FUNCTIONAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH STATE-DEPENDENT DELAY

MOHAMED ABDALLA DAR WISH AND SOTIRIS K. NTOUNYS

Department of Mathematics, Faculty of Science
Alexandria University at Damanhour, 22511 Damanhour, Egypt
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

ABSTRACT. In this paper we study the existence of solutions for the initial value problem for functional differential equations, as well as, for neutral functional differential equations of fractional order with state-dependent delay. The nonlinear alternative of Leray-Schauder type is the main tool in our analysis.

AMS (MOS) Subject Classification. 26A33, 26A42, 34K30

1. INTRODUCTION

The purpose of this paper is to study the existence of solutions for initial value problems (IVP for short) for both functional differential equations of fractional order with state-dependent delay

\begin{equation}
D^\beta y(t) = f(t, y_\rho(t,y_t)), \quad \text{for each } t \in J = [0,b], \quad 0 < \beta < 1,
\end{equation}

\begin{equation}
y(t) = \varphi(t), \quad t \in (-\infty, 0]
\end{equation}

as well as for neutral functional differential equations of fractional order with state-dependent delay

\begin{equation}
D^\beta [y(t) - g(t, y_\rho(t,y_t))] = f(t, y_\rho(t,y_t)), \quad \text{for each } t \in J,
\end{equation}

\begin{equation}
y(t) = \varphi(t), \quad t \in (-\infty, 0],
\end{equation}

where \( D^\beta \) is the standard Riemman-Liouville fractional derivative.

Here, \( f : J \times \mathcal{B} \to \mathbb{R}, \ g : J \times \mathcal{B} \to \mathbb{R} \) and \( \rho : J \times \mathcal{B} \to (-\infty, b] \) are appropriate given functions, \( \varphi \in \mathcal{B}, \ \varphi(0) = 0, \ g(0, \varphi) = 0 \) and \( \mathcal{B} \) is called a phase space that will be defined later (see Section 2).

For any function \( y \) defined on \((-\infty, b]\) and any \( t \in J \), we denote by \( y_t \) the element of \( \mathcal{B} \) defined by

\[ y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0]. \]
The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [12] (see also Kappel and Schappacher [19] and Schumacher [30]). For a detailed discussion on this topic we refer the reader to the book by Hino et al [18].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years, we refer to [2, 3, 5, 7, 13, 14, 15] and the references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [9, 17, 20, 22, 24, 25, 26, 27] and references therein.

Differential equations of fractional order play a very important role in describing some real-world problems. For example, some problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [8, 10, 17, 23, 27, 28, 29] and references therein. The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations, for example see [1, 20, 21, 26, 31].

Our approach is based on the nonlinear alternative of Leray-Schauder type [11]. These results can be considered as a contribution to this emerging field.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $\mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \}$, and let $C^0(\mathbb{R}^+)$ be the space of all continuous functions on $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R}_0^+)$ of all continuous real functions on $\mathbb{R}_0^+ = \{ x \in \mathbb{R} : x \geq 0 \}$, which later identify by abuse of notation, with the class of all $f \in C^0(\mathbb{R}^+)$ such that

$$\lim_{t \to 0^+} f(t) = f(0^+) \in \mathbb{R}.$$ 

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\},$$

where $| \cdot |$ denotes a suitable complete norm on $\mathbb{R}$.

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary orders, see [20, 25, 26, 27].
Definition 2.1. The fractional primitive of order $\beta > 0$ of a function $h : \mathbb{R}^+ \to \mathbb{R}$ of order $\beta \in \mathbb{R}^+$ is defined by

$$I^\beta_0 h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds,$$

provided the right hand side exists pointwise on $\mathbb{R}^+$. $\Gamma$ is the gamma function. For instance, $I^\beta_0 h$ exists for all $\beta > 0$, when $h \in C^0(\mathbb{R}^+_0) \cap L^1_{loc}(\mathbb{R}^+_0)$; note also that when $h \in C^0(\mathbb{R}^+_0)$ then $I^\beta_0 h \in C^0(\mathbb{R}^+_0)$ and moreover $I^\beta_0 h(0) = 0$.

Definition 2.2. The fractional derivative of order $\beta > 0$ of a continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\frac{d^\beta}{dt^\beta} h(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t (t-s)^{-\beta} h(s) ds = \frac{d}{dt} I^{1-\beta}_{a} h(t).$$

For our existence results we will need the following lemma.

Lemma 2.3. [6] Let $0 < \beta < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t \to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then $y$ is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(y(s)) ds,$$

if and only if, $y$ is a solution of the initial value problem for the fractional differential equation

$$D^\beta y(t) = h(y(t)), \ t \in (0, b], \quad y(0) = 0.$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [18]. More precisely, $\mathcal{B}$ will be a linear space of all functions from $(-\infty, 0]$ to $\mathbb{R}$ endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ satisfying the following axioms:

(A) If $y : (-\infty, b] \to \mathbb{R}$, $b > 0$, is continuous on $J$ and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

\begin{enumerate}
\item[(i)] $y_t \in \mathcal{B},$
\item[(ii)] $\|y_t\|_{\mathcal{B}} \leq K(t) \sup \{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}},$
\item[(iii)] $|y(t)| \leq H\|y_t\|_{\mathcal{B}},$
\end{enumerate}

where $H > 0$ is a constant, $K : [0, \infty) \to [1, \infty)$ is continuous, $M : [0, \infty) \to [1, \infty)$ is locally bounded and $H, K, M$ are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), $y_t$ is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A-2) The space $\mathcal{B}$ is complete.

The next lemma is a consequence of the phase space axioms and is proved in [13].
Lemma 2.4. Let $\varphi \in B$ and $I = (\gamma, 0]$ be such that $\varphi_t \in B$ for every $t \in I$. Assume that there exists a locally bounded function $J^\varphi : I \to [0, \infty)$ such that $\|\varphi_t\|_B \leq J^\varphi(t)\|\varphi\|_B$ for every $t \in I$. If $y : (\infty, b] \to \mathbb{R}$ is continuous on $J$ and $y_0 = \varphi$, then
\[
\|y_t\|_B \leq (M_b + J^\varphi(\max\{\gamma, -|s|\}))\|\varphi\|_B + K_b \sup\{|y(\theta)| : \theta \in [0, \max\{0, s\}]\}, \quad s \in (\gamma, b],
\]
where, we denoted $K_b = \sup_{t \in J} K(t)$ and $M_b = \sup_{t \in J} M(t)$.

Finally, we state the following generalization of Gronwall’s lemma for singular kernels, whose proof can be found in [16], Lemma 7.1.1, will be essential for our main results.

Lemma 2.5. Let $v : [0, b] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b]$ and there are constants $a > 0$ and $0 < \beta < 1$ such that
\[
v(t) \leq w(t) + a \int_0^t v(s) \frac{1}{(t - s)^\beta} ds,
\]
then, there exists a constant $K = K(\beta)$ such that
\[
v(t) \leq w(t) + Ka \int_0^t w(s) \frac{1}{(t - s)^\beta} ds,
\]
for every $t \in [0, b]$.

3. FDEs OF FRACTIONAL ORDER

In this section, the nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions of problem (1.1)–(1.2).

Let us start by defining what we mean by a solution of problem (1.1)–(1.2).

Definition 3.1. A function $y : (-\infty, b] \to \mathbb{R}$ is said to be a solution of (1.1)–(1.2) if $y_0 = \varphi, y_{\rho(s, y_s)} \in B$ for every $s \in J$ and
\[
y(t) = \frac{1}{\Gamma(\beta)} \int_0^t f(s, y_{\rho(s, y_s)}) \frac{1}{(t - s)^{1-\beta}} ds, \quad t \in J.
\]

In what follows we assume that $\rho : J \times B \to (-\infty, b]$ is continuous and $\varphi \in B$ and $f$ satisfies the following hypotheses:

(H1) $f$ is a continuous function;

(H2) There exist $p, q \in C(J, \mathbb{R}^+)$ such that
\[
|f(t, u)| \leq p(t) + q(t)\|u\|_B
\]
for $t \in J$ and each $u \in B$, and $\|I^\beta p\|_\infty < +\infty$;
(H3) The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in J \times B, \rho(s, \psi) \leq 0\}$ into $B$. Moreover, there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_B \leq J^\varphi(t)\|\varphi\|_B$ for every $t \in \mathcal{R}(\rho^-)$.

**Remark 3.2.** The hypothesis (H3) is adapted from [13], where we refer for remarks concerning this hypothesis.

**Theorem 3.3.** Assume that the hypotheses (H1)–(H3) hold. If $\rho(t, \psi) \leq t$ for every $(t, \psi) \in J \times B$, then the IVP (1.1)–(1.2) has at least one solution on $(-\infty, b]$.

**Proof.** Let $Y = \{u \in C(J, \mathbb{R}) : u(0) = \varphi(0) = 0\}$ endowed with the uniform convergence topology and $N : Y \to Y$ be the operator defined by

$$Ny(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_\rho(s, \bar{y}_s))}{(t-s)^{1-\beta}} \, ds, \quad t \in J,$$

where $\bar{y} : (-\infty, b] \to \mathbb{R}$ is such that $\bar{y}_0 = \varphi$ and $\bar{y} = y$ on $J$. From axiom (A) and our assumption on $\varphi$, we infer that $Ny(\cdot)$ is well defined and continuous.

Let $\bar{\varphi} : (-\infty, b] \to \mathbb{R}$ be the extension of $\varphi$ to $(-\infty, b]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on $J$ and $\bar{J}^\varphi = \sup\{J^\varphi : s \in \mathcal{R}(\rho^-)\}$.

We will prove that $N(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_J, Y)$ into $B_r(\bar{\varphi}|_J, Y)$.

**Step 1:** $N$ is continuous on $B_r(\bar{\varphi}|_J, Y)$.

This was proved in [13, p. 515, Step 3].

**Step 2:** The set $N(B_r(\bar{\varphi}|_J, Y))(t) = \{Ny(t) : y \in B_r(\bar{\varphi}|_J, Y)\}$ is relatively compact in $\mathbb{R}$ for every $t \in J$.

The case $t = 0$ is obvious. Let $0 < \varepsilon < t \leq b$. If $y \in B_r(\bar{\varphi}|_J, Y)$, from Lemma 2.4 follows that

$$\|\bar{y}_\rho(t, \bar{y}_t)\|_B \leq r^* = (M_b + \bar{J}^\varphi)\|\varphi\|_B + K_b r$$

and so

$$|(Ny)(t)| = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_\rho(s, \bar{y}_s))}{(t-s)^{1-\beta}} \, ds \leq \frac{1}{\Gamma(\beta)} \int_0^t \frac{p(s) + q(s)\|\bar{y}_\rho(s, \bar{y}_s)\|_B}{(t-s)^{1-\beta}} \, ds = \frac{b^\beta\|p\|_\infty}{\Gamma(\beta+1)} + \frac{b^\beta\|q\|_\infty}{\Gamma(\beta+1)} r^* := \ell$$

**Step 3:** $N$ maps bounded sets into equicontinuous sets of $Y$. 

Let \( t_1, t_2 \in [0, b], \ t_1 < t_2 \) and let \( B_r \) as in Step 2. Let \( y \in B_r \). Then for each \( t \in [0, b] \), we have

\[
|(Ny)(t_2) - (Ny)(t_1)| = \frac{1}{\Gamma(\beta)} \left| \int_{t_1}^{t_2} \left( (t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1} \right) f(s, \bar{y}_{\rho(s, \bar{y}_s)}) \, ds \right|
\]

As \( t_1 \to t_2 \) the right-hand side of the above inequality tends to zero. The equicontinuity for the cases \( t_1 < t_2 \leq 0 \) and \( t_1 \leq 0 \leq t_2 \) is obvious.

As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that \( N \) is continuous and completely continuous.

**Step 4: (A priori bounds).** We now show there exists an open set \( U \subseteq Y \) with \( y \neq \lambda N(y) \) for \( \lambda \in (0, 1) \) and \( y \in \partial U \).

Let \( y \in Y \) and \( y = \lambda N(y) \) for some \( 0 < \lambda < 1 \). Then for each \( t \in [0, b] \) we have

\[
y(t) = \lambda \left[ \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_{\rho(s, \bar{y}_s)})}{(t - s)^{1-\beta}} \, ds \right].
\]

This implies by (H2)

\[
|y(t)| = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_{\rho(s, \bar{y}_s)})}{(t - s)^{1-\beta}} \, ds 
\leq \frac{1}{\Gamma(\beta)} \int_0^t \frac{p(s) + q(s)\|f\|_B + K_b \sup \{|\bar{y}(s)| \in [0, t]|}{(t - s)^{1-\beta}} \, ds 
\leq \frac{b^2 \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{q\|q\|_\infty}{\Gamma(\beta)} \int_0^t \frac{(M_b + \bar{J}^\varphi)\|f\|_B + K_b \sup \{|\bar{y}(s)| \in [0, t]|}{(t - s)^{1-\beta}} \, ds,
\]

since \( \rho(s, \bar{y}_s) \leq s \) for every \( s \in J \). Here \( \bar{J}^\varphi = \sup \{J^\varphi(s) : s \in \mathcal{R}(\rho^-) \} \).

If \( \mu(t) = (M_b + \bar{J}^\varphi)\|f\|_B + K_b \sup \{|\bar{y}(s)| : s \in [0, \max\{0, \rho(s, \bar{y}_s)\}] \} \) then we obtain

\[
\mu(t) \leq \frac{b^2 \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{q\|q\|_\infty}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} \mu(s) \, ds.
\]
Then from Lemma 2.5, there exists $K = K(\beta)$ such that we have

$$|\mu(t)| \leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{\|q\|_\infty}{\Gamma(\beta)} K(\beta) \int_0^t (t - s)^{\beta - 1} \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} ds$$

$$= \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} \left(1 + \frac{\|q\|_\infty}{\Gamma(\beta)} K(\beta) \int_0^t (t - s)^{\beta - 1} ds\right)$$

$$\leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} \left(1 + \frac{\|q\|_\infty}{\Gamma(\beta + 1)} K(\beta) b^\beta\right).$$

Then

$$\|\mu\|_\infty \leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} \left(1 + \frac{\|q\|_\infty}{\Gamma(\beta + 1)} K(\beta) b^\beta\right) := M^*.$$ 

Set

$$U = \{y \in Y : \|y\|_\infty < M^* + 1\}.$$ 

$N : \overline{U} \to Y$ is continuous and completely continuous. From the choice of $U$, there is no $y \in \partial U$ such that $y = \lambda N(y)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [11], we deduce that $N$ has a fixed point $y$ in $U$. 

\[\square\]

4. **NFDEs of Fractional Order**

In this section we give existence results for the IVP (1.3)–(1.4).

**Definition 4.1.** A function $y : (-\infty, b] \to \mathbb{R}$ is said to be a solution of (1.3)–(1.4) if $y_0 = \varphi, y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and

$$y(t) = g(s, y_{\rho(s,y_s)}) + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s, y_{\rho(s,y_s)}) ds, \quad t \in J.$$ 

**Theorem 4.2.** Assume (H1)–(H2) and the following condition:

(H4) the function $g$ is continuous and completely continuous, and for any bounded set $Q$ in $\mathcal{B} \cap C([0, b], \mathbb{R})$, the set $\{t \to g(t, y_t) : y \in Q\}$ is equicontinuous in $C([0, b], \mathbb{R})$, and there exist constants $0 \leq K_0 d_1 < 1$, $d_2 \geq 0$ such that

$$|g(t, u)| \leq d_1 \|u\| + d_2, \quad t \in [0, b], \quad u \in \mathcal{B}.$$ 

If $\rho(t, \psi) \leq t$ for every $(t, \psi) \in J \times \mathcal{B}$, then the IVP (1.3)–(1.4) has at least one solution on $(-\infty, b]$.

**Proof.** Consider the operator $N_0 : C((0, b], \mathbb{R}) \to C((-\infty, b], \mathbb{R})$ defined by,

$$N_0(y)(t) = \begin{cases} 
\varphi(t), & \text{if } t \in (-\infty, 0], \\
\varphi(0) - g(0, \varphi) + g(t, y_{\rho(t,y_t)}) \\
\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s, y_{\rho(s,y_s)}) ds, & \text{if } t \in [0, b].
\end{cases}$$
In analogy to Theorem 3.3, we consider the operator \( N_1 : Y \rightarrow Y \) defined by

\[
(N_1 y)(t) = \begin{cases} 
0, & t \leq 0, \\
g(t, \bar{y}_{p(s,\bar{u})}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{p(s,\bar{u})}) \, ds, & t \in [0, b].
\end{cases}
\]

We shall show that the operator \( N_1 \) is continuous and completely continuous. Using (H4) it suffices to show that the operator \( N_2 : Y \rightarrow Y \) defined by,

\[
N_2(y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{p(s,\bar{u})}) \, ds, \quad t \in [0, b],
\]

is continuous and completely continuous. This was proved in Theorem 3.3.

We now show there exists an open set \( U \subseteq Y \) with \( y \neq \lambda N_1(y) \) for \( \lambda \in (0, 1) \) and \( y \in \partial U \).

Let \( y \in Y \) and \( y = \lambda N_1(y) \) for some \( 0 < \lambda < 1 \). Then

\[
y(t) = \lambda \left[ g(s, \bar{y}_{p(s,\bar{u})}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{p(s,\bar{u})}) \, ds \right], \quad t \in [0, b],
\]

and

\[
|y(t)| \leq d_1((M_b + \bar{J}^p)\|\varphi\|_B + K_b \sup \{ |y(s)| : s \in [0, t] \}) + d_2 \\
+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [p(s) + q(s)((M_b + \bar{J}^p)\|\varphi\|_B + K_b \sup \{ |y(s)| : s \in [0, t] \})] \, ds \\
\leq d_1((M_b + \bar{J}^p)\|\varphi\|_B + K_b \sup \{ |y(s)| : s \in [0, t] \}) + d_2 \\
+ \frac{b^2 \|p\|_\infty}{\Gamma(\beta+1)} + \frac{\|q\|_\infty}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}((M_b + \bar{J}^p)\|\varphi\|_B + K_b \sup \{ |y(s)| : s \in [0, t] \}) \, ds,
\]

for \( t \in (0, b] \). If \( \mu(t) = \sup \{ |y(s)| : s \in [0, t] \} \) then

\[
\mu(t) \leq d_1(M_b + \bar{J}^p)\|\varphi\|_B + d_1 K_b \mu(t) + d_2 \\
+ \frac{b^2 \|p\|_\infty}{\Gamma(\beta+1)} + \frac{b^2 \|q\|_\infty}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta-1}(M_b + \bar{J}^p)\|\varphi\|_B \\
+ \frac{\|q\|_\infty}{\Gamma(\beta)} K_b \int_0^t (t-s)^{\beta-1} \mu(s) \, ds \\
\leq d_1(M_b + \bar{J}^p)\|\varphi\|_B + d_1 K_b \mu(t) + d_2 \\
+ \frac{b^2 \|p\|_\infty}{\Gamma(\beta+1)} + \frac{b^2 \|q\|_\infty}{\Gamma(\beta+1)} t^{\beta-1}(M_b + \bar{J}^p)\|\varphi\|_B \\
+ K_b \frac{\|q\|_\infty}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mu(s) \, ds, \quad t \in (0, b],
\]
or

\[ \mu(t) \leq \frac{1}{1 - Kd_1} \left[ d_1(M_b + \bar{J}^\varphi)\|\varphi\|_B + d_2 + \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{b^\beta \|q\|_\infty}{\Gamma(\beta + 1)} (M_b + \bar{J}^\varphi)\|\varphi\|_B \right] \\
+ \frac{K_b}{1 - Kd_1} \frac{\|q\|_\infty}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \mu(s) \, ds, \quad t \in (0, b]. \]

Consequently

\[ \|\mu\|_\infty \leq \frac{1}{1 - Kd_1} \left[ d_1(M_b + \bar{J}^\varphi)\|\varphi\|_B + d_2 + \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{b^\beta \|q\|_\infty}{\Gamma(\beta + 1)} (M_b + \bar{J}^\varphi)\|\varphi\|_B \right] \\
+ \frac{K_b}{1 - Kd_1} \frac{\|q\|_\infty}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \mu(s) \, ds \]

and by Lemma 2.5, there exists \( K = K(\beta) \) such that

\[ \|\mu\|_\infty \leq \Lambda_1 + \Lambda_2 K(\beta) \int_0^t (t - s)^{\beta - 1} \Lambda_1 \, ds \leq \Lambda_1 + \Lambda_2 K(\beta) b^\beta := L^*, \]

where

\[ \Lambda_1 = \frac{1}{1 - Kd_1} \left[ d_1(M_b + \bar{J}^\varphi)\|\varphi\|_B + d_2 + \frac{b^\beta \|p\|_\infty}{\Gamma(\beta + 1)} + \frac{b^\beta \|q\|_\infty}{\Gamma(\beta + 1)} (M_b + \bar{J}^\varphi)\|\varphi\|_B \right], \]

\[ \Lambda_1 = \frac{K_b}{1 - Kd_1} \frac{\|q\|_\infty}{\Gamma(\beta)}. \]

Set

\[ U_1 = \{ y \in Y : \|y\|_\infty < L^* + 1 \}. \]

From the choice of \( U \) there is no \( y \in \partial U_1 \) such that \( y = \lambda N_1(y) \) for \( \lambda \in (0, 1) \). As a consequence of the nonlinear alternative of Leray-Schauder type [11], we deduce that \( N_1 \) has a fixed point \( y \) in \( U_1 \). Then \( N_1 \) has a fixed point, which is a solution of the IVP (1.3)--(1.4).

**Acknowledgement.** This work was completed when the first author was visiting the Department of Mathematics, Massachusetts Institute of Technology, USA. It is a pleasure for him to express gratitude for the warm hospitality. Also, the authors would like to thank the referee for his corrections and valuable remarks.

**REFERENCES**


