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APPROXIMATE CONTROLLABILITY OF IMPULSIVE DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In order to describe various real-world problems in physical and engineering sciences that are subject to abrupt changes at certain instants during the evolution process, impulsive differential equations has been used to describe the system model. In this article, the problem of approximate controllability for nonlinear impulsive neutral differential inclusions with nonlocal conditions is studied under the assumption that the corresponding linear control system is approximately controllable. Using a fixed point theorem for condensing multi-valued maps and semigroup theory, sufficient conditions are formulated and proved. Finally, an example is provided to illustrate the results obtained.

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1. INTRODUCTION

The concepts of controllability play an important role in analysis and design of control systems. Controllability of the deterministic systems in infinite dimensional spaces has been extensively studied. Several authors [4, 5, 6, 8, 9, 11, 12] studied the concept of exact controllability for systems represented by nonlinear evolutions equations, in which the authors effectively used fixed point technique. Anandhi and Voit [1] addressed the related problem of controllability, where the task is to steer a non-linear biochemical system, within a given time period, from an initial state to some target state, which may or may not be a steady state. On the other hand, many practical systems in physical and biological sciences have impulsive dynamical behaviours during the evolution process which can be modeled by impulsive differential equations. Differential equations involving impulse effects occur in many applications: pharmacokinetics, epidemiology, population dynamics, biological systems, fed-batch culture in fermentative production, bio-technology etc., [16, 17, 20, 27, 28]. For the basic theory of impulsive differential equations the reader can refer to Samoilenko and Perestyuk [25]. There are many papers that deal the exact controllability of impulsive differential inclusions and systems. Exact controllability of various types of nonlinear impulsive differential systems has been studied by several authors [3, 8]. Chang

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and Chalishajar [10] established sufficient conditions for the controllability of semilinear mixed Volterra-Fredholm-type integrodifferential inclusions using Bohnenblust-Karlin's fixed point theorem. Chang et al [9] established sufficient conditions for the controllability of evolution differential inclusions in Banach spaces. The exact controllability of first and second order functional semilinear differential inclusions with nonlocal conditions has been studied in [18, 19]. Chang [8] studied the exact controllability of impulsive functional systems with infinite delay in Banach spaces. Very recently, Abada et al [2] established exact controllability results for impulsive partial functional differential inclusions.

From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. In infinite-dimensional spaces the concept of exact controllability is usually too strong and, indeed has limited applicability (see [21] and references therein). Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications (see [14, 21] and references therein). Therefore, it is important, in fact necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear systems. There are few papers on the approximate controllability of the nonlinear systems under different conditions [14, 22, 23, 24] and references therein. Approximate controllability of first order functional differential equations with finite delay was considered in [14] with the aid of Schauder's fixed point theorem. Mahmudov [23] established a set of sufficient conditions for the approximate controllability of nonlinear evolution equations with nonlocal conditions in Hilbert spaces. Approximate controllability for semilinear deterministic and stochastic control systems can be found in Mahmudov [21].

On the other hand, the nonlocal Cauchy problem was considered by Byszewski [7] and the importance of nonlocal conditions in different fields has been discussed in [13] and the references therein. To the best of the authors knowledge, up to now approximate controllability problems for nonlinear neutral differential inclusions with impulses and nonlocal conditions have not been considered in the literature. In order to fill this gap, in this paper we study the approximate controllability of the impulsive neutral differential inclusions with nonlocal conditions described by

$$
\begin{cases}
\frac{d}{dt}[x(t) - G(t, x(h_1(t)))] \in -Ax(t) + (Bu)(t) + F(t, x(h_2(t))), \\
\qquad\qquad\qquad\qquad t \in J = [0, b], t \neq t_k \\
\Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, \dots, m, \\
x(0) + g(x) = x_0 \in X,
\end{cases}
\tag{1.1}
$$

where $-A : D(A) \subset X \to X$ is the infinitesimal generator of a analytic semigroup of uniformly bounded linear operators $T(t)$ on a Banach space X with norm $|\cdot|$, B is a bounded linear operator from a Banach space U into X, F is multi-valued map,

$\Delta x(t_k) = x(t_k^+)) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$. The functions G, F, g, I_k, h_1 and h_2 are given continuous functions to be specified later.

2. PRELIMINARIES

In this section, we state some definitions, notations and preliminary facts from the multi-valued analysis [15].

Let $C(J, X)$ is the Banach space of continuous functions from J into X with the norm $|\cdot|_J = \sup\{|\cdot| : t \in J\}$. $V(X)$ denotes the Banach space of bounded linear operators from X to X. We use the notations $P(X) = \{Y \in 2^X : Y \neq \Phi\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$. A multivalued map $F : X \to 2^X$ is said to be convex (closed) valued if $F(x)$ is convex(closed) for all $x \in X$. F is said to be bounded on bounded sets if $F(V) = \cup_{x \in V} F(x)$ is bounded in X for all $V \in P_b(X)$.

F is called upper semi-continuous (u.s.c) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open subset ∇ of X containing $F(x_0)$, then there exists an open neighbourhood $\hat{\Gamma}$ of x_0 such that $F(\hat{\Gamma}) \subset \nabla$. F is said to be completely continuous if $F(V)$ is relatively compact for every $V \in P_b(X)$. If the multi-valued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n) \to y_* \in F(x_*)$). We say that F has a fixed point if there is $x \in X$ such that $x \in F(x)$.

A multivalued map $F : J \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to R$ defined by $Y(t) = d(x, F(t)) = \inf\{|x - z| : z \in F(t)\}$ is measurable. An upper semi-continuous multi-valued map $F : X \to 2^X$ is said to be condensing if for any subset $V \subset X$ with $\beta(V) \neq 0$ we have $\beta(F(V)) < \beta(V)$, where β denotes the Kuratowski measure of non-compactness.

Define the fractional power A^β for $0 < \beta \leq 1$ as a closed linear operator on its domain $D(A^\beta)$. Moreover, the subspace $D(A^\beta)$ is dense in X. The closedness of A^β implies that $D(A^\beta)$ endowed with the graph norm $\|x\| = |x| + |A^\beta x|$ is a Banach space. Since A^β is invertible its graph norm $\|\cdot\|$ is equivalent to the norm $\|x\| = |A^\beta x|$. Thus $D(A^\beta)$ equipped with the norm $\|\cdot\|$ is a Banach space which we denote by X_β and $X_\beta \to X_\eta$ for $0 < \eta < \beta \leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)_{t\geq 0}\}$, for any $0 \leq \beta \leq 1$ there exists a positive constant C_β such that $\|(-A)^\beta T(t)\| \leq \frac{C_\beta}{t^\beta}$, $0 < t \leq b$.

We need the following lemmas for our subsequent discussion.

Lemma 2.1 ([26]). *Let X be a Banach Space. Let $F : J \times X \to P_{b,cl,c}(X)$ satisfies that*

(i) *For each* $x \in X$, $(t,x) \to F(t,x)$ *is measurable with respect to* t.

(ii) *For each* $t \in J$, $(t,x) \to F(t,x)$ *is u.s.c. with respect to* x.

(iii) *For each fixed* $x \in C(J,X)$, *the set*

$$S_{F,x} = \{y \in L^1(J,X) : y(t) \in F(t, x(h_2(t))) \text{ for a.e. } t \in J\}$$

is nonempty.

Let $\Upsilon : L^1(J,X) \to C(J,X)$ be a linear continuous mapping. Then the operator

$$\Upsilon \circ S_F : C(J,X) \to P_{cp,c}(C(J,X)), \quad x \to (\Upsilon \circ S_F)(x) = \Upsilon(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.2 ([15]). *Let* Ω *be a bounded and convex set in Banach space* X. $\psi : \Omega \to 2^\Omega \setminus \{\emptyset\}$ *is u.s.c., condensing multi-valued map. If for every* $x \in \Omega$, $\psi(x)$ *is a closed and convex set in* Ω, *the* ψ *has a fixed point in* Ω.

In order to define the solution of system (1.1), we shall consider the space Z

$$Z = \{x : [0,b] \to X_\beta : x_k \in C(J_k, X_\beta), \quad k = 0,1,\dots m,$$
$$\text{and there exist } x(t_k^-) \text{ and } x(t_k^+), \text{with } x(t_k^-) = x(t_k), x(t) + g(x) = x_0\},$$

which is a Banach space with the norm $|x|_Z = \max\{|x_k|_{J_k}\},\ k = 0,1,\dots m\}$.

Let $x_b(x_0;u)$ be the state value of (1.1) at terminal time b corresponding to the control u and the initial value $x_0 \in X$. Introduce the set

$$\Re(b,x_0) = \{x_b(x_0;u)(0) : u(\cdot) \in L_2(J,U)\},$$

which is called the reachable set of system (1.1) at terminal time b and its closure in X is denoted by $\overline{\Re(b,x_0)}$.

Definition 2.3. The system (1.1) is said to be approximately controllable on the interval J if $\overline{\Re(b,x_0)} = X$.

It is convenient at this point to define operators

$$\Gamma_0^b = \int_0^b T(b-s)BB^*T^*(b-s)ds$$

and

$$R(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1}.$$

(S_1) $\alpha R(\alpha,\Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

The assumption (S_1) holds if and only if the linear system

$$\begin{cases} x'(t) = Ax(t) + (Bu)(t), \quad t \in [0,b], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

is approximately controllable on J.

We assume that the following conditions hold:

(H_1) There exists a constant $\eta \in (0,1)$ such that $G : [0,b] \times X_\beta \to X_\eta$ satisfies the Lipschitz condition, that is there exists a constant $K > 0$ such that $\|A^\eta G(t_1,x_1) - A^\eta G(t_2,x_2)\| \leq K(|t_1 - t_2| + \|x_1 - x_2\|)$ for any $0 \leq t_1, t_2 \leq b$, $x_1, x_2 \in X_\beta$. Moreover, there exists a constant $K_1 > 0$ such that

$$\|A^\eta G(t,x)\| \leq K_1(\|x\| + 1), \ x \in X_\beta$$

(H_2) The multivalued map $F : J \times X_\beta \to P_{c,cp}(X)$ satisfies the following conditions:

(i) For each $t \in J$, the function $F(t,.) : X_\beta \to P_{c,cp}(X)$ is u.s.c. and for each $x \in X_\beta$, the function $F(\cdot,x)$ is measurable. And for each fixed $v \in Z$ the set

$$S_{F,v} = \{v \in L^1(J,X), \quad v(t) \in F(t,v) \quad for \quad a.e. t \in J\}$$

is nonempty.

(ii) For each $q > 0$, there exists a function $\lambda \in L^1(J,R^+)$ such that

$$\sup_{\|x\| \leq q} \|F(t,x)\| \leq \lambda(q), \qquad \text{for a.e. } t \in J,$$

and

$$\liminf_{q \to \infty} \int_0^b \frac{\lambda(q)}{q} dt = \delta < \infty,$$

where

$$\|F(t,x)\| = \sup\{\|v\| : v \in F(t,x)\}, \quad \|x\| = \sup_{0 \leq s \leq b} \|x(s)\|.$$

(H_3) $h_i \in C(J,J), i = 1,2, g \in C(E,X_\alpha)$, where $E = \{x : [0,a] \to X_\alpha : x_k \in C(J_k,X_\alpha) k = 1,2,\ldots,m\}$. $A^\beta g$ is a completely continuous map and there exist positive constants K_2 and K_3 such that

$$\|g(x)\| \leq K_2|x| + K_3, \quad \text{for all } x \in E.$$

(H_4) $I_k \in C(X_\beta, X_\beta)$ are completely continuous and there exist constants d_k such that $\|I_k(x)\| \leq d_k$, $k = 1, \ldots m$, for each $x \in X$.

(H_5) $T(t)$, $t > 0$ is compact

(H_6) The function $F : J \times X_\beta \to P_{c,cp}(X)$ is continuous and there exists $N > 0$ such that

$$\|F(t,x)\| \leq N, \quad \text{for all } (t,x) \in J \times X_\beta.$$

For convenience, let us introduce the following notations:

$$M = \|A^{-\eta}\|, \qquad M_1 = \max\{\|T(t)\| : 0 \leq t \leq b\}, \qquad M_2 = \|B\|,$$

$$K^* = \frac{b}{\alpha} M_1^2 M_2^2 \Bigg[\|x_b\| + M_1[\|x_0\| + K_3 + MK_1(r+1)] + K_1\left(M + \frac{C_{1-\eta}(b)^\eta}{\eta}\right) + M_1 \sum_{k=1}^m d_k \Bigg].$$

It will be shown that the system (1.1) is approximately controllable, if for all $\alpha > 0$ there exists a continuous function $x(\cdot) \in Z$ such that

$$\begin{aligned} u(t) &= B^*T^*(b-t)R(\alpha, \Gamma_0^b)p(x(\cdot)), \\ x(t) &= T(t)[x_0 - g(x) - G(0, x(h_1(0)))] + G(t, x(h_1(t))) \\ &\qquad + \int_0^t AT(t-s)G(t, x(h_1(s)))ds + \int_0^t T(t-s)[y(s) + Bu(s)]ds \\ &\qquad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad y \in S_{F,x}, \end{aligned}$$

where,

$$\begin{aligned} p(x(\cdot)) &= x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ &\qquad - \int_0^b AT(b-s)h(s, x_s)ds - \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-)). \end{aligned}$$

3. APPROXIMATE CONTROLLABILITY RESULT

Theorem 3.1. *Assume that conditions* (H_1)–(H_4) *are satisfied. Further, suppose that for all* $\alpha > 0$

$$\left(1 + \frac{b}{\alpha} M_1^2 M_2^2\right) \left[M_1 K_2 + K_1 \left(\frac{C_{1-\eta} b^\eta}{\eta} + M\right)\right] + M_1 \delta \le 1,$$

then the system (1.1) *has a solution on* J.

Proof. On the space Z, consider a set

$$Q = \{x \in Z; \|x(t)\| \le r, 0 \le t \le b\},$$

where r is a positive constant. Clearly, Q is a bounded, closed convex set in Z. For $\alpha > 0$, define the operator $\Psi : Z \to Z$ by $\Psi x(t) = z(t)$, where

$$\begin{aligned} z(t) &= T(t)[x_0 - g(x) - G(0, x(h_1(0)))] + G(t, x(h_1(t))) \\ &\qquad + \int_0^t AT(t-s)G(t, x(h_1(s)))ds + \int_0^t T(t-s)[y(s) + Bu(s)]ds \\ &\qquad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad y \in S_{F,x}, \\ u(t) &= B^*T^*(b-t)R(\alpha, \Gamma_0^b)p(x(\cdot)), \\ p(x(\cdot)) &= x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ &\qquad - \int_0^b AT(b-s)h(s, x_s)ds - \int_0^b T(b-s)y(s)ds \\ &\qquad - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-)). \end{aligned}$$

It is easy to see that $z(t) \in X_\beta$, by the assumptions on G, g, I_k and the fact $x_0 \in X_\beta$.

It will be shown that for all $\alpha > 0$ the operator $\Psi : Z \to 2^Z$ has a fixed point.

Step 1: For $\alpha > 0$, there exists $r > 0$ such that $\Psi(Q) \subset Q$. If this is not true, then there exists $\alpha > 0$ such that for every $r > 0$, there exist $x' \in Q$ and $t' \in J$ such that $r < \|\Psi x'(t')\|$. For such $\alpha > 0$, we find that

$$\begin{aligned}
r &< \|\Psi x'(t')\| \\
&\leq \|T(t)[x_0 - g(x) - G(0, x(h_1(0)))]\| + \|G(t, x(h_1(t)))\| \\
&\quad + \|\int_0^t AT(t-s)G(t, x(h_1(s)))ds\| + \|\int_0^t T(t-s)y(s)ds\| \\
&\quad + \|\int_0^t T(t-s)Bu(s)ds\| + \|\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\| \\
&\leq \|T(t)[x_0 - g(x) - A^{-\eta}A^{\eta}G(0, x(h_1(0)))]\| + \|A^{-\eta}A^{\eta}G(t, x(h_1(t)))\| \\
&\quad + \|\int_0^t A^{1-\eta}T(t-s)A^{\eta}G(t, x(h_1(s)))ds\| + \|\int_0^t T(t-s)y(s)ds\| \\
&\quad + \|\int_0^t T(t-s)Bu(s)ds\| + \|\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\| \\
&\leq M_1[\|x_0\| + K_2 r + K_3 + MK_1(r+1)] + MK_1(r+1) \\
&\quad + \int_0^t \frac{C_{1-\eta}}{(t-s)^{1-\eta}}K_1(r+1)ds + \int_0^t M_1\lambda(r)ds \\
&\quad + M_1\sum_{k=1}^m d_k + \|\int_0^t T(t-s)Bu(s)ds\|
\end{aligned}$$

Now

$$\begin{aligned}
&\|\int_0^t T(t-s)Bu(s)ds\| \\
&\leq \frac{b}{\alpha}M_1^2M_2^2\Big[\|x_b\| + M_1[\|x_0\| + K_2 r + K_3 + MK_1(r+1)] + MK_1(r+1) \\
&\qquad + \int_0^b \frac{C_{1-\eta}}{(b-s)^{1-\eta}}K_1(r+1)ds + \int_0^b M_1\lambda(r)ds + M_1\sum_{k=1}^m d_k] \\
&\leq \frac{b}{\alpha}M_1^2M_2^2\Big[\|x_b\| + M_1[\|x_0\| + K_3 + MK_1(r+1)] \\
&\qquad + K_1(M + \frac{C_{1-\eta}(b)^{\eta}}{\eta}) + M_1\sum_{k=1}^m d_k \\
&\qquad + [M_1K_2 + K_1\left(\frac{C_{1-\eta}b^{\eta}}{\eta} + M\right)] * r + M_1\int_0^b \lambda(r)ds\Big] \\
&\qquad \leq K^* + \frac{b}{\alpha}M_1^2M_2^2[M_1K_2 + K_1\left(\frac{C_{1-\eta}(b)^{\eta}}{\eta} + M\right)] * r + M_1\int_0^b \lambda(r)ds].
\end{aligned}$$

Therefore,

$$\begin{aligned} r \leq\ & K^*(1+\frac{b}{\alpha}M_1^2M_2^2) + \frac{b}{\alpha}M_1^2M_2^2\|x_b\| \\ & + (1+\frac{b}{\alpha}M_1^2M_2^2)\left[\left(M_1K_2 + K_1(\frac{C_{1-\eta}b^\eta}{\eta} + M)\right) * r + M_1\int_0^b \lambda(r)ds\right]. \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we get

$$\left(1+\frac{b}{\alpha}M_1^2M_2^2\right)\left[M_1K_2 + K_1\left(\frac{C_{1-\eta}b^\eta}{\eta} + M\right)\right] + M_1\delta \geq 1,$$

which is a contradiction to our assumption. Thus $\alpha > 0$, there exists $r > 0$ such that Ψ maps Q into itself.

Step 2: For each $\alpha > 0$, $\Psi(x)$ is convex for each $x \in Z$. Let $z_1, z_2 \in Z$, then there exist $y_1, y_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned} z_i(t) =\ & T(t)[x_0 - g(x) - G(0, x(h_1(0)))] + G(t, x(h_1(t))) \\ & + \int_0^t AT(t-s)G(t, x(h_1(s)))ds + \int_0^t T(t-s)y_i(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) + \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b \\ & - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ & - \int_0^b AT(b-s)G(t, x(h_1(s)))ds \\ & - \int_0^b T(b-s)y_i(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Big](\xi)d\xi, \quad i = 1, 2. \end{aligned}$$

Let $0 \leq \mu \leq 1$. then for each $t \in J$, we have

$$\begin{aligned} (\mu z_1 + (1-\mu)z_2)(t) =\ & T(t)[x_0 - g(x) - G(0, x(h_1(0)))] + G(t, x(h_1(t))) \\ & + \int_0^t AT(t-s)G(t, x(h_1(s)))ds + \int_0^t T(t-s)[\mu y_1(s) + (1-\mu)y_2(s)]ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) + \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\,[x_b \\ & - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ & - \int_0^b AT(b-s)G(t, x(h_1(s)))ds - \int_0^b T(b-s)[\mu y_1(s) + (1-\mu)y_2(s)]ds \\ & - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Big](\xi)d\xi. \end{aligned}$$

Since $S_{F,x}$ is convex, $\mu z_1 + (1-\mu)z_2 \in \Psi(x)$.

Step 3: $\Psi(x)$ is closed for each $x \in Z$. Let $\{w_n\}_{n\geq 0} \in \Psi(x)$ such that $w_n \to w$ in Z. Then there exists $y_n \in S_{F,x}$ such that for every $t \in J$,

$$\begin{aligned} w_n(t) &= T(t)[x_0 - g(x) - G(0, x(h_1(0)))] \\ &\quad + G(t, x(h_1(t))) + \int_0^t AT(t-s)G(t, x(h_1(s)))ds \\ &\quad + \int_0^t T(t-s)y_n(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) \\ &\quad + \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\ &\quad - G(b, x(h_1(b))) - \int_0^b AT(b-s)G(t, x(h_1(s)))ds \\ &\quad - \int_0^b T(b-s)y_n(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Big](\xi)d\xi. \end{aligned}$$

Since F has compact values, y_n converges to y in $L^1[J, X]$ and hence $y \in S_{F,x}$. Then for each $t \in J$, we have

$$\begin{aligned} w_n(t) \to w(t) &= T(t)[x_0 - g(x) - G(0, x(h_1(0)))] \\ &\quad + G(t, x(h_1(t))) + \int_0^t AT(t-s)G(t, x(h_1(s)))ds \\ &\quad + \int_0^t T(t-s)y(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) \\ &\quad + \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\ &\quad -G(b, x(h_1(b))) - \int_0^b AT(b-s)G(t, x(h_1(s)))ds \\ &\quad - \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Big](\xi)d\xi. \end{aligned}$$

Hence $y \in \Psi(x)$.

Step:4 We show that the operator Ψ is a u.s.c and a condensing multivalued map.

Let us decompose Ψ as $\Psi = \Psi_1 + \Psi_2$ where the operators Ψ_1, Ψ_2 are defined on Q by

$$(\Psi_1 x)(t) = G(t, x(h_1(t))) - T(t)G(0, x(h_1(0))) + \int_0^t AT(t-s)G(s, x(h_1(s)))ds,$$

$$\begin{aligned} (\Psi_2 x)(t) &= T(t)[x_0 - g(x)] + \int_0^t T(t-s)y(s)ds + \int_0^t T(t-s)Bu(s)ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

We now prove that Ψ_1 is a contraction operator and Ψ_2 is completely continuous operator. Let us consider $x, y \in Q$ and for each $t \in J$, we have

$$\begin{aligned}
&\|(\Psi_1 x)(t) - (\Psi_1 y)(t)\| \\
&\quad \leq \|G(t, x(h_1(t))) - G(t, y(h_1(t)))\| + \|T(t)[G(0, x(h_1(0))) - G(0, y(h_1(0)))]\| \\
&\qquad + \|\int_0^t AT(t-s)[G(s, x(h_1(s))) - G(s, y(h_1(s)))]ds\| \\
&\quad \leq \left[MK + MM_1K + \frac{KC_{1-\eta}b^\eta}{\eta}\right] \sup_{t\in J} \|x(s) - y(s)\| \leq M_0\|x(s) - y(s)\|,
\end{aligned}$$

where $K(M + MM_1 + \frac{C_{1-\eta}}{\eta}b^\eta) < 1$. Hence the map Ψ_1 is a contraction.

Next we prove that Ψ_2 is u.s.c and completely continuous. First we show that Ψ_2 is completely continuous. Let $\Psi_2(t) = {\Psi_2}^*(t) + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$. From (H_4), it is clear that $\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$ is completely continuous on Q. Hence it is enough, if we show that ${\Psi_2}^*(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)y(s)ds + \int_0^t T(t-s)Bu(s)ds$ is completely continuous on Q.

(I) ${\Psi_2}^*(Q)$ is clearly bounded.

(II) ${\Psi_2}^*(\cdot)$ is equicontinuous on Q. Let $z \in {\Psi_2}^*(x)$ and $x \in Q$, then there exists $y \in S_{F,x}$ such that for each $t \in J$,

$$z(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)y(s)ds + \int_0^t T(t-s)Bu(s)ds.$$

It is easy to see that $\|z(t) - z(0)\| = \|(T(t) - I)[A^\beta x_0 - A^\beta g(x)\| \to 0$ as $t \to 0^+$ uniformly on Q due to the complete continuity of $A^\beta g$ and the strong continuity of $T(t)$ at $t = 0$. Now let $t_1, t_2 \in J$, $0 < t_1 < t_2$ and $\epsilon > 0$ be small, then

$$\begin{aligned}
\|z(t_1) - z(t_2)\| &\leq \|T(t_1) - T(t_2)\|\|x_0 - g(x)\| \\
&+ \int_0^{t_1-\epsilon} \|[T(t_1 - s) - T(t_2 - s)]y(s)\|ds \\
&+ \int_{t_1-\epsilon}^{t_1} \|[T(t_1 - s) - T(t_2 - s)]y(s)\|ds + \int_{t_1}^{t_2} \|T(t_2 - s)y(s)\|ds \\
&+ \int_0^{t_1-\epsilon} \|[T(t_1 - \xi) - T(t_2 - \xi)]\|BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b \\
&- T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\
&- G(b, x(h_1(b))) - \int_0^b AT(b-s)G(s, x(h_1(s)))ds \\
&- \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Big](\xi)d\xi \\
&+ \int_{t_1-\epsilon}^{t_1} \|[T(t_1 - \xi) - T(t_2 - \xi)]\|BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b
\end{aligned}$$

$$
\begin{aligned}
&- T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\
&- G(b, x(h_1(b))) - \int_0^b AT(b-s)G(s, x(h_1(s)))ds \\
&- \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Bigg](\xi)d\xi \\
&+ \int_{t_1}^{t_2} \|T(t_2-\xi)\| BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Bigg[x_b \\
&- T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\
&- G(b, x(h_1(b))) - \int_0^b AT(b-s)G(s, x(h_1(s)))ds \\
&- \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Bigg](\xi)d\xi.
\end{aligned}
$$

Thus the right hand side of above does not depend on particular choices of $x(\cdot)$ and tends to zero as $t_1 - t_2 \to 0$, since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. This shows that ${\Psi_2}^*(\cdot)$ is equicontinuous on Q.

(III) $({\Psi_2}^*Q)(t)$ is relatively compact in X_β for every $t \in J$, where

$$({\Psi_2}^*Q)(t) = \{z(t) : z \in {\Psi_2}^*Q\}, \quad t \in J.$$

First we show that $V(t) = \{{\Psi_2}^*x(t) : x \in Q\}$ is relatively compact for $t \in J$. The case $t = 0$ is trivial. Let t, $0 < t \leq b$, be a fixed point and let ϵ be a given real number satisfying $0 < \epsilon < t$.

Define $({\Psi_2}^*x) = T(\epsilon)({\Psi_2}^*x)(t-\epsilon)$ where

$$
\begin{aligned}
({\Psi_2}^*x)(t-\epsilon) \;=\; & T(t-\epsilon)[x_0 - g(x)] + \int_0^{t-\epsilon} T(t-s-\epsilon)y(s)ds \\
& + \int_0^{t-\epsilon} T(t-s-\epsilon)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Bigg[x_b \\
& - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] \\
& - G(b, x(h_1(b))) - \int_0^b AT(b-s)G(s, x(h_1(s)))ds \\
& - \int_0^b T(b-s)y(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\Bigg]
\end{aligned}
$$

Since $T(t)$ is analytic and ${\Psi_2}^*$ is bounded on Q, the set $V_\epsilon(t) = \{({\Psi_2}^*x)(t), x(\cdot) \in Q\}$ is relatively compact in X_β. That is, a finite set $\{w_i, 1 \leq i \leq n\}$ in X exists such that

$$V_\epsilon(t) \subset \bigcup_{i=1}^n \tilde{N}(w_i, \tau/2),$$

where $\tilde{N}(w_i, \tau/2)$ is an open ball in X with center at w_i and radius $\tau/2$. On the other hand,

$$\|(\Psi_2{}^{*}x)(t) - (\Psi_2{}^{*\epsilon}x)(t)\| = \|\int_{t-\epsilon}^{t} T(t-s)[Bv(s) + y(s)]ds\| \leq \tau/2.$$

Consequently,

$$V(t) \subset \bigcup_{i=1}^{n} \tilde{N}(w_i, \tau).$$

Hence for each $t \in J$, $V(t)$ is relatively compact in X_β.

As a consequence of (I)–(III) and together with the Arzela-Ascoli theorem we can conclude that $\Psi_2{}^{*}$ is completely continuous and hence, so does the operator Ψ_2.

(IV) Ψ_2 has a closed graph.

Let $x^n \to x^*$, $x^n \in Q$, $z_n \in \Psi_2(x^n)$ and $z_n \to z_*$. We shall prove that $z_* \in \Psi_2(x^*)$. Since $z_n \in \Psi_2(x^n)$, there exists $y_n \in S_{F,x_n}$, such that

$$\begin{aligned} z_n(t) &= T(t)[x_0 - g(x^n)] + \int_0^t T(t-s)y_n(s)ds \\ &+ \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^n) - G(0, x^n(h_1(0)))] \\ &- G(b, x^n(h_1(b))) - \int_0^b AT(b-s)G(s, x^n(h_1(s)))ds - \int_0^b T(b-s)y_n(s)ds \\ &- \sum_{k=1}^{m} T(b-t_k)I_k(x^n(t_k^-))\Big](\xi)d\xi + \sum_{0<t_k<t} T(t-t_k)I_k(x^n(t_k^-)). \end{aligned}$$

We must prove that there exists $y_* \in S_{F,x^*}$, such that for each $t \in J$

$$\begin{aligned} z_*(t) &= T(t)[x_0 - g(x^*)] + \int_0^t T(t-s)y_*(s)ds \\ &+ \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^*) - G(0, x^*(h_1(0)))] \\ &- G(b, x^*(h_1(b))) - \int_0^b AT(b-s)G(s, x^*(h_1(s)))ds - \int_0^b T(b-s)y_*(s)ds \\ &- \sum_{k=1}^{m} T(b-t_k)I_k(x^*(t_k^-))\Big](\xi)d\xi + \sum_{0<t_k<t} T(t-t_k)I_k(x^*(t_k^-)). \end{aligned}$$

Since I_k, $k = 1, 2, \ldots, m$, and g are continuous, we obtain that

$$\begin{aligned} \Bigg(& z_n(t) - T(t)[x_0 - g(x^n)] - \sum_{0<t_k<t} T(t-t_k)I_k(x^n(t_k^-)) \\ &- \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha, \Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^n)] \\ &- G(0, x^n(h_1(0))) - G(b, x^n(h_1(b))) - \int_0^b AT(b-s)G(s, x^n(h_1(s)))ds \end{aligned}$$

$$- \int_0^b T(b-s)y_n(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x^n(t_k^-))\Big](\xi)d\xi\Bigg)$$

$$- \Bigg(z_*(t) - T(t)[x_0 - g(x^*)] + \sum_{0<t_k<t} T(t-t_k)I_k(x^*(t_k^-))$$

$$+ \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha,\Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^*)]$$

$$- G(0, x^*(h_1(0))) - G(b, x^*(h_1(b))) - \int_0^b AT(b-s)G(s, x^*(h_1(s)))ds$$

$$- \int_0^b T(b-s)y_*(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x^*(t_k^-))\Big](\xi)d\xi\Bigg) \to 0 \quad \text{as } n \to \infty.$$

Consider the linear continuous operator $\Upsilon : L^1(J,X) \to C(J,X), y \to \Upsilon(y)(t) = \int_0^t T(t-s)\Big[y(s) + BB^*T^*(b-t)R(\alpha,\Gamma_0^b)(\int_0^b T(b-\xi)y(\xi)d\xi)(s)\Big]ds$. Clearly it follows from Lemma 2.1 that $\Upsilon \cdot S$ is a closed graph operator. Moreover we have that

$$\Bigg(z_n(t) - T(t)[x_0 - g(x^n)] - \sum_{0<t_k<t} T(t-t_k)I_k(x^n(t_k^-))$$

$$- \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha,\Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^n)]$$

$$- G(0, x^n(h_1(0))) - G(b, x^*(h_1(b))) - \int_0^b AT(b-s)G(s, x^n(h_1(s)))ds$$

$$- \int_0^b T(b-s)y_n(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x^n(t_k^-))\Big](\xi)d\xi\Bigg) \in \Upsilon(S_{F,x^n})$$

Since $y_n \to y^*$, it follows from Lemma 2.1, that

$$\Bigg(z_*(t) - T(t)[x_0 - g(x^*)] + \sum_{0<t_k<t} T(t-t_k)I_k(x^*(t_k^-))$$

$$+ \int_0^t T(t-\xi)BB^*T^*(b-t)R(\alpha,\Gamma_0^b)\Big[x_b - T(b)[x_0 - g(x^*)]$$

$$- G(0, x^*(h_1(0))) - G(b, x^*(h_1(b))) - \int_0^b AT(b-s)G(s, x^*(h_1(s)))ds$$

$$- \int_0^b T(b-s)y_*(s)ds - \sum_{k=1}^m T(b-t_k)I_k(x^*(t_k^-))\Big](\xi)d\xi\Bigg) \in \Upsilon(S_{F,x^*}).$$

Therefore, Ψ_2 has a closed graph. Also Ψ_2 is a completely continuous multi-valued map with compact value and hence Ψ_2 is u.s.c.

On the other hand Ψ_1 is a contraction. So $\Psi = \Psi_1 + \Psi_2$ is u.s.c. and condensing. Hence, by Lemma 2.4, there exists a fixed point $x(\cdot)$ on Q. Thus the problem (1) has a solution on J. $\square$

Theorem 3.2. *Assume assumptions* $(S_1), (H_5)$ *and* (H_6) *are satisfied. Then the system* (1.1) *is approximately controllable on* J.

Proof. Let $x^\alpha(\cdot)$ be a fixed point of Ψ in Q. Any fixed point of Ψ is a mild solution of (1.1) under the control

$$u^\alpha(t) = B^*T^*(b-t)R(\alpha, \Gamma_0^b)p(x^\alpha)$$

and satisfies

$$\begin{aligned} x^\alpha(b) = x_b + \alpha R(\alpha, \Gamma_0^b)\{x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ - \int_0^b AT(b-s)G(s, x(h_1(s)))ds - \int_0^b T(b-s)y^\alpha(s)ds \\ - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-))\}, \end{aligned}$$

where

$$y^\alpha \in S_{F,x^\alpha} = \{y \in L^1(J, X); y^\alpha(t) \in F(t, x^\alpha(h_2(t))) \quad \text{for a.e. } t \in J\}.$$

By the condition (H_1) and (H_6)

$$\int_0^b \|y^\alpha(s)\|^2 ds \le bN^2$$

$$\int_0^b \|A^\eta G(s, x^\alpha(h_1(s)))\|^2 ds \le b[K_1\|x\| + 1]^2$$

and consequently the sequences $\{y^\alpha(s)\}$ and $\{A^\eta G(s, x^\alpha(h_1(s)))\}$ is bounded in $L_2(J, X)$. Thus there are subsequences, still denoted by $\{y^\alpha(s)\}$ and $\{A^\eta G(s, x^\alpha(h_1(s)))\}$, that converge weakly to say $y(s)$ and $h(s)$ in $L_2(J, X)$ respectively. Now, from the compactness of the operators $l(t) \to \int_0^t T(t-s)l(s)ds : L_2(J, X) \to C(J, X)$ and $l(t) \to \int_0^t A^\eta T(t-s)l(s)ds : L_2(J, X) \to C(J, X)$, we obtain that

$$\begin{aligned} \|p(x^\alpha) - w\| = \|\int_0^b T(b-s)[y^\alpha(s) - y(s)]ds\| \\ + \|\int_0^b T(b-s)[A^\eta G(s, x^\alpha(h_1(s))) - h(s)]\| \to 0 \end{aligned} \tag{3.1}$$

as $\alpha \to 0^+$, where

$$\begin{aligned} w = x_b - T(b)[x_0 - g(x) - G(0, x(h_1(0)))] - G(b, x(h_1(b))) \\ - \int_0^b AT(b-s)G(s, x^\alpha(h_1(s)))ds - \int_0^b T(b-s)y^\alpha(s)ds \\ - \sum_{k=1}^m T(b-t_k)I_k(x(t_k^-)). \end{aligned}$$

Then

$$\|x^\alpha(b) - x_b\| = \|\alpha R(\alpha, \Gamma_0^b)p(x^\alpha)\|$$

$$\leq \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|\alpha R(\alpha, \Gamma_0^b)\| \|p(x_\alpha) - w\|$$
$$\leq \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|p(x^\alpha) - w\| \to 0$$

as $\alpha \to 0^+$. This proves the approximate controllability of (1.1). $\square$

Remark 3.3. If $F(t, x(h_2(t)))$ is a single-valued map, then one can establish the approximate controllability of the impulsive neutral differential inclusions with nonlocal conditions by suitably introducing the technique of single valued map defined in [22].

Example 3.4. To illustrate the obtained result, we consider the following example.

Let $X = L^2[0, \pi]$ and let $A : X \to X$ be defined as follows $Aw = -w''$ with domain $D(A) = \{w \in X : w', w'' \in X, w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X which is analytic, compact and self-adjoint, the eigenvalues are $-n^2, n \in N$ with corresponding normalized eigenvectors $z_n(y) = (2/\pi)^{1/2} \sin(ny)$. Moreover the following hold:

(i) $\{z_n : n \in N\}$ is an orthonormal basis of X.
(ii) If $w \in D(A)$ then $Aw = -\sum_{n=1}^{\infty} n^2 < w, z_n > z_n$.
(iii) The operator $(A)^{1/2}$ is given as $(A)^{1/2}w = \sum_{n=1}^{\infty} n\langle w, z_n\rangle z_n$ on the space $D((A)^{1/2}) = \{w \in X : \sum_{n=1}^{\infty} n\langle w, z_n\rangle z_n \in X\}$.

Consider the following impulsive partial functional differential inclusion with nonlocal condition

$$(3.2) \qquad \begin{cases} \dfrac{\partial}{\partial t}\left[z(t,y) - \displaystyle\int_0^\pi a(t,y,\eta) z(\sin t, \eta) d\eta\right] \in -\dfrac{\partial^2 z(t,y)}{\partial y^2} \\ +\mu(t,y) + \hat{g}(t, \frac{\partial}{\partial y} z(\sin t, y)), \quad y \in [0,\pi], \quad t \neq t_k, \quad k = 1, \dots, m \end{cases}$$

subject to the conditions

$$z(t,0) = z(t,\pi) = 0,$$
$$z(t_k^+, y) - z(t_k^-, y) = \bar{I}_k(z(t_k^-, y))$$
$$z(0,y) = z_0(y) + \sum_{i=1}^{p} \hat{c}_i z(t_i, y),$$

where $z_0(y) \in X$, $\bar{I}_k \in C(R,R)$, $a : J \times [0,\pi] \times [0,\pi] \to R$ and $\hat{g} : J \times R \to R$ are continuous functions. Here $\hat{c}_i$, $i = 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \cdots \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

Here we choose $\beta = \frac{1}{2}$. For $t \in J, \psi \in X_{1/2}$ define $F(t,\psi)(\cdot) = \hat{g}(t, \psi'(\cdot))$, $G(t,\psi)(\cdot) = \int_0^\pi a(t,\cdot,\eta)\psi(\eta)d\eta$. Define the bounded linear control operator $B : X \to X$ by $Bu(t)(y) = \mu(t,y)$ and $g(x)(y) = \sum_{i=1}^{p} \hat{c}_i z(t_i, y)$. Let $h_1(t) = h_2(t) = \sin t$ and there exists a constant s_k such that $\|\bar{I}_k(w)\| \leq s_k$.

Hence (3.2) can be expressed as (1.1) with A, g, B, I_k, F and G as defined above. It is well known that A generates compact semigroup $T(t)$ in X and is given by $T(t)w = \sum_{n=1}^{\infty} e^{-n^2}\langle w, z_n\rangle z_n$. Because of the compactness of the semigroup $T(t)$ generated by A, the associated linearized system of (3.2) is not exactly controllable but it is approximately controllable [21]. Thus by Theorem 3.2 the system (3.2) is approximately controllable.

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