}_{(1)}),$$

the time scale product rule, and system (3.17) for $i = 1, 2$. Then it follows that the pair $\mathrm{Sin}^{\pm}$ and $\mathrm{Cos}^{\pm}$ solves the system (3.20) and $\mathrm{Sin}^{\pm}(a) = 0$, $\mathrm{Cos}^{\pm}(a) = I$.

Next we show identity (3.21). From the definition of Sin^{+} and Cos^{+}, from the first identity in (3.5) for $i = 1$, and from the second identity in (3.6) for $i = 2$ we get

$$\begin{aligned} \mathrm{Sin}^{+}\,(\mathrm{Sin}^{+})^{T} + \mathrm{Cos}^{+}\,(\mathrm{Cos}^{+})^{T} &= \mathrm{Sin}_{(1)}\,(\mathrm{Cos}_{(2)}^{T}\,\mathrm{Cos}_{(2)} + \mathrm{Sin}_{(2)}^{T}\,\mathrm{Sin}_{(2)})\,\mathrm{Sin}_{(1)}^{T} \\ &\quad + \mathrm{Cos}_{(1)}\,(\mathrm{Cos}_{(2)}^{T}\,\mathrm{Cos}_{(2)} + \mathrm{Sin}_{(2)}^{T}\,\mathrm{Sin}_{(2)})\,\mathrm{Cos}_{(1)}^{T} \\ &= \mathrm{Cos}_{(1)}\,\mathrm{Cos}_{(1)}^{T} + \mathrm{Sin}_{(1)}\,\mathrm{Sin}_{(1)}^{T} = I. \end{aligned}$$

The other identities in (3.21) are shown in analogous way. Similarly, by using (3.5) and (3.6) for $i = 1, 2$ one can show that all the identities in (3.22) hold true. ☐

Remark 3.17. The properties in (3.21) and (3.22) of solutions $\mathrm{Sin}^{\pm}$ and $\mathrm{Cos}^{\pm}$ of system (3.20) mirror the properties in (3.5) and (3.6) of normalized conjoined bases of $(\mathcal{S})$. However, the two pairs $\binom{\mathrm{Sin}^{+}}{\mathrm{Cos}^{+}}$ and $\binom{\mathrm{Sin}^{-}}{\mathrm{Cos}^{-}}$ are not conjoined bases of their corresponding systems, because these systems are not symplectic.

Remark 3.18. In the continuous time case the assertion of Theorem 3.16 is proven in [14, Theorem 1]. On the other hand, the discrete form is new. The details can be found in [33, Theorem 3.14].

When the two systems in (3.17) are the same, Theorem 3.16 yields the following.

Corollary 3.19. *Assume that* $\mathcal{P}$ *and* $\mathcal{Q}$ *satisfy* (3.1) *and* (3.2). *Then the system*

$$\begin{aligned} X^{\Delta} &= \quad \mathcal{P}X + \mathcal{Q}U + X\mathcal{P}^T + U\mathcal{Q}^T + \mu\,[\mathcal{P}\,(X\mathcal{P}^T + U\mathcal{Q}^T) + \mathcal{Q}\,(-X\mathcal{Q}^T + U\mathcal{P}^T)], \\ U^{\Delta} &= -\mathcal{Q}X + \mathcal{P}U - X\mathcal{Q}^T + U\mathcal{P}^T + \mu\,[-\mathcal{Q}\,(X\mathcal{P}^T + U\mathcal{Q}^T) + \mathcal{P}\,(-X\mathcal{Q}^T + U\mathcal{P}^T)] \end{aligned}$$

with the initial conditions $X(a) = 0$ *and* $U(a) = I$ *possesses the solution*

$$X = 2\,\mathrm{Sin}\,\mathrm{Cos}^T \quad \text{and} \quad U = \mathrm{Cos}\,\mathrm{Cos}^T - \mathrm{Sin}\,\mathrm{Sin}^T,$$

where Sin *and* Cos *are the matrix functions in Definition 3.6. Moreover, the above matrices* X *and* U *commute, i.e.,* $XU = UX$.

Proof. The statement follows from Theorem 3.16 in which we take $\mathcal{P}_{(1)} = \mathcal{P}_{(2)} = \mathcal{P}$, $\mathcal{Q}_{(1)} = \mathcal{Q}_{(2)} = \mathcal{Q}$, and $\mathrm{Sin}_{(1)} = \mathrm{Sin}_{(2)} = \mathrm{Sin}$, $\mathrm{Cos}_{(1)} = \mathrm{Cos}_{(2)} = \mathrm{Cos}$. Finally, from (3.5) and (3.6) we get that $XU - UX = 0$. □

The previous corollary can be viewed as the n-dimensional analogy of the double angle formulae for scalar continuous time trigonometric functions. In the continuous time case, the content of Corollary 3.19 is [20, Theorem 1.1]. On the other hand, in the discrete case this result is new, see [33, Corollary 3.16]. The details are here omitted.

Corollary 3.20. *For all* $t \in [a,b]_{\mathbb{T}}$ *we have the identities*

$$\mathrm{Sin}_{(1)}\,\mathrm{Sin}_{(2)}^T = \frac{1}{2}\,(\mathrm{Cos}^- - \mathrm{Cos}^+), \tag{3.23}$$

$$\mathrm{Cos}_{(1)}\,\mathrm{Cos}_{(2)}^T = \frac{1}{2}\,(\mathrm{Cos}^- + \mathrm{Cos}^+), \tag{3.24}$$

$$\mathrm{Sin}_{(1)}\,\mathrm{Cos}_{(2)}^T = \frac{1}{2}\,(\mathrm{Sin}^- + \mathrm{Sin}^+). \tag{3.25}$$

Proof. Subtracting the two equations in (3.19) we obtain $\mathrm{Cos}^- - \mathrm{Cos}^+ = 2\,\mathrm{Sin}_{(1)}\,\mathrm{Sin}_{(2)}^T$ from which formula (3.23) follows. Similarly, we have $\mathrm{Cos}^- + \mathrm{Cos}^+ = 2\,\mathrm{Cos}_{(1)}\,\mathrm{Cos}_{(2)}^T$ and $\mathrm{Sin}^- + \mathrm{Sin}^+ = 2\,\mathrm{Sin}_{(1)}\,\mathrm{Cos}_{(2)}^T$ from which we get (3.24) and (3.25). □

In the scalar continuous time case, identities (3.23)–(3.25) have the form of elementary trigonometric identities. The next definition is a natural time scale matrix extension of the scalar trigonometric tangent and cotangent functions. It extends the discrete matrix tangent and cotangent functions known in [2, pg. 42] to time scales.

Definition 3.21. Whenever $\mathrm{Cos}(t)$, resp. $\mathrm{Sin}(t)$, is invertible we define the matrix-valued function *tangent* (we write Tan), resp. *cotangent* (we write Cotan), by

$$\mathrm{Tan}(t) := \mathrm{Cos}^{-1}(t)\ \mathrm{Sin}(t), \quad \text{resp.} \quad \mathrm{Cotan}(t) := \mathrm{Sin}^{-1}(t)\ \mathrm{Cos}(t).$$

Theorem 3.22. *Whenever* $\operatorname{Tan}(t)$ *is defined we get*

$$\operatorname{Tan}^T(t) = \operatorname{Tan}(t), \tag{3.26}$$

$$\operatorname{Cos}^{-1}(t)\operatorname{Cos}^{T-1}(t) - \operatorname{Tan}^2(t) = I. \tag{3.27}$$

Moreover, if $\operatorname{Cos}(t)$ *and* $\operatorname{Cos}^\sigma(t)$ *are invertible, then*

$$\operatorname{Tan}^\Delta(t) = [\operatorname{Cos}^\sigma(t)]^{-1}\mathcal{Q}(t)\ \operatorname{Cos}^{T-1}(t). \tag{3.28}$$

Proof. From (3.6) it follows that

$$\operatorname{Tan}^T - \operatorname{Tan} = \operatorname{Cos}^{-1}(\operatorname{Cos}\operatorname{Sin}^T - \operatorname{Sin}\operatorname{Cos}^T)\operatorname{Cos}^{T-1} = 0,$$

while from (3.5) and (3.26) we get

$$I = \operatorname{Cos}(\operatorname{Cos}^{-1}\operatorname{Sin}\operatorname{Sin}^T\operatorname{Cos}^{T-1} + I)\operatorname{Cos}^T = \operatorname{Cos}(\operatorname{Tan}^2 + I)\operatorname{Cos}^T,$$

which can be written as equation (3.27). In order to show (3.28) we note that if $\operatorname{Cos}(t_0)$ and $\operatorname{Cos}^\sigma(t_0)$ are invertible, then $\operatorname{Tan}^\Delta(t_0)$ exists and, by (3.5), (2.2), (3.10), and (3.26), we get

$$\begin{aligned}\operatorname{Tan}^\Delta &= (\operatorname{Cos}^{-1}\operatorname{Sin})^\Delta = -(\operatorname{Cos}^\sigma)^{-1}\operatorname{Cos}^\Delta\operatorname{Cos}^{-1}\operatorname{Sin} + (\operatorname{Cos}^\sigma)^{-1}\operatorname{Sin}^\Delta\\ &= (\operatorname{Cos}^\sigma)^{-1}\mathcal{Q}\,(-\operatorname{Sin}\operatorname{Cos}^{-1}\operatorname{Sin} + \operatorname{Cos}) = (\operatorname{Cos}^\sigma)^{-1}\mathcal{Q}\operatorname{Cos}^{T-1}.\end{aligned}$$

Therefore (3.28) is established. □

Similar results as in Theorem 3.22 can be shown for the matrix function cotangent.

Theorem 3.23. *Whenever* $\operatorname{Cotan}(t)$ *is defined we get*

$$\operatorname{Cotan}^T(t) = \operatorname{Cotan}(t), \tag{3.29}$$

$$\operatorname{Sin}^{-1}(t)\ \operatorname{Sin}^{T-1}(t) - \operatorname{Cotan}^2(t) = I. \tag{3.30}$$

Moreover, if $\operatorname{Sin}(t)$ *and* $\operatorname{Sin}^\sigma(t)$ *are invertible, then*

$$\operatorname{Cotan}^\Delta(t) = -[\operatorname{Sin}^\sigma(t)]^{-1}\mathcal{Q}(t)\ \operatorname{Sin}^{T-1}(t). \tag{3.31}$$

Proof. It is analogous to the proof of Theorem 3.22. □

Remark 3.24. In the scalar case $n = 1$ identities (3.26) and (3.29) are trivial. In the scalar continuous time case, identities (3.27), (3.30), (3.28), and (3.31) take the form

$$\frac{1}{\cos^2(s)} - \tan^2(s) = 1, \quad \frac{1}{\sin^2(s)} - \operatorname{cotan}^2(s) = 1, \quad \text{with } s = \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau,$$

$$\left(\tan\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{\mathcal{Q}(t)}{\cos^2 s}, \quad \left(\operatorname{cotan}\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{-\mathcal{Q}(t)}{\sin^2 s},$$

compare with Remark 3.7 (iii). The discrete versions of these identities can be found in [2, Corollary 6 and Lemma 12].

Remark 3.25. In the continuous time case with $\mathcal{Q}(t) \equiv I$, i.e., when system (CTS) is $X' = U$, $U' = -X$ and hence it represents the second order matrix equation $X'' + X = 0$, the matrix functions Sin, Cos, Tan, and Cotan satisfy

$$\operatorname{Sin}' = \operatorname{Cos}, \quad \operatorname{Cos}' = -\operatorname{Sin}, \quad \operatorname{Tan}' = \operatorname{Cos}^{-1}\operatorname{Cos}^{T-1}, \quad \operatorname{Cotan}' = -\operatorname{Sin}^{-1}\operatorname{Sin}^{T-1}.$$

The first two equalities follow from the definition of Sin and Cos, while the last two equalities are simple consequences of (3.28) and (3.31).

Next, similarly to the definitions of the time scale matrix functions $\operatorname{Sin}_{(i)}$, $\operatorname{Cos}_{(i)}$, $\operatorname{Sin}^{\pm}$, and $\operatorname{Cos}^{\pm}$ from (3.17)–(3.19) we define the following functions

$$\begin{aligned} \operatorname{Tan}_{(i)}(t) &:= \operatorname{Cos}_{(i)}^{-1}(t)\ \operatorname{Sin}_{(i)}(t), & \operatorname{Cotan}_{(i)}(t) &:= \operatorname{Sin}_{(i)}^{-1}(t)\ \operatorname{Cos}_{(i)}(t), \\ \operatorname{Tan}^{\pm}(t) &:= [\operatorname{Cos}^{\pm}(t)]^{-1}\operatorname{Sin}^{\pm}(t), & \operatorname{Cotan}^{\pm}(t) &:= [\operatorname{Sin}^{\pm}(t)]^{-1}\operatorname{Cos}^{\pm}(t). \end{aligned}$$

Remark 3.26. It is natural that the matrix-valued functions $\operatorname{Tan}^{\pm}$ have similar properties as the function Tan. In particular, the first identity in (3.22) implies that $\operatorname{Tan}^{\pm}$ are symmetric. Similarly, the functions $\operatorname{Cotan}^{\pm}$ are also symmetric.

The results of the following theorem are new even in the special case of continuous and discrete time, see [33, Theorem 3.26].

Theorem 3.27. *For all $t \in [a,b]_{\mathbb{T}}$ such that all involved functions are defined we have (suppressing the argument t)*

$$\operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)} = \operatorname{Tan}_{(1)}\left(\operatorname{Cotan}_{(2)} \pm \operatorname{Cotan}_{(1)}\right)\operatorname{Tan}_{(2)}, \tag{3.32}$$

$$\operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)} = \operatorname{Cos}_{(1)}^{-1}\ \operatorname{Sin}^{\pm}\ \operatorname{Cos}_{(2)}^{T-1}, \tag{3.33}$$

$$\operatorname{Cotan}_{(1)} \pm \operatorname{Cotan}_{(2)} = \operatorname{Cotan}_{(1)}\left(\operatorname{Tan}_{(2)} \pm \operatorname{Tan}_{(1)}\right)\operatorname{Cotan}_{(2)}, \tag{3.34}$$

$$\operatorname{Cotan}_{(1)} \pm \operatorname{Cotan}_{(2)} = \pm\operatorname{Sin}_{(1)}^{-1}\ \operatorname{Sin}^{\pm}\ \operatorname{Sin}_{(2)}^{T-1}, \tag{3.35}$$

$$\operatorname{Tan}^{\pm} = \operatorname{Cos}_{(2)}^{T-1}\left(I \mp \operatorname{Tan}_{(1)}\operatorname{Tan}_{(2)}\right)^{-1}\left(\operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)}\right)\operatorname{Cos}_{(2)}^{T}, \tag{3.36}$$

$$\operatorname{Cotan}^{\pm} = \operatorname{Sin}_{(2)}^{T-1}\left(\operatorname{Cotan}_{(2)} \pm \operatorname{Cotan}_{(1)}\right)^{-1}\left(\operatorname{Cotan}_{(1)}\ \operatorname{Cotan}_{(2)} \mp I\right)\operatorname{Sin}_{(2)}^{T}. \tag{3.37}$$

Proof. For identity (3.32) we have

$$\begin{aligned} \operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)} &= \operatorname{Cos}_{(1)}^{-1}\operatorname{Sin}_{(1)}\left(\operatorname{Sin}_{(2)}^{-1}\operatorname{Cos}_{(2)} \pm \operatorname{Sin}_{(1)}^{-1}\operatorname{Cos}_{(1)}\right)\operatorname{Cos}_{(2)}^{-1}\operatorname{Sin}_{(2)} \\ &= \operatorname{Tan}_{(1)}\left(\operatorname{Cotan}_{(2)} \pm \operatorname{Cotan}_{(1)}\right)\operatorname{Tan}_{(2)}. \end{aligned}$$

The equations in (3.33) follow by the symmetry of $\operatorname{Tan}_{(2)}$, i.e.,

$$\operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)} = \operatorname{Cos}_{(1)}^{-1}\left(\operatorname{Sin}_{(1)}\operatorname{Cos}_{(2)}^{T} \pm \operatorname{Cos}_{(1)}\operatorname{Sin}_{(2)}^{T}\right)\operatorname{Cos}_{(2)}^{T-1} = \operatorname{Cos}_{(1)}^{-1}\ \operatorname{Sin}^{\pm}\ \operatorname{Cos}_{(2)}^{T-1}.$$

The proofs of identities (3.34) and (3.35) are simialar to the proofs of (3.32) and (3.33). Next, upon transposing identity (3.33) we get

$$\operatorname{Tan}_{(1)} \pm \operatorname{Tan}_{(2)} = \operatorname{Cos}_{(2)}^{-1}\left(\operatorname{Tan}^{\pm}\right)^{T}\left(\operatorname{Cos}^{\pm}\right)^{T}\operatorname{Cos}_{(1)}^{T-1},$$

from which we eliminate $\mathrm{Tan}^{\pm}$. That is, with the symmetry of $\mathrm{Tan}^{\pm}$ and $\mathrm{Tan}_{(i)}$ we have

$$\begin{aligned}\mathrm{Tan}^{\pm} &= (\mathrm{Cos}^{\pm})^{-1}\,\mathrm{Cos}_{(1)}\,(\mathrm{Tan}_{(1)}^{T} \pm \mathrm{Tan}_{(2)}^{T})\,\mathrm{Cos}_{(2)}^{T}\\ &= [\mathrm{Cos}_{(1)}(I \mp \mathrm{Cos}_{(1)}^{-1}\,\mathrm{Sin}_{(1)}\,\mathrm{Sin}_{(2)}^{T}\,\mathrm{Cos}_{(2)}^{T-1})\,\mathrm{Cos}_{(2)}^{T}]^{-1}\,\mathrm{Cos}_{(1)}(\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)})\,\mathrm{Cos}_{(2)}^{T}\\ &= \mathrm{Cos}_{(2)}^{T-1}\,(I \mp \mathrm{Tan}_{(1)}\,\mathrm{Tan}_{(2)})^{-1}\,(\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)})\,\mathrm{Cos}_{(2)}^{T}\,.\end{aligned}$$

Therefore, the formulas in (3.36) are established. The identities in (3.37) follow from (3.36) by noticing that $\mathrm{Tan}^{\pm}\,\mathrm{Cotan}^{\pm} = I$ and by using the symmetry of $\mathrm{Cotan}_{(i)}$. $\square$

Consider the system (TTS) in the scalar continuous time case with $\mathcal{P}(t) \equiv 0$ and $\mathcal{Q}(t) \equiv 1$, or equivalently system (CTS) with $Q(t) \equiv 1$. Then the identities in (3.32)–(3.37) have the form of elementary trigonometric identities.

4. TIME SCALE HYPERBOLIC SYSTEMS

In this section we define time scale matrix hyperbolic functions and prove analogous results as for the trigonometric functions in the previous section. In particular, we derive time scale matrix extensions of several identities which are known for the continuous time scalar hyperbolic functions. The proofs are similar to the corresponding proofs for the trigonometric case and therefore mostly they will be omitted. We wish to remark that some results from this section have previously been derived in the unpublished paper [28] by Z. Pospíšil. We now present these results for completeness and clear comparison with the corresponding trigonometric results established in Section 3, as well as derive several new formulas for time scale matrix hyperbolic functions.

Consider the system ($\mathcal{S}$) with the matrix $\mathcal{S}(t) = \begin{pmatrix} \mathcal{P}(t) & \mathcal{Q}(t) \\ \mathcal{Q}(t) & \mathcal{P}(t) \end{pmatrix}$, where the coefficients satisfy the identities

$$\mathcal{Q}^{T} - \mathcal{Q} + \mu\,(\mathcal{Q}^{T}\mathcal{P} - \mathcal{P}^{T}\mathcal{Q}) = 0, \tag{4.1}$$

$$\mathcal{P} + \mathcal{P}^{T} + \mu\,(\mathcal{P}^{T}\mathcal{P} - \mathcal{Q}^{T}\mathcal{Q}) = 0, \tag{4.2}$$

see also [28, pg. 9] and [24, Theorem 8].

Definition 4.1 (Time scale hyperbolic system)**.** The system

$$X^{\Delta} = \mathcal{P}(t)\,X + \mathcal{Q}(t)\,U, \quad U^{\Delta} = \mathcal{Q}(t)\,X + \mathcal{P}(t)\,U, \tag{THS}$$

where the matrices $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ satisfy identities (4.1) and (4.2) for all $t \in [a, \rho(b)]_{\mathbb{T}}$, is called a *time scale hyperbolic system* (THS).

Remark 4.2. The above time scale hyperbolic system is in general defined through two coefficient matrices $\mathcal{P}$ and $\mathcal{Q}$. However, in the continuous time case we can use the same transformation as in Remark 3.3 and write the hyperbolic system (THS) in the form of (CHS). Similarly, by using the same arguments as in Remark 3.4, in the discrete case we can write the above hyperbolic system as (DHS).

Remark 4.3. In the discrete case it is known that the matrix $\mathcal{P}_k$ is necessarily invertible for all $k \in [a,b]_{\mathbb{Z}}$, see [19, identity (12)] or [32, Remark 67]. Similarly, in the general time scale setting we have that identity (4.2) implies $(I+\mu\mathcal{P}^T)\,(I+\mu\mathcal{P}) = I + \mu^2\,\mathcal{Q}^T\mathcal{Q} > 0$, that is, the matrix $I+\mu\mathcal{P}$ is invertible. And then (4.1) yields that $\mathcal{Q}\,(I+\mu\mathcal{P})^{-1}$ is symmetric.

Lemma 4.4. *The pair* $\binom{X}{U}$ *solves the time scale hyperbolic system* (THS) *if and only if the pair* $\binom{U}{X}$ *solves the same hyperbolic system.*

Following [28, Definition 2.1], we next define the time scale matrix hyperbolic functions. See also the discrete version in [19, Definition 3.1] or [32, Definition 32].

Definition 4.5. Let $s \in [a,b]_{\mathbb{T}}$ be fixed. We define the $n\times n$ matrix valued functions *hyperbolic sine* (denoted by Sinh_s) and *hyperbolic cosine* (denoted Cosh_s) by

$$\mathrm{Sinh}_s(t) := X(t), \quad \mathrm{Cosh}_s(t) := U(t),$$

where the pair $\binom{X}{U}$ is the principal solution of system (THS) at s, that is, it is given by the initial conditions $X(s) = 0$ and $U(s) = I$. We suppress the index s when $s = a$, i.e., we denote $\mathrm{Sinh} := \mathrm{Sinh}_a$ and $\mathrm{Cosh} := \mathrm{Cosh}_s$.

Remark 4.6. (i) The matrix functions Sinh_s and Cosh_s are n-dimensional analogs of the scalar hyperbolic functions $\sinh(t-s)$ and $\cosh(t-s)$.

(ii) When $n=1$ and $\mathcal{P}=0$ and $\mathcal{Q}=p$ with $p \in \mathrm{C_{rd}}$, the matrix functions $\mathrm{Sinh}_s(t)$ and $\mathrm{Cosh}_s(t)$ reduce exactly to the scalar time scale hyperbolic functions $\sinh_p(t,s)$ and $\cosh_p(t,s)$ from [9, Definition 3.17].

(iii) In the continuous time scalar case and when $\mathcal{P}=0$, i.e., system (THS) is the same as (CHS), we have $\mathrm{Sinh}(t) = \sinh\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$ and $\mathrm{Cosh}(t) = \cosh\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$, see [22, pg. 12]. Similar formulas hold for the discrete scalar case, see [19, equations (27)–(28)].

Since the solutions $\binom{\mathrm{Cosh}}{\mathrm{Sinh}}$ and $\binom{\mathrm{Sinh}}{\mathrm{Cosh}}$ form normalized conjoined bases of (THS),

$$\hat{\Psi}(t) := \begin{pmatrix} \mathrm{Cosh}(t) & \mathrm{Sinh}(t) \\ \mathrm{Sinh}(t) & \mathrm{Cosh}(t) \end{pmatrix}$$

is a fundamental matrix of (THS). Therefore, every solution $\binom{X}{U}$ of (THS) has the form

$$X(t) = \mathrm{Cosh}(t)\,X(a) + \mathrm{Sinh}(t)\,U(a) \quad \text{and} \quad U(t) = \mathrm{Sinh}(t)\,X(a) + \mathrm{Cosh}(t)\,U(a)$$

for all $t \in [a,b]_{\mathbb{T}}$. As a consequence of formulas (2.4) and (2.5) we get for solutions of time scale hyperbolic systems the following, see also [28, Theorem 2.1].

Corollary 4.7. *For all $t \in [a,b]_{\mathbb{T}}$ the identities*

$$\operatorname{Cosh}^T \operatorname{Cosh} - \operatorname{Sinh}^T \operatorname{Sinh} = I = \operatorname{Cosh} \operatorname{Cosh}^T - \operatorname{Sinh} \operatorname{Sinh}^T, \tag{4.3}$$

$$\operatorname{Cosh}^T \operatorname{Sinh} = \operatorname{Sinh}^T \operatorname{Cosh}, \quad \operatorname{Cosh} \operatorname{Sinh}^T = \operatorname{Sinh} \operatorname{Cosh}^T \tag{4.4}$$

hold, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have the identities

$$\operatorname{Cosh}^\sigma \operatorname{Cosh}^T - \operatorname{Sinh}^\sigma \operatorname{Sinh}^T = I + \mu\mathcal{P}, \quad \operatorname{Sinh}^\sigma \operatorname{Cosh}^T - \operatorname{Cosh}^\sigma \operatorname{Sinh}^T = \mu\mathcal{Q}.$$

Now we establish a matrix analog of the formula $\cosh^2(t) - \sinh^2(t) = 1$, see also [28, Theorem 2.1], as well as the formulas from [28, Theorem 2.5].

Corollary 4.8. *For all $t \in [a,b]_{\mathbb{T}}$ the identity*

$$\|\operatorname{Cosh}\|_F^2 - \|\operatorname{Sinh}\|_F^2 = n$$

holds, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have

$$\operatorname{Cosh}^\Delta \operatorname{Cosh}^T - \operatorname{Sinh}^\Delta \operatorname{Sinh}^T = \mathcal{P}, \quad \operatorname{Sinh}^\Delta \operatorname{Cosh}^T - \operatorname{Cosh}^\Delta \operatorname{Sinh}^T = \mathcal{Q}.$$

Remark 4.9. It follows from identity (4.3) that the matrix $\operatorname{Cosh}(t)$ is invertible for all $t \in [a,b]_{\mathbb{T}}$. Moreover, if $\operatorname{Sinh}(t)$ is invertible at some t, then from (4.3) and (4.4) we get

$$\operatorname{Cosh}^{-1} = \operatorname{Cosh}^T - \operatorname{Sinh}^T \operatorname{Cosh}^{T-1} \operatorname{Sinh}^T, \quad \operatorname{Sinh}^{-1} = \operatorname{Cosh}^T \operatorname{Sinh}^{T-1} \operatorname{Cosh}^T - \operatorname{Sinh}^T.$$

The following additive formulas are established in [28, Theorem 2.2]. They are proven in a similar way to the formulas in Theorem 3.13.

Theorem 4.10. *For $t, s \in [a,b]_{\mathbb{T}}$ we have*

$$\operatorname{Sinh}_s(t) = \operatorname{Sinh}(t) \operatorname{Cosh}^T(s) - \operatorname{Cosh}(t) \operatorname{Sinh}^T(s), \tag{4.5}$$

$$\operatorname{Cosh}_s(t) = \operatorname{Cosh}(t) \operatorname{Cosh}^T(s) - \operatorname{Sinh}(t) \operatorname{Sinh}^T(s), \tag{4.6}$$

$$\operatorname{Sinh}(t) = \operatorname{Sinh}_s(t) \operatorname{Cosh}(s) + \operatorname{Cosh}_s(t) \operatorname{Sinh}(s), \tag{4.7}$$

$$\operatorname{Cosh}(t) = \operatorname{Cosh}_s(t) \operatorname{Cosh}(s) + \operatorname{Sinh}_s(t) \operatorname{Sinh}(s). \tag{4.8}$$

With respect to Remark 4.6 for the scalar continuous time case, identities (4.5)–(4.8) are matrix analogies of elementary hyperbolic identities.

Interchanging the parameters t and s in (4.5) and (4.6) we get [28, formula (34)].

Corollary 4.11. *Let $t, s \in [a,b]_{\mathbb{T}}$. Then*

$$\operatorname{Sinh}_s(t) = -\operatorname{Sinh}_t^T(s) \quad \text{and} \quad \operatorname{Cosh}_s(t) = \operatorname{Cosh}_t^T(s). \tag{4.9}$$

Remark 4.12. In the scalar continuous time case and when $\mathcal{Q}(t) \equiv 1$ and $s = 0$, the formulas in (4.9) show that $\sinh(t) = -\sinh(-t)$ and $\cosh(t) = \cosh(-t)$. So we can see that Corollary 4.11 gives the matrix analogies of the statement about the parity for the scalar functions hyperbolic sine and hyperbolic cosine.

Now we use the same approach as for the time scale trigonometric functions to obtain generalized sum and difference formulas for solutions of two time scale hyperbolic systems. Hence, we consider the following two time scale hyperbolic systems

$$X^{\Delta} = \mathcal{P}_{(i)}(t)\, X + \mathcal{Q}_{(i)}(t)\, U, \quad U^{\Delta} = \mathcal{Q}_{(i)}(t)\, X + \mathcal{P}_{(i)}(t)\, U \tag{4.10}$$

with initial conditions $X_{(i)}(a) = 0$ and $U_{(i)}(a) = I$, where $i = 1, 2$. Denote by $\mathrm{Sinh}_{(i)}(t)$ and $\mathrm{Cosh}_{(i)}(t)$ the corresponding matrix hyperbolic sine and hyperbolic cosine functions from Definition 4.5. If we set

$$\mathrm{Sinh}^{\pm}(t) := \mathrm{Sinh}_{(1)}(t)\, \mathrm{Cosh}^{T}_{(2)}(t) \pm \mathrm{Cosh}_{(1)}(t)\, \mathrm{Sinh}^{T}_{(2)}(t), \tag{4.11}$$

$$\mathrm{Cosh}^{\pm}(t) := \mathrm{Cosh}_{(1)}(t)\, \mathrm{Cosh}^{T}_{(2)}(t) \pm \mathrm{Sinh}_{(1)}(t)\, \mathrm{Sinh}^{T}_{(2)}(t), \tag{4.12}$$

then similarly to Theorem 3.16 we have the following.

Theorem 4.13. *Assume that $\mathcal{P}_{(i)}$ and $\mathcal{Q}_{(i)}$ satisfy* (4.1) *and* (4.2). *The pair* $\mathrm{Sinh}^{\pm}$ *and* $\mathrm{Cosh}^{\pm}$ *solves the system*

$$\begin{aligned} X^{\Delta} &= \mathcal{P}_{(1)}X + \mathcal{Q}_{(1)}U + X\mathcal{P}^{T}_{(2)} \pm U\mathcal{Q}^{T}_{(2)} \\ &\quad + \mu\, [\mathcal{P}_{(1)}\, (X\mathcal{P}^{T}_{(2)} \pm U\mathcal{Q}^{T}_{(2)}) + \mathcal{Q}_{(1)}\, (\pm X\mathcal{Q}^{T}_{(2)} + U\mathcal{P}^{T}_{(2)})], \\ U^{\Delta} &= \mathcal{Q}_{(1)}X + \mathcal{P}_{(1)}U \pm X\mathcal{Q}^{T}_{(2)} + U\mathcal{P}^{T}_{(2)} \\ &\quad + \mu\, [\mathcal{Q}_{(1)}\, (X\mathcal{P}^{T}_{(2)} \pm U\mathcal{Q}^{T}_{(2)}) + \mathcal{P}_{(1)}\, (\pm X\mathcal{Q}^{T}_{(2)} + U\mathcal{P}^{T}_{(2)})] \end{aligned}$$

with the initial conditions $X(a) = 0$ and $U(a) = I$. Moreover, for all $t \in [a, b]_{\mathbb{T}}$ we have

$$\mathrm{Cosh}^{\pm}(\mathrm{Cosh}^{\pm})^{T} - \mathrm{Sinh}^{\pm}(\mathrm{Sinh}^{\pm})^{T} = I = (\mathrm{Cosh}^{\pm})^{T}\, \mathrm{Cosh}^{\pm} - (\mathrm{Sinh}^{\pm})^{T}\, \mathrm{Sinh}^{\pm}, \tag{4.13}$$

$$\mathrm{Sinh}^{\pm}(\mathrm{Cosh}^{\pm})^{T} = \mathrm{Cosh}^{\pm}(\mathrm{Sinh}^{\pm})^{T}, \quad (\mathrm{Sinh}^{\pm})^{T}\, \mathrm{Cosh}^{\pm} = (\mathrm{Cosh}^{\pm})^{T}\, \mathrm{Sinh}^{\pm}. \tag{4.14}$$

Remark 4.14. An analogous statement as in Remark 3.17 now applies to the solutions $\binom{\mathrm{Sinh}^{+}}{\mathrm{Cosh}^{+}}$ and $\binom{\mathrm{Sinh}^{-}}{\mathrm{Cosh}^{-}}$. Namely, these two pairs are not conjoined bases of their corresponding systems, because these systems are not symplectic.

Remark 4.15. In the continuous time case, the assertion of Theorem 4.13 can be found in [22, Theorem 4.2]. For the discrete time hyperbolic systems this result is new, see the details in [33, Theorem 4.13].

When the two systems in (4.10) are the same, Theorem 4.13 yields the following.

Corollary 4.16. *Assume that $\mathcal{P}$ and $\mathcal{Q}$ satisfy* (4.1) *and* (4.2). *Then the system*

$$\begin{aligned} X^{\Delta} &= \mathcal{P}X + \mathcal{Q}U + X\mathcal{P}^{T} + U\mathcal{Q}^{T} + \mu\, [\mathcal{P}\, (X\mathcal{P}^{T} + U\mathcal{Q}^{T}) + \mathcal{Q}\, (X\mathcal{Q}^{T} + U\mathcal{P}^{T})], \\ U^{\Delta} &= \mathcal{Q}X + \mathcal{P}U + X\mathcal{Q}^{T} + U\mathcal{P}^{T} + \mu\, [\mathcal{Q}\, (X\mathcal{P}^{T} + U\mathcal{Q}^{T}) + \mathcal{P}\, (X\mathcal{Q}^{T} + U\mathcal{P}^{T})] \end{aligned}$$

with the initial conditions $X(a) = 0$ and $U(a) = I$ possesses the solution

$$X = 2\, \mathrm{Sinh}\, \mathrm{Cosh}^{T} \quad \textit{and} \quad U = \mathrm{Cosh}\, \mathrm{Cosh}^{T} + \mathrm{Sinh}\, \mathrm{Sinh}^{T},$$

where Sinh *and* Cosh *are the matrix functions in Definition 4.5. Moreover, the above matrices* X *and* U *commute, i.e.,* $XU = UX$*.*

The previous corollary can be viewed as the n-dimensional analogy of the double angle formulae for scalar continuous time hyperbolic functions. In the continuous time case the content of Corollary 4.16 can be found in [22, Corollary 1]. In the discrete case we get a new result, namely [33, Corollary 4.15].

Now we can prove as in Corollary 3.20 the following identities.

Corollary 4.17. *For all* $t \in [a, b]_{\mathbb{T}}$ *we have the identities*

$$\mathrm{Sinh}_{(1)}\ \mathrm{Sinh}_{(2)}^{T} = \frac{1}{2}\,(\mathrm{Cosh}^{+} - \mathrm{Cosh}^{-}), \tag{4.15}$$

$$\mathrm{Cosh}_{(1)}\ \mathrm{Cosh}_{(2)}^{T} = \frac{1}{2}\,(\mathrm{Cosh}^{+} + \mathrm{Cosh}^{-}), \tag{4.16}$$

$$\mathrm{Sinh}_{(1)}\ \mathrm{Cosh}_{(2)}^{T} = \frac{1}{2}\,(\mathrm{Sinh}^{+} + \mathrm{Sinh}^{-}). \tag{4.17}$$

In the scalar continuous time case identities (4.15)–(4.17) have the form of elementary hyperbolic identities. The next definition of time scale matrix hyperbolic tangent and cotangent functions is from [28, Definition 2.2]. It extends the discrete matrix hyperbolic tangent and hyperbolic cotangent functions known in [19, Definition 3.2] to time scales. Recall that the matrix function Cosh is invertible for all $t \in [a, b]_{\mathbb{T}}$, see Remark 4.9.

Definition 4.18. We define the matrix-valued function *hyperbolic tangent* (we write Tanh) and, whenever $\mathrm{Sinh}(t)$ is invertible the matrix-valued function *hyperbolic cotangent* (we write Cotanh), by

$$\mathrm{Tanh}(t) := \mathrm{Cosh}^{-1}(t)\ \mathrm{Sinh}(t) \quad \text{and} \quad \mathrm{Cotanh}(t) := \mathrm{Sinh}^{-1}(t)\ \mathrm{Cosh}(t).$$

Similarly to Theorems 3.22 and 3.23 we can establish the symmetry of the functions Tanh and Cotanh. The following two results can be found in [28, Theorems 2.4, 2.5].

Theorem 4.19. *The following identities hold true*

$$\mathrm{Tanh}^{T}(t) = \mathrm{Tanh}(t), \tag{4.18}$$

$$\mathrm{Cosh}^{-1}(t)\,\mathrm{Cosh}^{T-1}(t) + \mathrm{Tanh}^{2}(t) = I, \tag{4.19}$$

$$\mathrm{Tanh}^{\Delta}(t) = [\mathrm{Cosh}^{\sigma}(t)]^{-1}\,\mathcal{Q}(t)\ \mathrm{Cosh}^{T-1}(t). \tag{4.20}$$

Theorem 4.20. *Whenever* $\mathrm{Cotanh}(t)$ *is defined we get*

$$\mathrm{Cotanh}^{T}(t) = \mathrm{Cotanh}(t), \tag{4.21}$$

$$\mathrm{Cotanh}^{2}(t) - \mathrm{Sinh}^{-1}(t)\ \mathrm{Sinh}^{T-1}(t) = I. \tag{4.22}$$

Moreover, if $\mathrm{Sinh}(t)$ *and* $\mathrm{Sinh}^{\sigma}(t)$ *are invertible, then*

$$\mathrm{Cotanh}^{\Delta}(t) = -[\mathrm{Sinh}^{\sigma}(t)]^{-1}\, \mathcal{Q}(t)\, \mathrm{Sinh}^{T-1}(t). \tag{4.23}$$

Remark 4.21. In the scalar case $n = 1$ identities (4.18) and (4.21) are trivial. In the scalar continuous time case identities (4.19), (4.22), (4.20), and (4.23) take the form

$$\frac{1}{\cosh^2(s)} + \tanh^2(s) = 1, \quad \mathrm{cotanh}^2(s) - \frac{1}{\sinh^2(s)} = 1, \quad \text{with} \quad s = \int_a^t \mathcal{Q}(\tau)\, \mathrm{d}\tau,$$
$$\left(\tanh \int_a^t \mathcal{Q}(\tau)\, \mathrm{d}\tau\right)' = \frac{\mathcal{Q}(t)}{\cosh^2 s}, \quad \left(\mathrm{cotanh} \int_a^t \mathcal{Q}(\tau)\, \mathrm{d}\tau\right)' = \frac{-\mathcal{Q}(t)}{\sinh^2 s},$$

compare with Remark 4.6 (iii). The discrete versions of these identities can be found in [19, Theorem 3.4] or [32, Theorem 89].

Next, similarly to the definitions of the time scale matrix functions $\mathrm{Sinh}_{(i)}$, $\mathrm{Cosh}_{(i)}$, $\mathrm{Sinh}^{\pm}$, and $\mathrm{Cosh}^{\pm}$ from (4.10)–(4.12) we define

$$\mathrm{Tanh}_{(i)}(t) := \mathrm{Cosh}_{(i)}^{-1}(t)\, \mathrm{Sinh}_{(i)}(t), \qquad \mathrm{Cotanh}_{(i)}(t) := \mathrm{Sinh}_{(i)}^{-1}(t)\, \mathrm{Cosh}_{(i)}(t),$$
$$\mathrm{Tanh}^{\pm}(t) := [\mathrm{Cosh}^{\pm}(t)]^{-1}\, \mathrm{Sinh}^{\pm}(t), \qquad \mathrm{Cotanh}^{\pm}(t) := [\mathrm{Sinh}^{\pm}(t)]^{-1}\, \mathrm{Cosh}^{\pm}(t).$$

Remark 4.22. As in Remark 3.26 we conclude that the first identities from (4.14) imply the symmetry of the functions $\mathrm{Tanh}^{\pm}$. Similarly, the functions $\mathrm{Cotanh}^{\pm}$ are also symmetric.

As it was the case for the trigonometric functions in Theorem 3.27, the results of the following theorem are new even in the special case of continuous and discrete time, see also [33, Theorem 4.25].

Theorem 4.23. *For all* $t \in [a, b]_{\mathbb{T}}$ *such that all involved functions are defined we have (suppressing the argument* t*)*

$$\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)} = \mathrm{Tanh}_{(1)}\, (\mathrm{Cotanh}_{(2)} \pm \mathrm{Cotanh}_{(1)})\, \mathrm{Tanh}_{(2)}, \tag{4.24}$$
$$\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)} = \mathrm{Cosh}_{(1)}^{-1}\, \mathrm{Sinh}^{\pm}\, \mathrm{Cosh}_{(2)}^{T-1}, \tag{4.25}$$
$$\mathrm{Cotanh}_{(1)} \pm \mathrm{Cotanh}_{(2)} = \mathrm{Cotanh}_{(1)}\, (\mathrm{Tanh}_{(2)} \pm \mathrm{Tanh}_{(1)})\, \mathrm{Cotanh}_{(2)}, \tag{4.26}$$
$$\mathrm{Cotanh}_{(1)} \pm \mathrm{Cotanh}_{(2)} = \pm\, \mathrm{Sinh}_{(1)}^{-1}\, \mathrm{Sinh}^{\pm}\, \mathrm{Sinh}_{(2)}^{T-1}, \tag{4.27}$$
$$\mathrm{Tanh}^{\pm} = \mathrm{Cosh}_{(2)}^{T-1}\, (I \pm \mathrm{Tanh}_{(1)}\, \mathrm{Tanh}_{(2)})^{-1}\, (\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)})\, \mathrm{Cosh}_{(2)}^{T}, \tag{4.28}$$
$$\mathrm{Cotanh}^{\pm} = \mathrm{Sinh}_{(2)}^{T-1}\, (\mathrm{Cotanh}_{(2)} \pm \mathrm{Cotanh}_{(1)})^{-1} \times \times\, (\mathrm{Cotanh}_{(1)}\, \mathrm{Cotanh}_{(2)} \pm I)\, \mathrm{Sinh}_{(2)}^{T}. \tag{4.29}$$

Consider now the system (THS) in the scalar continuous time case with $\mathcal{P}(t) \equiv 0$ and $\mathcal{Q}(t) \equiv 1$, or equivalently system (CHS) with $Q(t) \equiv 1$. Then the identities (4.24)–(4.29) have the form of elementary hyperbolic identities.

5. CONCLUDING REMARKS

In this paper we extended to the time scale matrix case several identities known in particular for the scalar continuous time trigonometric and hyperbolic functions. Namely, for trigonometric functions these are the identity $\cos^2(t) + \sin^2(t) = 1$ in Corollary 3.10, and the identities displayed in Theorems 3.13, 3.22, and 3.27, Remarks 3.15 and 3.24, and Corollaries 3.19 and 3.20. For hyperbolic functions these are the identity $\cosh^2(t) - \sinh^2(t) = 1$ in Corollary 4.8, and the identities displayed in Theorems 4.10 and 4.23, Remarks 4.12 and 4.21, and Corollary 4.17.

On the other hand, there are still several trigonometric and hyperbolic identities which we could not extend to the general time scale matrix case. For example, these are the identities $\sin x \pm \sin y = 2 \sin \frac{x\pm y}{2} \cos \frac{x\mp y}{2}$, as well as other corresponding identities for the sum or difference of scalar trigonometric and hyperbolic functions. When $y = 0$ in the above identity, we get $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$. The right-hand side is similar to the solution $X(t)$ in Corollary 3.19, but the left-hand side is not the matrix function Sin, because the corresponding system is not a trigonometric system (TTS), see Remark 3.17.

Furthermore, as for the time scale versions of the identities

$$\left\{ \begin{aligned} \sin(x+y)\sin(x-y) &= \sin^2 x - \sin^2 y, \\ \cos(x+y)\cos(x-y) &= \cos^2 x - \sin^2 y, \\ \sinh(x+y)\sinh(x-y) &= \sinh^2 x - \sinh^2 y, \\ \cosh(x+y)\cosh(x-y) &= \sinh^2 x + \cosh^2 y, \end{aligned} \right. \tag{5.1}$$

in the *scalar case* on an arbitrary time scale we can calculate the products

$$\left\{ \begin{aligned} \mathrm{Sin}^+\,\mathrm{Sin}^- &= \mathrm{Sin}^2_{(1)} - \mathrm{Sin}^2_{(2)}, & \mathrm{Sinh}^+\,\mathrm{Sinh}^- &= \mathrm{Sinh}^2_{(1)} - \mathrm{Sinh}^2_{(2)}, \\ \mathrm{Cos}^+\,\mathrm{Cos}^- &= \mathrm{Cos}^2_{(1)} - \mathrm{Sin}^2_{(2)}, & \mathrm{Cosh}^+\,\mathrm{Cosh}^- &= \mathrm{Sinh}^2_{(1)} + \mathrm{Cosh}^2_{(2)}, \end{aligned} \right. \tag{5.2}$$

because the cross terms cancel due to the commutativity. However, in the *general case* the matrix products $\mathrm{Sin}^+\,\mathrm{Sin}^-$, $\mathrm{Cos}^+\,\mathrm{Cos}^-$, $\mathrm{Sinh}^+\,\mathrm{Sinh}^-$, and $\mathrm{Cosh}^+\,\mathrm{Cosh}^-$ corresponding to the left-hand side of (5.1) do not simplify as in (5.2), since the matrix multiplication is not commutative.

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REFERENCES

[1] C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[2] D. R Anderson, Discrete trigonometric matrix functions, *Panamer. Math. J.* **7** (1997), no. 1, 39–54.

[3] J. H. Barrett, A Prüfer transformation for matrix differential systems, *Proc. Amer. Math. Soc.* **8** (1957), 510–518.

[4] D. S. Bernstein, *Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press, Princeton, 2005.

[5] P. Bohl, Über eine Differentialgleichung der Störungstheorie, *J. Reine Angew. Math.* **131** (1906), 268–321.

[6] M. Bohner, O. Došlý, Trigonometric transformations of symplectic difference systems, *J. Differential Equations* **163** (2000), no. 1, 113–129.

[7] M. Bohner, O. Došlý, The discrete Prüfer transformations, *Proc. Amer. Math. Soc.* **129** (2001), no. 9, 2715–2726.

[8] M. Bohner, O. Došlý, Trigonometric systems in oscillation theory of difference equations, in: "Dynamic Systems and Applications", Proceedings of the Third International Conference on Dynamic Systems and Applications (Atlanta, GA, 1999), Vol. 3, pp. 99–104, Dynamic, Atlanta, GA, 2001.

[9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[10] M. Bohner, A. Peterson, editors, *Advances in Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2003.

[11] Z. Došlá, D. Škrabáková, Phases of second order linear difference equations and symplectic systems, *Math. Bohem.* **128** (2003), no. 3, 293–308.

[12] O. Došlý, On transformations of self-adjoint linear differential systems, *Arch. Math. (Brno)* **21** (1985), no. 3, 159–170.

[13] O. Došlý, Phase matrix of linear differential systems, *Časopis pro pěstování matematiky* **110** (1985), 183–192.

[14] O. Došlý, On some properties of trigonometric matrices, *Časopis pro pěstování matematiky* **112** (1987), no. 2, 188–196.

[15] O. Došlý, Riccati matrix differential equation and classification of disconjugate differential systems, *Arch. Math. (Brno)* **23** (1987), no. 4, 231–241.

[16] O. Došlý, The Bohl Transformation and its applications, in: "2004–Dynamical Systems and Applications", Proceedings of the International Conference (Antalya, 2004), pp. 371–385, GBS Publishers & Distributors, Delhi, 2005.

[17] O. Došlý, S. Hilger, R. Hilscher, Symplectic dynamic systems, in: "Advances in Dynamic Equations on Time Scales", M. Bohner and A. Peterson, editors, pp. 293–334, Birkhäuser, Boston, 2003.

[18] O. Došlý, Š. Pechancová, Trigonometric recurrence relations and tridiagonal trigonometric matrices, *Internat. J. Difference Equ.* **1** (2006), no. 1, 19–29.

[19] O. Došlý, Z. Pospíšil, Hyperbolic transformation and hyperbolic difference systems, *Fasc. Math.* **32** (2001), 25–48.

[20] G. J. Etgen, A note on trigonometric matrices, *Proc. Amer. Math. Soc.* **17** (1966), 1226–1232.

[21] G. J. Etgen, Oscillatory properties of certain nonlinear matrix differential systems of second order, *Trans. Amer. Math. Soc.* **122** (1966), 289–310.

[22] K. Filakovský, *Properties of Solutions of Hyperbolic System* (in Czech), MSc thesis, University of J. E. Purkyn, Brno, 1987.

[23] S. Hilger, Analysis on measure chains – A unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56.

[24] S. Hilger, Matrix Lie theory and measure chains, in: "Dynamic Equations on Time Scales", R. P. Agarwal, M. Bohner, and D. O'Regan, editors, *J. Comput. Appl. Math.* **141** (2002), no. 1–2, 197–217.

[25] R. Hilscher, V. Zeidan, Time scale symplectic systems without normality, *J. Differential Equations* **230** (2006), no. 1, 140–173.

[26] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.

[27] Z. Pospíšil, Hyperbolic sine and cosine functions on measure chains, in: "Proceedings of the Third World Congress of Nonlinear Analysts, Part 2" (Catania, 2000), *Nonlinear Anal.* **47** (2001), no. 2, 861–872.

[28] Z. Pospíšil, Hyperbolic matrix systems, functions and transformations on time scales, author preprint, October 2001.

[29] Z. Pospíšil, An inverse problem for matrix trigonometric and hyperbolic functions on measure chains, in: "Colloquium on Differential and Difference Equations, CDDE 2002" (Brno), pp. 205–211, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., Vol. 13, Masaryk Univ., Brno, 2003.

[30] W. T. Reid, A Prüfer transformation for differential systems, *Pacific J. Math.* **8** (1958), 575–584.

[31] W. T. Reid, *Ordinary Differential Equations*, John Willey & Sons, New York, 1971.

[32] P. Zemánek, *Discrete Symplectic Systems* (in Czech), MSc thesis, Masaryk University, Brno, 2007.

[33] P. Zemánek, Discrete trigonometric and hyperbolic systems: An overview, in: "Ulmer Seminare", University of Ulm, Ulm, submitted.