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ABSOLUTE AND INPUT-TO-STATE STABILITIES OF NONAUTONOMOUS SYSTEMS WITH CAUSAL MAPPINGS

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ABSTRACT. We consider systems governed by the scalar equation

$$\sum_{k=0}^{n} a_k(t)x^{(n-k)}(t) = [Fx](t) \quad (t \geq 0),$$

where $a_0 \equiv 1$; $a_k(t)$ $(k = 1, \ldots, n)$ are positive continuous functions and F is a causal mapping. We also consider the case when F depends on the input. Such equations include differential, integro-differential and other traditional equations. It is assumed that all the roots $r_k(t)$ $(k = 1, \ldots, n)$ of the polynomial $z^n + a_1(t)z^{n-1} + \cdots + a_n(t)$ are real and negative for all $t \geq 0$. Exact explicit conditions for the absolute and input-to-state stabilities of the considered systems are established.

Key words: nonlinear nonautonomous system, causal operators, absolute stability, input-to-state stability

AMS (MOS) subject classification: 34K20, 34K99, 93D05, 93D25

1. INTRODUCTION AND MAIN DEFINITIONS

We consider systems governed by the scalar equation

$$\sum_{k=0}^{n} a_k(t)x^{(n-k)}(t) = [Fx](t) \quad (t > 0) \tag{1.1}$$

where $a_0 \equiv 1$; $a_k(t)$ $(k = 1, \ldots, n)$ are positive continuous functions bounded on $[0, \infty)$ and F is a causal mapping. Below we recall the definition of the causal mapping and present the relevant examples.

Equation (1.1) includes various differential, integro-differential and other traditional equations. For the details see the excellent book [2]. The stability theory of nonlinear equations with causal mappings is at an early stage of development. The basic method for the stability analysis is the direct Liapunov method. But finding the Liapunov functionals for equations with causal mappings is a difficult mathematical problem.

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In the present paper we establish exact explicit conditions for the absolute and input-to-state stabilities of equation (1.1). To the best of our knowledge these stabilities for equations with causal mappings and nonautonomous linear parts were not explored in the available literature.

The literature on the absolute and input-to-state stabilities of continuous systems is very rich, cf. [16, 20, 22] and references therein. The classical results were developed in the interesting papers [12, 13, 14, 19, 24]. Mainly the systems with autonomous and periodic linear parts were considered, cf. [23, 18].

A deep investigation of linear causal operators is presented in the book [15]. The papers [1, 4] also should be mentioned. In the paper [4], the existence and uniqueness of local and global solutions to the Cauchy problem for equations with causal operators in a Banach space are established. In the paper [1] it is proved that the input-output stability of vector equations with causal operators is equivalent to the causal invertibility of causal operators.

The approach suggested below enables us to consider various classes of systems from a unified point of view.

Recall the definition of the causal mapping. To this end, for a positive $T \le \infty$, let E be a Banach space of functions defined on $[0, T]$ with the unit operator I. For all $\tau \in [0, T)$ and $w \in E$, let the projections P_τ be defined by

$$(P_\tau w)(t) = \begin{cases} w(t) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases} .$$

In addition, $P_T = I$.

Definition 1.1. A mapping $F : E \to E$ satisfying the condition

$$P_\tau F P_\tau = P_\tau F \quad (\tau \in [0, T]) \tag{1.2}$$

will be called a causal mapping (operator).

This definition was introduced in the excellent books [5, 21] (see also the papers [10, 11]); it is somewhat different from the definition of the causal operator suggested in [2].

Let us point an example of a causal mapping. Denote by $B(0, T)$ the set of scalar measurable functions defined and bounded on $[0, T]$. Consider in $B(0, T)$ the operator

$$(Fw)(t) = \tilde{f}(t, w(t)) + \int_0^t k(t, s, w(s))ds \quad (w \in B(0, T))$$

with a continuous kernel k, defined on $[0, T]^2 \times \mathbb{R}$ and a continuous function $\tilde{f}$: $[0, T] \times \mathbb{R} \to \mathbb{R}$. For each $\tau \in [0, T)$, we have

$$(P_\tau F w)(t) = f_\tau(t, w(t)) + \int_0^t k_\tau(t, s, w(s))ds$$

where

$$k_\tau(t,s,w(s)) = \begin{cases} k(t,s,w(s)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}$$

$(0 \le s \le t)$, and

$$f_\tau(t,w(t)) = \begin{cases} \tilde f(t,w(t)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}.$$

Thus (1.2) holds and the considered mapping is causal. Note that, the integral operator

$$w \to \int_0^c k(t,s,w(s))ds$$

with a fixed positive $c \le T$ is not causal.

2. THE BASIC LEMMA

Put $R_+ = [0,\infty]$. *Everywhere below, F is a continuous causal mapping acting in* $B(R_+)$. Consider the equation

$$x(t) = f(t) + \int_0^t K(t,t_1)(Fx)(t_1)dt_1 \quad (t > 0), \tag{2.1}$$

where $K : [0 \le s \le t < \infty) \to \mathbb{R}$ is a measurable kernel and $f \in B(R_+)$ is given.

A solution of (2.1) *is a scalar function x defined on R_+ which is bounded for any finite $\tau > 0$ and satisfies* (2.1) *a.e. on R_+.*

Introduce the sup-norm

$$\|v\| = \sup_{t\ge 0} |v(t)| \quad (v \in B(R_+)).$$

It is assumed that there are constants q and l, such that

$$\|Fv\| \le q\,\|v\| + l \quad (v \in B(R_+)). \tag{2.2}$$

In $B(R_+)$ introduce the operator V_K by

$$(V_K v)(t) = \int_0^t K(t,t_1)v(t_1)dt_1 \quad (t > 0; \quad v \in B(R_+))$$

assuming that V_K is bounded in $B(R_+)$. The following lemma plays an essential role below.

Lemma 2.1. *Let V_K be compact in $B(0,\tau)$ for each finite τ, and the conditions* (2.2), *and*

$$q\|V_K\| < 1 \tag{2.3}$$

hold. Then (2.1) *has at least one solution. Moreover any solution x of* (2.1) *satisfies the inequality*

$$\|x\| \le \frac{\|V_K\| l + \|f\|}{1 - q\|V_K\|}.$$

To prove this lemma we need the following simple result.

Lemma 2.2. *If condition* (2.2) *holds, then for all* $\tau \geq 0$ *and* $w \in B(0,\tau)$, *we have*

$$\|Fw\|_{B(0,\tau)} \leq q\, \|w\|_{B(0,\tau)} + l$$

where

$$\|v\|_{B(0,\tau)} = \sup_{0\leq t\leq \tau} |v(t)| \quad (v \in B(0,\tau)).$$

So $\|v\|_{B(0,\infty)} = \|v\|$.

Proof. From (1.2) and (2.2) it follows that

$$\|Fw\|_{B(0,\tau)} = \|P_\tau Fw\| = \|P_\tau F P_\tau w\| \leq \|F P_\tau w\| \leq q\|P_\tau w\| + l = q\|w\|_{B(0,\tau)} + l,$$

as claimed. □

Proof of Lemma 2.1: On $B(0,T)$, $T < \infty$ let us define the mapping Φ by $(\Phi w)(t) = f(t) + (V_K F w)(t)$ for a $w \in B(0,T)$. Hence, according to the previous lemma, for any $r > 0$, large enough,

$$\|\Phi w\|_{B(0,T)} \leq \|w\|_{B(0,T)} + \|V_K\|_{B(0,T)}(q\|w\|_{B(0,T)} + l) \leq r \quad (\|w\|_{B(0,T)} \leq r).$$

So Φ maps a bounded set of $B(0,T)$ into itself. Now the existence of a solution $x(t)$ is due to the Schauder Fixed Point Theorem, since V_K is compact. Furthermore, from (2.1) it follows that

$$\|x\|_{B(0,T)} \leq \|f\|_{B(0,T)} + \|V_K\|_{B(0,T)}(q\|x\|_{B(0,T)} + l).$$

Now (2.3) implies the required result. □

3. ABSOLUTE STABILITY

Consider equation (1.1) where F is a causal mapping acting in $B(R_+)$. Take the initial conditions

(3.1) $$x^{(k)}(0) = x_{0k} \;\; (x_{0k} \in \mathbb{R}; \quad k = 0, \dots, n-1).$$

Rewrite (1.1) as

(3.2) $$x(t) = z(t) + \int_0^t G(t,s)[Fx](s)ds$$

where $G(t,s)$ is the Green function of the Cauchy problem (the fundamental solution) to the linear homogeneous equation

(3.3) $$\sum_{k=0}^{n} a_k(t) z^{(n-k)}(t) = 0 \quad (t \geq 0),$$

and $z(t)$ is a solution the problem (3.1), (3.3). A continuous solution of the integral equation (3.2) will be called *a mild solution of problem (1.1), (3.1)*. In this section it is assumed that

$$\|Fw\| \le q\|w\| \quad (w \in B(R_+)). \tag{3.4}$$

We will say that *equation* (1.1) *is absolutely stable in the class of nonlinearities* (3.4), if problem (1.1), (3.1) has at least one mild solution, and there is a positive constant M which do not depend on a concrete form of F (but which depend on q), such that

$$\|x\| \le M \max_{k=0,\dots,n} |x_{k0}|$$

for any mild solution $x(t)$ of (1.1), (3.1).

Introduce the polynomial

$$P(z,t) = \sum_{k=0}^{n} a_k(t) z^{n-k} \quad (z \in \mathbb{C}).$$

Let all the roots $r_k(t)$ $(k = 1, \dots, n)$ of polynomial $P(z,t)$ for each $t \ge 0$ be real and

$$r_k(t) \le -\beta \quad (t \ge 0; \quad k = 1, \dots, n) \tag{3.5}$$

with a constants $\beta > 0$. The aim of this paper is to prove the following theorem.

Theorem 3.1. *Let all the roots of polynomial $P(z,t)$ be real for each $t \ge 0$ and the conditions* (3.5) *and*

$$q < \beta^n \tag{3.6}$$

hold. Then equation (1.1) *is absolutely stable in the class of nonlinearities* (3.4).

This theorem is proved in the next section.

Let us consider the is sharpness of Theorem 1.1. To this end let us consider the equation

$$\sum_{k=0}^{n} c^k C_n^k x^{(n-k)}(t) = qx(t) \quad (t > 0) \tag{3.7}$$

with a constant $c > 0$. Here C_n^k are the binomial coefficients. The sufficient and necessary stability conditions of this equation is the Hurwitzness of the polynomial $(z + c)^n - q$ or the inequality

$$c^n > q. \tag{3.8}$$

But in the considered case $\beta = c$. Thus, in the case of equation (3.7) the conditions of Theorem 3.1 are necessary.

4. PROOF OF THEOREM 3.1

Let $L^p(R_+)$, $p \geq 1$ be the space of real functions defined on R_+ with the finite norms

$$\|f\|_p = [\int_0^\infty |f(t)|^p]^{1/p} \quad (1 \leq p < \infty),$$

and

$$\|f\|_\infty = \text{ess} \sup_{t \geq 0} |f(t)|.$$

So if f is bounded, then $\|f\|_\infty = \|f\|$. Consider the problem

$$\sum_{k=0}^{n} a_k(t) D^{n-k} v(t) = f(t) \quad (D := \frac{d}{dt}; t > 0) \tag{4.1}$$

with a continuous function $f \in L^p(R_+)$ and the zero initial conditions

$$v^{(k)}(0) = 0 \; (k = 0, 1, \ldots, n-1). \tag{4.2}$$

For a fixed $p \geq 1$ introduce the set

$$Dom\ (E) := \{w \in L^p(R_+) : w^{(j)} \in L^p(R_+) \quad (j = 1, 2, \ldots, n),$$

$$w^{(k)}(0) = 0 \quad (k = 0, 1, \ldots, n-1)\}.$$

Lemma 4.1. *Let $a_0 \equiv 1$; $a_k(t)$ $(k = 1, \ldots, n)$ be positive continuous functions bounded on $[0, \infty)$. Let condidion (3.5) hold and f be a continuous function from $L^p(R_+)$. Then problem (4.1), (4.2) has a unique solution $v \in Dom(E)$. Moreover,*

$$\|v\|_p \leq \frac{\|f\|_p}{\beta^n}.$$

Proof. Define on $Dom\ (E)$ the operator E by

$$Eu(t) := P(t, D)u = \sum_{k=0}^{n} a_k(t) D^{n-k} u(t) \quad (u \in Dom\ (E)).$$

So problem (4.1), (4.2) can be written as $Ev = f$. Since the coefficients of equation (4.1) are bounded, the roots of $P(z, t)$ are bounded on R_+. Thus,

$$r_k(t) \geq -\alpha \quad (t \geq 0; \quad k = 1, 2, \ldots, n)$$

for a finite positive number α. Define on $Dom\ (E)$ also the operator E_0 by

$$E_0 u(t) := (D + \alpha)^n u(t) = (\frac{d}{dt} + \alpha)^n u(t).$$

Then the inverses to E and E_0 satisfy the relations

$$E^{-1} = E_0^{-1} E_0 E^{-1} = E_0^{-1} (E E_0^{-1})^{-1}. \tag{4.3}$$

Below we check that the inverses really exist.

By the Laplace transform for any bounded continuous function y defined on R_+ we have

$$E_0^{-1}y(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{\lambda t}\tilde{y}(\lambda)}{(\lambda+\alpha)^n}\,d\lambda$$

where $\tilde{y}$ is the Laplace transform to y. Set $h := EE_0^{-1}y$. Then

$$h(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{\lambda t}P(\lambda,t)\tilde{y}(\lambda)d\lambda}{(\lambda+\alpha)^n}.$$

Hence,

$$h(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\tilde{y}(\lambda)\prod_{k=1}^{n}\frac{\lambda - r_k(t)}{\lambda+\alpha}\,d\lambda.$$

Put

$$F(t,\nu) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\tilde{y}(\lambda)\prod_{k=1}^{n}\frac{\lambda - r_k(\nu)}{\lambda+\alpha}\,d\lambda \quad (t,\nu \geq 0). \tag{4.4}$$

Thus

$$F(t,t) = h(t).$$

We can write out

$$F(t,\nu) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\tilde{y}(\lambda)\prod_{k=1}^{n}\left(1 - \frac{\alpha + r_k(\nu)}{\lambda+\alpha}\right)\,d\lambda \quad (t,\nu \geq 0).$$

Then by the convolution property,

$$F(t,\nu) = (K_1(\nu) * K_2(\nu) * \cdots * K_n(\nu) * y)(t) \tag{4.5}$$

where for a continuous u,

$$(K_j(\nu) * u)(t) = u(t) - (\alpha + r_j(\nu))\int_0^t e^{-\alpha(t-s)}u(s)ds,$$

since $e^{-\alpha t}$ is the Laplace original to $(\lambda+\alpha)^{-1}$. So

$$(K_j(t) * u)(t) = u(t) - (\alpha + r_j(t))\int_0^t e^{-\alpha(t-s)}u(s)ds.$$

Therefore

$$|(K_j(t) * u)(t)| \geq |u(t)| - (\alpha - \beta)\int_0^\infty |u(s)|e^{-\alpha(t-s)}ds \tag{4.6}$$

since $-r_j(\nu) > \beta$ $(\nu \geq 0)$. From (4.5) it follows that

$$h(t) = F(t,t) = (K_1(t) * K_2(t) * \cdots * K_n(t) * y)(t).$$

Put

$$y_j(t) = (K_{j+1}(t) * cdots * K_n(t) * y)(t) \;\; (j = 1, \dots, n-1).$$

Then

$$h(t) = (K_1(t) * y_1)(t) = y_1(t) - (\alpha - r_1(t))\int_0^t e^{-\alpha(t-s)}y_1(s)ds$$

and

$$y_j(t) = (K_{j+1}(t) * y_{j+1})(t) \quad (j = 1, \dots, n-1); \quad y_n = y.$$

By (4.6) we have

$$\|y_j\|_p \geq \|y_{j+1}\|_p(1-(\alpha-\beta)\int_0^\infty e^{-\alpha s}ds) = \|y_{j+1}\|_p\left(1+\frac{\beta-\alpha}{\alpha}\right) = \|y_{j+1}\|_p\frac{\beta}{\alpha}.$$

Thus,

$$\|y\|_p \leq \|y_{n-1}\|_p\frac{\alpha}{\beta} \leq \|y_{n-2}\|_p\frac{\alpha^2}{\beta^2} \leq \cdots \leq \|y_1\|_p\frac{\alpha^{n-1}}{\beta^{n-1}} \leq \|h\|_p\frac{\alpha^n}{\beta^n}.$$

This means that the operator $T := EE_0^{-1}$ satisfies the inequality

$$\|Ty\|_p\frac{\alpha^n}{\beta^n} \geq \|y\|_p. \tag{4.7}$$

Let us prove that T is invertible. Indeed, from the (4.4), by the convolution property it follows,

$$h(t) = F(t,t) = y(t) + \int_0^t W(t,t-s)y(s)ds$$

where

$$W(\nu,t) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\frac{(\lambda+\alpha)^n - P(\lambda,\nu)}{(\lambda+\alpha)^n}d\lambda \quad (\nu \geq 0).$$

Furthermore, on the space $L^p(0,\tau)$ with a positive $\tau < \infty$, introduce the Volterra operator V by

$$(Vw)(t) = \int_0^t W(t,t-s)w(s)ds.$$

Then $y - Vy = Ty = h$. By the Neumann series,

$$T^{-1} = (I-V)^{-1}h = \sum_{k=0}^\infty V^k.$$

Here I is the unit operator. Note that the Neumann series of any Volterra operator with a continuous kernel converges in the norm of L^p on each finite segment, since the spectral radius of that operator is equal to zero. Taking into account that continuous functions are dense in L^p, by (4.7) we get the inequality

$$\|T^{-1}\|_p = \|(EE_0^{-1})^{-1}\|_p \leq \frac{\alpha^n}{\beta^n}. \tag{4.8}$$

Furthermore, take into account that for any continuous $y \in L^p(R_+)$,

$$E_0^{-1}y(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{\lambda t}\tilde{y}(\lambda)}{(\lambda+\alpha)^n}\,d\lambda = \int_0^t Q(t-s)y(s)ds \ \ (y)$$

where

$$Q(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{\lambda t}}{(\lambda+\alpha)^n}\,d\lambda.$$

By the Cauchy formula for derivatives, we have

$$Q(t) = \frac{t^{n-1}}{(n-1)!}e^{-\alpha t} \quad (t \geq 0).$$

Hence,

$$\|E_0^{-1}y\|_p \leq \|y\|_p \sup_{t\geq 0}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}e^{-\alpha(t-s)}ds = \|y\|_p\int_0^\infty \frac{s^{n-1}}{(n-1)!}e^{-\alpha s}ds = \|y\|_p\frac{1}{\alpha^n}.$$

So

$$\|E_0^{-1}\|_p \leq \frac{1}{\alpha^n}.$$

Now (4.3) and (4.8) imply

$$\|E^{-1}\|_p \leq \|E_0^{-1}\|_p \|(EE_0^{-1})^{-1}\|_p \leq \frac{1}{\beta^n}.$$

This proves the required result. □

Recall that $\|\cdot\|$ means the sup-norm on R_+.

Lemma 4.2. *Let all the roots of polynomial $P(z,t)$ be real and the conditions* (2.2), (3.5) *and* (3.6) *hold. Then problem* (1.1), (3.1) *has at least one mild solution. Moreover any mild solution x of* (1.1), (3.1) *satisfies the inequality*

$$\|x\| \leq c_2(l + \max_k |x_{0k}|)$$

where the constant c_2 does not depend on l and the initial conditions.

Proof. The well known Theorem III.5.1 [3] asserts that if a nonhomogeneous linear differential equation has a bounded solution for any bounded right-hand part and the zero initial conditions, then the corresponding homogeneous equation is exponentially stable. So by Lemma 4.1, a solution z of the linear problem (3.1), (3.3) is subject to the inequality

$$|z(t)| \leq c_1 \max_k |x_{0k}| e^{-\epsilon t} \quad (t \geq 0,\ c_1, \epsilon = const > 0).$$

Moreover, according to (3.2) by Lemma 4.1 we have $\|V_K\| \leq 1/\beta^n$. This and Lemma 2.1 prove the required result. □

The assertion of Theorem 3.1 follows from the previous lemma with $l = 0$.

5. EXPONENTIAL AND INPUT-TO-STATE STABILITIES

5.1. **The general case.** *Equation* (1.1) *is said to be absolutely exponentially stable* in the class of nonlinearities (3.4) if there is are positive constants M and ν which do not depend on a concrete form of F, such that

$$|x(t)| \leq Me^{-\nu t} \max_{k=0,\dots,n} |x_{k0}|$$

for any mild solution $x(t)$ of (1.1), (3.1) provided (3.4) holds.

Assume that the condition

$$\lim_{\epsilon \to 0} \sup_{t \geq 0} e^{\epsilon t} |[F(e^{-\epsilon t} w)](t)| \leq q\|w\| \quad (w \in C(R_+)) \tag{5.1}$$

holds. Substituting

$$x(t) = y(t)e^{-\epsilon t} \tag{5.2}$$

in (1.1) with an $\epsilon > 0$ we arrive at the equation

$$\sum_{k=0}^{n} a_{\epsilon,k}(t) D^{n-k} y(t) = [F_\epsilon y](t) \quad (t > 0) \tag{5.3}$$

where $a_{\epsilon,k}(t) \to a_k(t)$ in the sup-norm as $\epsilon \to 0$ and

$$[F_\epsilon y] = e^{\epsilon t}[F e^{-\epsilon t} y].$$

According to (5.1) $\|F_\epsilon y\| \le \epsilon_1 + \|y\|$ where $\epsilon_1 \to 0$ as $\epsilon \to 0$. Applying Theorem 3.1 to equation (5.3) with $\epsilon > 0$ small enough, under conditions (3.5) and (3.6) we have

$$\|y\| \le const \max_{k=0,\dots,n-1} |x_{k0}|$$

Now (5.2) implies

Theorem 5.1. *Let all the roots of polynomial $P(z,t)$ be real and the conditions* (3.5), (3.6) *and* (5.1) *hold. Then equation* (1.1) *is absolutely exponentially stable in the class of nonlinearities* (3.4).

Furthermore, let $\tilde{U}$ be a Banach space of functions defined on R_+. For example, $\tilde{U} = L^p(R_+)$. Consider the equation

$$P(D,t)x(t) = [F(u,x)](t). \tag{5.4}$$

Here x is the state, $u \in \tilde{U}$ is the input, $F(u,\cdot)$ for each $u \in \tilde{U}$ is a causal mapping acting in $L^2(R_+)$.

Equation (5.4) *is said to be globally bounded input-to-bounded state stable (globally BIBS stable), if the zero initial conditions*

$$x^{(k)}(0) = 0 \quad (k = 0, \dots, n-1).$$

and $u \in \tilde{U}$ imply that (5.4) *has at least one mild solution, and every mild solution is bounded.*

Suppose that there are constants q and q_U, such that

$$\|F(u,w)\| \le q\|w\| + q_U |u|_{\tilde{U}} \quad (w \in C(R_+); \quad u \in \tilde{U}) \tag{5.5}$$

where $|u|_{\tilde{U}}$ is the norm of space $\tilde{U}$. Now Lemma 4.2 implies

Corollary 5.2. *For each $u \in \tilde{U}$, let a mapping $F(u,\cdot) : B(R_+) \to B(R_+)$ be continuous causal. If, in addition, all the roots of polynomial $P(z,t)$ be real and the conditions* (3.5), (3.6) *and* (5.5) *hold, then equation* (5.3) *is globally BIBS-stable.*

5.2. **Nonlinear differential equation.** Consider the differential equation

$$\sum_{k=0}^{n} a_k(t) D^{n-k} x(t) = F_0(x,t) \quad (t > 0) \tag{5.6}$$

where $F_0 : \mathbb{R} \times R_+ \to \mathbb{R}$ is continuous and satisfies the condition

$$|F_0(v,t)| \leq q|v| \quad (v \in \mathbb{R}; \quad t \geq 0). \tag{5.7}$$

Then, clearly,

$$\lim_{\epsilon \to 0} \sup_{t \geq 0} e^{\epsilon t} |[F_0(e^{-\epsilon t} w(t))| \leq \lim_{\epsilon \to 0} \sup_{t \geq 0} e^{\epsilon t} q e^{-\epsilon t} |w(t)| = q\|w\| \quad (w \in C(R_+)).$$

Thus, by Theorem 5.1, we get

Corollary 5.3. *Under conditions* (3.5) *and* (3.6)*, equation* (5.6) *is absolutely exponentially stable in the class of nonlinearities* (5.7)*.*

Similarly the BIBS stability of ordinary differential equation can be considered.

Concluding Remarks. The paper proposes explicit conditions that provide the absolute stability and input-to-state stability of equation (1.1). These conditions enable us to avoid constructing the Liapunov functions (functionals). Our results are exact: the suggested sufficient conditions became necessary in appropriate situations. The notion of the causal operator allows us to consider various systems from the unified point of view.

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