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EXISTENCE AND UNIFORM DECAY FOR A NONLINEAR VISCOELASTIC EQUATION WITH STRONG DAMPING AND NONLINEAR BOUNDARY MEMORY DAMPING TERM

JONG YEOUL PARK AND JUNG AE KIM

Department of Mathematics, College of Science, Pusan National University,
Kumjung, Pusan 609-735, Republic of Korea
jyepark@pusan.ac.kr
Department of Mathematical Sciences, KAIST, Guseong-dong 373-1, Yuseong-gu,
Daejeon, 305-701, South Korea
jakim@kaist.ac.kr

ABSTRACT. In this paper, we prove the existence of the solution to the nonlinear viscoelastic equation with strong damping and nonlinear boundary memory damping term. Moreover, we discuss the uniform decay of the solution.

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1. INTRODUCTION

This manuscript is devoted to the existence and uniform decay rates of the energy of solutions for the nonlinear viscoelastic problem with strong damping and nonlinear boundary memory damping term:

$$
\begin{cases}
|u_t|^\rho u_{tt} - \beta\Delta u_{tt} - \Delta u - \Delta u_t = 0 & \text{in } \Omega \times (0,\infty), \\
u = 0 & \text{on } \Gamma_1 \times (0,\infty), \\
\beta \frac{\partial u_{tt}}{\partial \nu} + \frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + u = \int_0^t g(t-\tau)|u_t(\tau)|^\gamma u_t(\tau)d\tau & \text{on } \Gamma_0 \times (0,\infty), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{for } x \in \Omega,
\end{cases}
\tag{1.1}
$$

where Ω is a bounded domain of R^n with $\mathbb{C}^2$ boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures and ν denotes the unit outer normal vector pointing towards Ω. Here γ, ρ is a real number such that

$$
0 < \gamma, \quad \rho \le \frac{1}{n-2} \quad \text{if } n \ge 3 \text{ or } \gamma, \quad \rho > 0 \text{ if } n = 1, 2,
\tag{1.2}
$$

$\beta \ge 0$ and g represents the kernel of the memory term which will be assumed to decay exponentially.

Problem related to the equation

$$
f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0
\tag{1.3}
$$

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are interesting not only from the point of view of PDE general theory, but also due to its applications in Mechanics. For instance, when the material density, $f(u_t)$ is equal to 1, Equation (1.3) describes the extensional vibrations of thin rods, see Love [13] for the physical details. When the material density $f(u_t)$ is not constant, we are dealing with a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity.

On the other hand, it is important to observe that similar equations to the one given in (1.3) arise in the study of viscoelastic plate with memory, more precisely

$$u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t g(t-\tau)\Delta^2 u(\tau) d\tau = 0. \tag{1.4}$$

The existence of global weak solutions to problem (1.3), in the degenerate case, that is, when we have the equation

$$K(x,t)u_{tt} - \Delta u + F(u) - \Delta u_t = 0$$

and K can vanish, was studied by Ferreira and Pereira in [6]. More recently, Ferreira and Rojas Medar [7] studied the existence of weak solutions to problem (1.1) when $g = 0$, in non-cylindrical domains. However, no uniform decay result was presented in Reference [6] and in Reference [7] only an existence result was studied.

Concerning the study of plates, there is a substantial number of papers dealing with Equation (1.4). In this direction, we can cite the work of Lagnese [11], who studied the viscoelastic plate equation and showed that the energy decays to zero as time goes to infinity by introducing a dissipative mechanism on the boundary of the system and the work of Munoz Rivera et al. [18], who proved that the first and second order energy, associated with the solutions of the viscoelastic plate equation, decay exponentially provided the kernel of the memory also decays exponentially, that is, when the unique dissipation mechanism is given by the relaxation function. The combination of memory effects and dissipative mechanisms was already introduced by the authors for the wave equation in the works [1, 2, 4, 5, 8–10, 14–17, 20].

To the best of our knowledge, this is the first result dealing with Equation (1.3) subject to viscoelastic effects and presenting uniform decay rates. Therefore, our results are interesting to be studied even considering a nonlinear memory damping terms acting in the boundary.

In order to obtain the existence of global solutions to problem (1.1), we use the Faedo Galerkin method and in order to get the uniform decay rates of the energy

$$E(t) = \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{\beta}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|u(t)\|_{\Gamma_0}^2$$

we use the perturbed energy method, see Zuazua [20].

Our paper is organized as follows. In Section 2 we give some notations, assumptions and state our main result. In Section 3 we obtain global existence for weak solutions and in Section 4 we derive the uniform decay of the energy.

2. ASSUMPTIONS AND MAIN RESULT

We begin by introducing some notations that will be used throughout this work. Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product and the corresponding norm

$$(u,v)=\int_\Omega u(x)v(x)dx,\quad (u,v)_{\Gamma_0}=\int_{\Gamma_0} u(x)v(x)d\Gamma,$$

$$\|u\|_{p,\Gamma_0}^p=\int_{\Gamma_0}|u(x)|^p dx,\quad \|u\|_\infty=\|u\|_{L^\infty(\Omega)}.$$

Let $V=\{u\in H^1(\Omega); u=0 \text{ on } \Gamma_1\}$.

Throughout the article, we assume always that the function $g(\cdot)$ satisfies the following conditions:

(H.1) g; $R^+\to R^+$ be a positive and bounded $\mathbb{C}^1$ function such that

$$1-\int_0^\infty g(s)ds=l>0.$$

(H.2) There exists a positive constants m_0, m_1 such that

$$-m_0 g(t)\le g'(t)\le -m_1 g(t),\quad \forall\, t\ge 0.$$

(H.3) Condition (H.2) implies the following condition of $|g'|$:

There exists a positive constant m_2 such that

$$|g'(t)|\le m_2 g(t),\quad \forall\, t\in[0,t_0].$$

We recall that the energy related with problem (1.1) is given by

$$E(t)=\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{\beta}{2}\|\nabla u'(t)\|^2+\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{2}\|u(t)\|_{\Gamma_0}^2. \tag{2.1}$$

Now we are in a position to state our main result.

Theorem 2.1. *Let u_0, $u_1\in V$. Under assumptions* (H.1)–(H.3)*, suppose that γ, ρ satisfy hypothesis* (1.2) *with $\rho\ge\gamma$ and $\beta>0$. Then, problem* (1.1) *possesses at least a strong solution $u:\Omega\times(0,\infty)\to\mathbb{R}$ in the class*

$$u\in L^\infty(0,\infty;V),\quad u'\in L^\infty(0,\infty;V),\quad u''\in L^2(0,\infty;V). \tag{2.2}$$

Moreover, the energy determined by the solution u possesses the following decay:

$$E(t)\le 3l^{-1}E(0)\exp\Big(-\frac{\varepsilon}{2}C_2 t\Big),\quad \text{for all } t\ge 0 \quad \text{and } \varepsilon\in(0,\varepsilon_0],$$

where $C_2=C_2(\rho,E(0),\beta)$ and $\varepsilon_0=\varepsilon_0(\rho,E(0),m_1,\|g\|_{L^1(0,\infty)})$ are positive constants.

Remark. When $g = 0$, following the calculations of Section 4, we obtain exponential decay rates given by

$$E(t) \leq 3E(0)\exp\Big(-\frac{\varepsilon}{2}C_2 t\Big), \quad \text{for all } t \geq 0 \quad \text{and } \varepsilon \in (0, \varepsilon_0],$$

where $C_2 = C_2(\rho, E(0))$ and $\varepsilon_0 = \varepsilon_0(\rho, E(0))$.

3. PROOF OF THEOREM 2.1

In this section we are going to show the existence of solution for problem (1.1) using Faedo-Galerkin's approximation. For this end we represent by $\{w_j\}_{j\in\mathbb{N}}$ a basis in V which is orthonormal in $L^2(\Omega)$, by V_m the finite dimensional subspace of V generated by the first m vectors.

Next we define $u_m(t) = \Sigma_{j=1}^m g_{jm}(t)w_j$, where $u_m(t)$ is the solution of the following Cauchy problem:

$$\begin{aligned}(3.1)\qquad &(|u_m'|^\rho u_m'', w) + \beta(\nabla u_m'', \nabla w) + (\nabla u_m, \nabla w) + (\nabla u_m', \nabla w) + (u_m, w)_{\Gamma_0}\\ &\quad = \int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), w)_{\Gamma_0} d\tau, \quad w \in V_m.\end{aligned}$$

with the initial conditions,

$$\begin{aligned}(3.2)\qquad & u_m(0) = u_{0m} = \sum_{j=1}^m (u_0, w_j)w_j \to u_0 \quad \text{in } V,\\ & u_m'(0) = u_{1m} = \sum_{j=1}^m (u_1, w_j)w_j \to u_1 \quad \text{in } V.\end{aligned}$$

Note that we can solve the system (3.1) and (3.2) by Picard's iteration method. In fact, the problems (3.1) and (3.2) have a unique solution on some interval $[0, T_m)$. The extension of the solution to the whole interval $[0, \infty)$ is a consequence of the first estimate we are going to obtain below.

3.1. **A Priori Estimate I.** Considering $w = u_m'(t)$ in (3.1), assumption (H.3) yields

$$\begin{aligned}(3.3)\qquad &\frac{d}{dt}\Big(\frac{1}{\rho+2}\|u_m'(t)\|_{\rho+2}^{\rho+2} + \frac{\beta}{2}\|\nabla u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{2}\|u_m(t)\|_{\Gamma_0}^2\\ &\quad + \frac{1}{\gamma+2}g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \int_0^t g(t-\tau)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2}d\tau\Big) + \|\nabla u_m'(t)\|^2\\ &= \int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), u_m'(t))_{\Gamma_0}d\tau + \frac{1}{\gamma+2}g'(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\\ &\quad + g(t)(|u_m(t)|^\gamma u_m(t), u_m'(t))_{\Gamma_0} + \int_0^t g'(t-\tau)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2}d\tau\\ &\quad + g(0)\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\\ &\leq \int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), u_m'(t))_{\Gamma_0}d\tau + \frac{m_2}{\gamma+2}g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\end{aligned}$$

$$+ g(t)(|u_m(t)|^\gamma u_m(t), u_m'(t))_{\Gamma_0} + m_2 \int_0^t g(t-\tau)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2} d\tau$$
$$+ g(0)\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}.$$

Note that Hölder's inequality and Young's inequality [3] yields

$$\begin{aligned}(|u_m'(\tau)|^\gamma u_m'(\tau), u_m'(t))_{\Gamma_0} &\le \int_{\Gamma_0} |u_m'(\tau)|^{\gamma+1}|u_m'(t)|d\Gamma \\ &\le \Big(\int_{\Gamma_0} |u_m'(\tau)|^{\gamma+2} d\Gamma\Big)^{\frac{\gamma+1}{\gamma+2}} \Big(\int_{\Gamma_0} |u_m'(t)|^{\gamma+2} d\Gamma\Big)^{\frac{1}{\gamma+2}} \\ &= \|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+1}\|u_m'(t)\|_{\gamma+2,\Gamma_0} \\ &\le C_1(\eta)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2},\end{aligned} \tag{3.4}$$

where $C_1(\eta) = (\frac{\gamma+1}{\gamma+2})(\frac{1}{\eta(\gamma+2)})^{\frac{1}{\gamma+1}}$, $p = \frac{\gamma+2}{\gamma+1}$, $q = \gamma+2$.

Using (3.4), we get

$$\begin{aligned}&\int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), u_m'(t))_{\Gamma_0} d\tau \\ &\quad\le \int_0^t g(t-\tau)\{C_1(\eta)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\} d\tau \\ &\quad= C_1(\eta)\int_0^t g(t-\tau)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2} d\tau + \eta\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\int_0^t g(\tau)d\tau.\end{aligned} \tag{3.5}$$

Since $\rho \ge \gamma$, $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ and therefore we can obtain

$$\eta\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\int_0^t g(\tau)d\tau \le C_2(\eta)\int_0^t g(\tau)d\tau + \eta\int_0^t g(\tau)d\tau \|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2}, \tag{3.6}$$

where $C_2(\eta) = (\frac{\rho-\gamma}{\rho+2})(k\eta(\frac{\gamma+2}{\eta(\rho+2)})^{\frac{\rho+2}{\gamma+2}})^{\frac{\rho+2}{\rho-\gamma}}$, k is a Sobolev embedding's constant.

Therefore, (3.5) and (3.6) yield

$$\begin{aligned}&\int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), u_m'(t))_{\Gamma_0} d\tau \\ &\quad\le C_1(\eta)\int_0^t g(t-\tau)\|u_m'(\tau)\|_{\gamma+2,\Gamma_0}^{\gamma+2} d\tau + C_2(\eta)\int_0^t g(\tau)d\tau \\ &\quad\quad+\eta\int_0^t g(\tau)d\tau\|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2}.\end{aligned} \tag{3.7}$$

Similarly applying Hölder's inequality, Young's inequality and the result $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$, we have

$$\begin{aligned}g(t)(|u_m(t)|^\gamma u_m(t), u_m'(t))_{\Gamma_0} &\le g(t)\int_{\Gamma_0} |u_m(t)|^{\gamma+1}|u_m'(t)|d\Gamma \\ &\le g(t)\Big(\int_{\Gamma_0} |u_m(t)|^{\gamma+2} d\Gamma\Big)^{\frac{\gamma+1}{\gamma+2}} \Big(\int_{\Gamma_0} |u_m'(t)|^{\gamma+2} d\Gamma\Big)^{\frac{1}{\gamma+2}} \\ &= g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+1}\|u_m'(t)\|_{\gamma+2,\Gamma_0} \\ &\le C_3(\eta)g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta g(t)\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\end{aligned} \tag{3.8}$$

$$\leq C_3(\eta)g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + C_4(\eta)g(t) + \eta g(t)\|u'_m(t)\|^{\rho+2}_{\rho+2,\Gamma_0},$$

where $C_3(\eta) = (\frac{\gamma+1}{\gamma+2})(\frac{1}{\eta(\gamma+2)})^{\frac{1}{\gamma+1}}$, $C_4(\eta) = (\frac{\rho-\gamma}{\rho+2})(k\eta(\frac{\gamma+2}{\eta(\rho+2)})^{\frac{\rho+2}{\gamma+2}})^{\frac{\rho+2}{\rho-\gamma}}$.

Therefore, (3.3), (3.7) and (3.8) give

$$(3.9)\quad \begin{aligned} &\frac{d}{dt}\Big(\frac{1}{\rho+2}\|u'_m(t)\|^{\rho+2}_{\rho+2} + \frac{\beta}{2}\|\nabla u'_m(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{2}\|u_m(t)\|^2_{\Gamma_0} \\ &+\frac{1}{\gamma+2}g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \int_0^t g(t-\tau)\|u'_m(\tau)\|^{\gamma+2}_{\gamma+2,\Gamma_0}d\tau\Big) + \|\nabla u'_m(t)\|^2 \\ &\leq (C_1(\eta)+m_2)\int_0^t g(t-\tau)\|u'_m(\tau)\|^{\gamma+2}_{\gamma+2,\Gamma_0}d\tau + g(0)\|u'_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\ &+\Big(C_3(\eta)+\frac{m_2}{\gamma+2}\Big)g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + C_4(\eta)g(t) \\ &+C_2(\eta)\int_0^t g(\tau)d\tau + \eta g(t)\|u'_m(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \eta\int_0^t g(\tau)d\tau\|u'_m(t)\|^{\rho+2}_{\rho+2,\Gamma_0}. \end{aligned}$$

Integrating (3.9) over $[0,t]$, choosing $\eta > 0$ sufficiently small , the result $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ and employing Gronwall's lemma we obtain the first estimate:

$$(3.10)\quad \begin{aligned} &\|u'_m(t)\|^{\rho+2}_{\rho+2} + \|\nabla u'_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \|u_m(t)\|^2_{\Gamma_0} + g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\ &\qquad + \int_0^t g(t-\tau)\|u'_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}d\tau + \int_0^t \|\nabla u'_m(\tau)\|^2 d\tau \leq L_1, \end{aligned}$$

where $L_1 > 0$ is independent of m, u_0, u_1.

3.2. **A Priori Estimate II.** Substituting $w = u''_m(t)$ in (3.1), using Young's inequality and the continuity of the trace operator $\gamma_0 : H^1(\Omega) \hookrightarrow L^2(\Gamma)$ for $1 \leq q \leq \frac{2n-2}{n-2}$, it holds that

$$(3.11)\quad \begin{aligned} &\int_\Omega |u'_m(t)|^\rho |u''_m(t)|^2 dx + \beta\|\nabla u''_m(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u'_m(t)\|^2 \\ &= -(\nabla u_m(t), \nabla u''_m(t)) - (u_m(t), u''_m(t))_{\Gamma_0} \\ &\quad + \int_0^t g(t-\tau)(|u'_m(\tau)|^\gamma u'_m(\tau), u''_m(t))_{\Gamma_0} d\tau \\ &\leq 2\eta\|\nabla u''_m(t)\|^2 + C_5(\eta)L_1 + \int_0^t g(t-\tau)(|u'_m(\tau)|^\gamma u'_m(\tau), u''_m(t))_{\Gamma_0} d\tau. \end{aligned}$$

Now, taking into account that $\frac{\gamma+1}{2\gamma+2} + \frac{1}{2} = 1$, using the generalized Hölder inequality, Young's inequality and the continuity of the trace operator $\gamma_0 : H^1(\Omega) \hookrightarrow L^2(\Gamma)$ for $1 \leq q \leq \frac{2n-2}{n-2}$, we obtain

$$(3.12)\quad \begin{aligned} (|u'_m(\tau)|^\gamma u_m(\tau), u''_m(t))_{\Gamma_0} &\leq \Big(\int_{\Gamma_0} |u'_m(\tau)|^{2\gamma+2} d\tau\Big)^{\frac{\gamma+1}{2\gamma+2}} \Big(\int_{\Gamma_0} |u''_m(t)|^2 d\tau\Big)^{\frac{1}{2}} \\ &\leq C_6(\eta)\|\nabla u'_m(\tau)\|^{2\gamma+2} + \eta\|\nabla u''_m(t)\|^2 \\ &\leq C_6(\eta)L_1^{\gamma+1} + \eta\|\nabla u''_m(t)\|^2. \end{aligned}$$

Thus from (3.12), we get

$$
\begin{aligned}
(3.13)\qquad & \int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), u_m''(t))_{\Gamma_0} d\tau \\
&\le \int_0^t g(t-\tau)\{C_6(\eta)L_1^{\gamma+1} + \eta\|\nabla u_m''(t)\|^2\}d\tau \\
&\le C_6(\eta)L_1^{\gamma+1}\|g\|_{L^1(0,\infty)} + \eta\|\nabla u_m''(t)\|^2\|g\|_{L^1(0,\infty)}.
\end{aligned}
$$

Combining estimate (3.11)–(3.13), we get

$$
\begin{aligned}
(3.14)\quad & \int_\Omega |u_m'(t)|^\rho |u_m''(t)|^2 dx + (\beta - 2\eta - \eta\|g\|_{L^1(0,\infty)})\|\nabla u_m''(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m'(t)\|^2 \\
&\le C_5(\eta)L_1 + C_6(\eta)L_1^{\gamma+1}\|g\|_{L^1(0,\infty)}.
\end{aligned}
$$

Integrating (3.14) over $[0,t]$, we infer

$$
\begin{aligned}
&\int_0^t \int_\Omega |u_m'(s)|^\rho |u_m''(s)|^2 dx ds + (\beta - 2\eta - \eta\|g\|_{L^1(0,\infty)}) \int_0^t \|\nabla u_m''(s)\|^2 ds + \frac{1}{2}\|\nabla u_m'(t)\|^2 \\
&\le C_7(\eta) + C_6(\eta)L_1^{\gamma+1}\|g\|_{L^1(0,\infty)}T,
\end{aligned}
$$

where $C_7(\eta)$ is a positive constant which depends on η and T.

From the last inequality choosing $\eta > 0$ small enough we obtain the second estimate:

$$
(3.15)\qquad \|\nabla u_m'(t)\|^2 + \int_0^t \|\nabla u_m''(s)\|^2 ds \le L_2,
$$

where $L_2 > 0$ is independent of m, u_0, u_1.

The estimates (3.10) and (3.15) are sufficient to pass to the limit in the linear terms of problem (3.1). Next we are going to consider the nonlinear ones.

3.3. **Analysis of the nonlinear terms.** From the above estimate (3.10) and (3.15), we have that

$$
(3.16)\qquad u_\mu \rightharpoonup u \quad \text{weak star in } L^\infty(0,T;V),
$$

$$
(3.17)\qquad u_\mu' \rightharpoonup u' \quad \text{weak star in } L^\infty(0,T;V),
$$

$$
(3.18)\qquad u_\mu'' \rightharpoonup u'' \quad \text{weakly in } L^2(0,T;V).
$$

From the first estimate, we deduce

$$
\begin{aligned}
(3.19)\qquad \||u_\mu'|^\rho u_\mu'\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|u_\mu'(t)\|^{2(\rho+1)}_{2(\rho+1)} dt \\
&\le C\int_0^T \|\nabla u_\mu'(t)\|^{2(\rho+1)} dt \le CTL_1^{\rho+1},
\end{aligned}
$$

where $C > 0$ comes from embedding $H^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$.

On the other hand, from Aubin-Lions theorem, see Lions [12] we deduce that there exists a subsequence of (u_μ), still represented by the same notation, such that

$$u'_\mu \to u' \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Therefore,

$$|u'_\mu|^\rho u'_\mu \to |u'|^\rho u' \quad \text{a.e. in } \Omega \times (0,T). \tag{3.20}$$

Combining (3.19), (3.20) and owing to Lions lemma, we deduce

$$|u'_\mu|^\rho u'_\mu \rightharpoonup |u'|^\rho u' \quad \text{weak in } L^2(0,T;L^2(\Omega)). \tag{3.21}$$

Also, from the first estimate, we have that

$$(u_\mu) \quad \text{is bounded in } L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)), \tag{3.22}$$

$$(u'_\mu) \quad \text{is bounded in } L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)). \tag{3.23}$$

From (3.22) and (3.23), taking into consideration that the injection $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we deduce that there exists a subsequence of (u_μ), still represented by the same notation, such that

$$u_\mu \to u \text{ a.e. on } \Sigma_0 \quad \text{and} \quad u'_\mu \to u' \text{ a.e. on } \Sigma_0. \tag{3.24}$$

and therefore

$$|u'_\mu|^\gamma u'_\mu \to |u'|^\gamma u' \quad \text{a.e. on } \Sigma_0. \tag{3.25}$$

On the other hand, from the first estimate we obtain

$$\Big(\int_0^t g(t-\tau)|u'_\mu|^\gamma u'_\mu\Big) \quad \text{is bounded in } L^2(\Sigma_0). \tag{3.26}$$

Combining (3.25) and (3.26), we deduce that

$$\int_0^t g(t-\tau)|u'_\mu|^\gamma u'_\mu d\tau \to \int_0^t g(t-\tau)|u'|^\gamma u' d\tau \quad \text{weakly in } L^2(\Sigma_0). \tag{3.27}$$

Multiplying (3.1) by $\theta \in D(0,T)$ (here $D(0,t)$ means the space of functions in C^∞ with compact support in $(0,T)$) and integrating the obtained result over $(0,T)$, it holds that

$$\begin{aligned} -\frac{1}{\rho+1}&\int_0^T (|u'_m(t)|^\rho u'_m(t), w)\theta'(t)dt + \int_0^T (\nabla u_m(t), \nabla w)\theta(t)dt \\ &+\beta\int_0^T (\nabla u''_m(t), \nabla w)\theta(t)dt + \int_0^T (\nabla u'_m(t), \nabla w)\theta(t)dt \\ &+\int_0^T (u_m(t), w)_{\Gamma_0}\theta(t)dt \\ =&\int_0^T\int_0^t g(t-\tau)(|u'_m(\tau)|^\gamma u'_m(\tau), w)_{\Gamma_0}\theta(t)d\tau dt, \quad \forall w \in V_m. \end{aligned} \tag{3.28}$$

Convergences (3.16)–(3.18), (3.21)–(3.23) and (3.27) are sufficient to pass to the limit in (3.28) in order to obtain

$$|u'|^\rho u'' - \beta\Delta u'' - \Delta u - \Delta u' = 0 \quad \text{in } L^2_{loc}(0,\infty;H^{-1}(\Omega)).$$

This completes the proof of the existence of solutions of (1.1). The uniqueness is obtained in a usual way, so we omit the proof here. □

4. UNIFORM DECAY

In this section we prove the exponential decay for weak solutions of problem (1.1).

We define the energy $E(t)$ of problem (1.1) by

$$E(t) = \frac{1}{\rho+2}\|u'(t)\|^{\rho+2}_{\rho+2} + \frac{\beta}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|u(t)\|^2_{\Gamma_0}.$$

Then the derivative of the energy is given by

$$E'(t) = -\|\nabla u'(t)\|^2 + \int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u(t))_{\Gamma_0} d\tau.$$

Defining

$$(g\Box u)(t) = \int_0^t g(t-\tau)\||u'(\tau)|^\gamma u'(\tau) - u(t)|^2_{\Gamma_0} d\tau, \tag{4.1}$$

a simple computation give us

$$\begin{aligned}(g\Box u)'(t) &= \int_0^t g'(t-\tau)\||u'(\tau)|^\gamma u'(\tau) - u(t)|^2_{\Gamma_0} d\tau \\ &\quad +(\frac{d}{dt}\|u(t)\|^2_{\Gamma_0})\int_0^t g(\tau)d\tau - 2\int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u'(t))_{\Gamma_0} d\tau \\ &= (g'\Box u)(t) - 2\int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u'(t))_{\Gamma_0} d\tau \\ &\quad + \frac{d}{dt}\{\|u(t)\|^2_{\Gamma_0}\int_0^t g(\tau)d\tau\} - g(t)\|u(t)\|^2_{\Gamma_0}.\end{aligned} \tag{4.2}$$

Thus we have

$$\begin{aligned}&\int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u'(t))_{\Gamma_0} d\tau \\ &= -\frac{1}{2}(g\Box u)'(t) + \frac{1}{2}(g'\Box u)(t) + \frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|^2_{\Gamma_0}\int_0^t g(\tau)d\tau\Big) - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_0}.\end{aligned} \tag{4.3}$$

Define the modified energy by

$$\begin{aligned}e(t) &= \frac{1}{\rho+2}\|u'(t)\|^{\rho+2}_{\rho+2} + \frac{\beta}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 \\ &\quad + \frac{1}{2}(g\Box u)(t) + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|u(t)\|^2_{\Gamma_0}.\end{aligned} \tag{4.4}$$

Then

$$e'(t) = -\|\nabla u'(t)\|^2 + \frac{1}{2}(g'\Box u)(t) - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_0}. \tag{4.5}$$

We observe that in view of assumption (H.1) we have $e(t) \geq 0$ and according to assumption (H.2) we deduce that $e'(t) \leq 0$.

On the other hand, we note that from assumption (H.1)

$$
\begin{aligned}
E(t) &= \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{\beta}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|u(t)\|_{\Gamma_0}^2 \\
&\leq \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{\beta}{2}\|\nabla u'(t)\|^2 \\
&\quad + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2l}(1 - \int_0^t g(\tau)d\tau)\|u(t)\|_{\Gamma_0}^2 \\
&\leq l^{-1}e(t)
\end{aligned}
\tag{4.6}
$$

and therefore it is enough to obtain the desired exponential decay for the modified energy $e(t)$ which will be done below.

We define the perturbed energy by

$$
e_\varepsilon(t) = e(t) + \varepsilon\Psi(t) \tag{4.7}
$$

where

$$
\Psi(t) = \frac{1}{\rho+1}(|u'(t)|^\rho u'(t), u(t)) + \beta(\nabla u'(t), \nabla u(t)). \tag{4.8}
$$

Proposition 4.1. *There exists $C_1 = C_1(\rho, E(0), \beta)$ a positive constant such that*

$$
|e_\varepsilon(t) - e(t)| \leq \varepsilon C_1 e(t), \quad \forall t \geq 0 \quad \text{and} \quad \forall \varepsilon > 0.
$$

Proof. From Young's inequality, we deduce

$$
|\Psi(t)| \leq \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{(\rho+1)^{-1}}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} + \beta^{\frac{1}{2}}\Big(\frac{1}{2}\|\nabla u(t)\|^2 + \frac{\beta}{2}\|\nabla u'(t)\|^2\Big).
$$

Now, considering the embedding $H^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ and taking (4.4) into account, it holds that

$$
\begin{aligned}
|\Psi(t)| &\leq \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + C\frac{(\rho+1)^{-1}}{\rho+2}\|\nabla u(t)\|^{\rho+2} + \beta^{\frac{1}{2}}(\frac{1}{2}\|\nabla u(t)\|^2 + \frac{\beta}{2}\|\nabla u'(t)\|^2) \\
&\leq e(t) + C\frac{(\rho+1)^{-1}}{\rho+2}2^{\frac{\rho+2}{2}}e(0)^{\frac{\rho}{2}}e(t) + \beta^{\frac{1}{2}}e(t),
\end{aligned}
$$

where C comes from the inequality $\|v\|_{\rho+2} \leq C\|\nabla v\|$ for all $v \in V$.

Then, $|e_\varepsilon(t) - e(t)| \leq \varepsilon C_1 e(t)$, where $C_1 = 1 + C\frac{(\rho+1)^{-1}}{\rho+2}2^{\frac{\rho+2}{2}}e(0)^{\frac{\rho}{2}} + \beta^{\frac{1}{2}}$.

This completes the proof. □

Proposition 4.2. *There exist $C_2 = C_2(\rho, E(0), \beta)$ and $\varepsilon_1 = \varepsilon_1(\rho, m_1, \|g\|_{L^1(0,\infty)})$ positive constants such that*

$$
e_\varepsilon'(t) \leq -\varepsilon C_2 e(t), \quad \forall\, t \geq 0 \quad \text{and} \quad \forall\, \varepsilon \in (0, \varepsilon_1].
$$

Proof. Taking the derivative of $\Psi(t)$ defined in (4.8) and using the problem (1.1), we have

$$
\begin{aligned}
\Psi'(t) &= (|u'(t)|^\rho u''(t), u(t)) + \frac{1}{\rho+1}(|u'(t)|^\rho u'(t), u'(t)) \\
&\quad +\beta(\nabla u''(t), \nabla u(t)) + \beta\|\nabla u'(t)\|^2 \\
&= \frac{1}{\rho+1}(|u'(t)|^\rho u'(t), u'(t)) + \beta\|\nabla u'(t)\|^2 - \|\nabla u(t)\|^2 \\
&\quad -(\nabla u'(t), \nabla u(t)) - \|u(t)\|^2_{\Gamma_0} + \int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u(t))_{\Gamma_0} d\tau \\
&= -e(t) + C(\rho)(|u'(t)|^\rho u'(t), u'(t)) + \frac{3}{2}\beta\|\nabla u'(t)\|^2 - \frac{1}{2}\|\nabla u(t)\|^2 \\
&\quad -\frac{1}{2}\|u(t)\|^2_{\Gamma_0} + \frac{1}{2}(g\Box u)(t) - \frac{1}{2}\int_0^t g(\tau)d\tau\|u(t)\|^2_{\Gamma_0} \\
&\quad -(\nabla u'(t), \nabla u(t)) + \int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u(t))_{\Gamma_0} d\tau,
\end{aligned} \tag{4.9}
$$

where $C(\rho) = \frac{(2\rho+3)}{[(\rho+1)(\rho+2)]}$.

Next, we will analyse terms on the right-hand side of (4.9).

Estimate for $I_1 = C(\rho)(|u'(t)|^\rho u'(t), u'(t))$.

We have

$$
\begin{aligned}
|I_1| &\le C(\rho)\|u'(t)\|^{\rho+1}_{2(\rho+1)}\|u'(t)\| \\
&\le \eta\|\nabla u'(t)\|^{2(\rho+1)} + C(\rho,\eta)\|\nabla u'(t)\|^2 \\
&\le 2^{\rho+1}\beta^{-(\rho+1)}\eta[e(0)]^\rho e(t) + C(\rho,\eta)\|\nabla u'(t)\|^2,
\end{aligned} \tag{4.10}
$$

where $\eta > 0$ is an arbitrary positive constant.

Estimate for $I_2 = \int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau), u(t))_{\Gamma_0} d\tau$.

We have

$$
\begin{aligned}
|I_2| &= \int_0^t g(t-\tau)(|u'(\tau)|^\gamma u'(\tau) - u(t), u(t))_{\Gamma_0} d\tau + \int_0^t g(t-\tau)\|u(t)\|^2_{\Gamma_0} d\tau \\
&\le \eta\|u(t)\|^2_{\Gamma_0} + \frac{1}{4\eta}(\int_0^t g(t-\tau)||u'(\tau)|^\gamma u'(\tau) - u(t)|_{\Gamma_0} d\tau)^2 \\
&\quad + \int_0^t g(t-\tau)\|u(t)\|^2_{\Gamma_0} d\tau \\
&\le \eta\|u(t)\|^2_{\Gamma_0} + \frac{1}{4\eta}\|g\|_{L^1(0,\infty)}(g\Box u)(t) + (\int_0^t g(\tau)d\tau)\|u(t)\|^2_{\Gamma_0}.
\end{aligned} \tag{4.11}
$$

Estimate for $I_3 = (\nabla u'(t), \nabla u(t))$.

Analogously, we have

$$
|I_3| \le \frac{1}{4\eta}\|\nabla u'(t)\|^2 + 2\eta e(t). \tag{4.12}
$$

Combining (4.9)–(4.12), we infer

$$\Psi'(t) \le -(1-\eta L)e(t) + M(\eta)\|\nabla u'(t)\|^2 - \frac{1}{2}\|\nabla u(t)\|^2 \\ -(\frac{1}{2} - \eta + \frac{1}{2}\|g\|_{L^1(0,\infty)})\|u(t)\|^2_{\Gamma_0} + N(\eta)(g\Box u)(t), \tag{4.13}$$

where $L = 2^{\rho+1}\beta^{-(\rho+1)}[e(0)]^{\rho+2}$, $M(\eta) = C(\rho,\eta)+\frac{3}{2}\beta+\frac{1}{4\eta}$ and $N(\eta) = \frac{1}{2}+\frac{1}{4\eta}\|g\|_{L^1(0,\infty)}$. Keeping in mind that $0 < \|g\|_{L^1(0,\infty)} < 1$ (see assumption (H.1)), then $\frac{1}{2}+\frac{1}{2}\|g\|_{L^1(0,\infty)} > 0$ and consequently considering $\eta > 0$ sufficiently small such that

$$C_2 = 1 - \eta L > 0 \quad \text{and} \quad \frac{1}{2} + \frac{1}{2}\|g\|_{L^1(0,\infty)} - \eta \ge 0$$

from (4.13) we deduce

$$\Psi'(t) \le -C_2 e(t) + M(\eta)\|\nabla u'(t)\|^2 + N(\eta)(g\Box u)(t) - \frac{1}{2}\|\nabla u(t)\|^2. \tag{4.14}$$

On the other hand, from (H.2), (4.5), (4.7) and (4.14) we deduce

$$\begin{aligned} e'_\varepsilon(t) &= e'(t) + \varepsilon\Psi'(t) \\ &\le -\varepsilon C_2 e(t) - (1-\varepsilon M(\eta))\|\nabla u'(t)\|^2 - (\frac{m_1}{2} - \varepsilon N(\eta))(g\Box u)(t) \\ &\quad - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_0} - \frac{\varepsilon}{2}\|\nabla u(t)\|^2. \end{aligned} \tag{4.15}$$

Defining $\varepsilon_1 = \min\{\frac{1}{M(\eta)}, \frac{m_1}{2N(\eta)}\}$ and considering $\varepsilon \in (0,\varepsilon_1]$, we conclude, from (4.15) that

$$e'_\varepsilon(t) \le -\varepsilon C_2 e(t).$$

This completes the proof. □

4.1. **Continuing the Proof of Theorem 2.1.** Let $\varepsilon_0 = \min\left\{\varepsilon_1, \frac{1}{2C_1}\right\}$, where C_1 is given in Proposition 4.1.

Consider $\varepsilon \in (0,\varepsilon_0]$. From Proposition 4.1 we obtain

$$(1-\varepsilon C_1)e(t) \le e_\varepsilon(t) \le (1+\varepsilon C_1)e(t).$$

Since $\varepsilon \le \frac{1}{2C_1}$, then

$$\frac{1}{2}e(t) \le e_\varepsilon(t) \le \frac{3}{2}e(t) \le 2e(t) \quad \text{for all} \quad t \ge 0. \tag{4.16}$$

From (4.16) and Proposition 4.2 we get

$$e'_\varepsilon(t) \le -\frac{\varepsilon}{2}C_2 e_\varepsilon(t), \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \varepsilon \in (0,\varepsilon_0].$$

Consequently,

$$e_\varepsilon(t) \le e_\varepsilon(0)\exp\Big(-\frac{\varepsilon}{2}C_2 t\Big)$$

and taking (4.16) into account, we get

$$e(t) \le 3e(0)\exp\Big(-\frac{\varepsilon}{2}C_2 t\Big),$$

for all $t \ge 0$ and $\varepsilon \in (0,\varepsilon_0]$.

Then

$$E(t) \le l^{-1}e(t) \le 3l^{-1}E(0)\exp\Big(-\frac{\varepsilon}{2}C_2 t\Big),$$

for all $t \ge 0$ and $\varepsilon \in (0, \varepsilon_0]$.

This conclude the Proof of Theorem 2.1. □

Finally, when $\beta = 0$, we will show the existence and the uniform decay of solution for problem (4.18) using same method of Theorem 2.1.

Corollary 4.3. *Let us consider $u_0, u_1 \in V \cap H^{3/2}(\Omega)$ verifying the compatibility conditions*

$$\begin{aligned} &\Delta u_0 + \Delta u_1 = 0 \quad on \ \ \Omega, \\ &\frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + u_0 = 0 \quad on \ \ \Gamma_0 \end{aligned} \tag{4.17}$$

and under assumptions (H.1)–(H.3)*, suppose that $\rho > 1$, γ satisfy hypothesis* (1.2)*. Then, problem*

$$\begin{aligned} &|u_t|^\rho u_{tt} - \Delta u - \Delta u_t = 0 \quad in \ \ \Omega \times (0,\infty), \\ &\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + u = \int_0^t g(t-\tau)|u_t(\tau)|^\gamma u_t(\tau)d\tau \quad on \ \ \Gamma_0 \times (0,\infty), \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad for \ \ x \in \Omega \end{aligned} \tag{4.18}$$

possesses at least a strong solution $u : \Omega \times (0,\infty) \to \mathbb{R}$ in the class (2.2)*. Moreover, the energy determined by the solution u possesses the following decay:*

$$E(t) \le 3l^{-1}E(0)\exp\Big(-\frac{\varepsilon}{2}C_2^* t\Big), \quad for\ all \ \ t \ge 0 \ \ and \ \ \varepsilon \in (0, \varepsilon_0], \tag{4.19}$$

where $C_2^* = C_2^*(\rho, E(0))$ and $\varepsilon_0 = \varepsilon_0(\rho, E(0), m_1, \|g\|_{L^1(0,\infty)})$ are positive constants.

Proof. We define $u_m(t) = \Sigma_{j=1}^m g_{jm}(t)w_j$, where $u_m(t)$ is the solution of the following Cauchy problem:

$$\begin{aligned} &(|u_m'|^\rho u_m'', w) + (\nabla u_m, \nabla w) + (\nabla u_m', \nabla w) + (u_m, w)_{\Gamma_0} \\ &\quad = \int_0^t g(t-\tau)(|u_m'(\tau)|^\gamma u_m'(\tau), w)_{\Gamma_0} d\tau, \quad w \in V_m \end{aligned} \tag{4.20}$$

with the initial conditions (3.2). Then, we can know the problems (4.20) have a unique solution on some interval $[0, T_m)$. Applying similar to the Priori Estimate I of Theorem 2.1, we have the the first estimate:

$$\begin{aligned} &\|u_m'(t)\|_{\rho+2}^{\rho+2} + \|\nabla u_m(t)\|^2 + \|u_m(t)\|_{\Gamma_0}^2 + g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \\ &\quad + \int_0^t g(t-\tau)\|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} d\tau + \int_0^t \|\nabla u_m'(\tau)\|^2 d\tau \le L_1^*, \end{aligned} \tag{4.21}$$

where $L_1^* > 0$ is independent of m, u_0, u_1.

Next, we are estimating $u''_m(0)$ in the L^2-norm. Considering $t = 0$ and $w = u''_m(0)$ in (4.20), we conclude

$$(|u_1|^\rho u''_m(0), u''_m(0)) = (\Delta u_0 + \Delta u_1, u''_m(0)) + (u_0, u''_m(0))_{\Gamma_0}.$$

The above identity, initial conditions (3.2) and (4.17) yield

$$\|u''_m(0)\| \le L_2^*, \tag{4.22}$$

where $L_2^* > 0$ is independent of m, u_0, u_1.

Differentiating (4.20) and substituting w by $u''_m(t)$, using (H.3),we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u'_m(t)|^\rho |u''_m(t)|^2 dx + \|\nabla u'_m(t)\|^2 + \|u'_m(t)\|^2_{\Gamma_0}\Big) \\ &\quad + \|\nabla u''_m(t)\|^2 + \frac{\rho}{2}\int_\Omega |u'_m(t)|^{\rho-1}|u''_m(t)|^3 dx \\ &= \int_0^t g'(t-\tau)(|u'_m(\tau)|u'_m(\tau), u''_m(t))_{\Gamma_0} d\tau \\ &\le m_2 \int_0^t g(t-\tau)(|u'_m(\tau)|u'_m(\tau), u''_m(t))_{\Gamma_0} d\tau. \end{aligned} \tag{4.23}$$

Thus from (3.13), we get

$$\begin{aligned} &m_1 \int_0^t g(t-\tau)(|u'_m(\tau)|u'_m(\tau), u''_m(t))_{\Gamma_0} d\tau \\ &\quad \le m_1 C(\eta)(2L_1^*)^{\gamma+1}\|g\|_{L^1(0,\infty)} + m_1\eta\|u''_m(t)\|^2_{\Gamma_0}\|g\|_{L^1(0,\infty)}. \end{aligned} \tag{4.24}$$

Combining the estimates (4.22)-(4.24), we see that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u'_m(t)|^\rho |u''_m(t)|^2 dx + \|\nabla u'_m(t)\|^2 + \|u'_m(t)\|^2_{\Gamma_0}\Big) + \|\nabla u''_m(t)\|^2 \\ &\quad \le m_1 C(\eta)(2L_1^*)^{\gamma+1}\|g\|_{L^1(0,\infty)} + m_1\eta\|u''_m(t)\|^2_{\Gamma_0}\|g\|_{L^1(0,\infty)}. \end{aligned} \tag{4.25}$$

Integrating (4.25) over $[0, t]$ and Gronwall's lemma we infer the second estimate:

$$\int_\Omega |u'_m(t)|^\rho |u''_m(t)|^2 dx + \|\nabla u'_m(t)\|^2 + \|u'_m(t)\|^2_{\Gamma_0} + \int_0^t \|\nabla u''_m(t)\|^2 d\tau \le L_3^*, \tag{4.26}$$

where $L_3^* > 0$ is independent of m, u_0, u_1.

By using (4.24) and (4.25), repeating the procedure similar to the proof of section 3.3, we can get the existence result (2.2) of solutions of the problem (4.18). Also, since the proof of the uniform decay of the problem (4.18) is similar to the proof of Theorem 2.1, we can easily obtain the uniform decay result (4.19) of solutions of the problem (4.18). □

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