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APPLICABLE FIXED POINT THEORY IN FUNCTIONAL DIFFERENTIAL EQUATIONS ON UNBOUNDED INTERVALS

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ABSTRACT. In this article, we discuss the applications of some fixed point theorems to various types of functional differential equations for proving the existence as well as global attractivity and ultimate positivity of solutions on unbounded intervals under some usual natural conditions. Our hypotheses and claims have also been explained with the help of some natural realizations.

AMS (MOS) Subject Classification. 39A10.

Keywords. Functional differential equation; Fixed point theorem; Attractive solutions; Ultimately positive solutions.

1. INTRODUCTION

The applicable fixed point theory is a core part of the subject of nonlinear functional analysis and has its origin in the works of Schauder [20, page 56], Banas [20, page 17] and Tarski [20, page 506]. See Granas and Dugundji [15], Deimling [6], Zeidler [20] and the references therein. A fixed point theorem useful for applications to other areas of mathematics such as theory of differential and integral equations, approximation and optimization theory, control theory, economics and game theory etc., is classified as applicable fixed point theorem and the collection of such applicable fixed point theorems is the applicable fixed point theory. The theory of functional differential equations is not escaped from the use of fixed point theorems in which the fixed point theorems have been used in variety of ways to prove the existence results for various types of nonlinear functional differential equations.

Most of applications of the fixed point theorems to nonlinear problems of any dynamical systems are existential in nature. However, now it is clear that the fixed point theory is also useful in obtaining the different characterizations of the solutions. See Heikkilä and Lakshmikantham [17], Burton and Zhang [4], Banas and Dhage [2], Dhage [8, 9, 10] and the references given therein. The method of applications of fixed point theorems to functional differential equations consists of following main steps. Since integrals are easier to handle than differentials, first the given functional differential equation is converted into an equivalent integral equation via theory of

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differential and integral calculus and then so obtained integral equation is written in the form of operator equation in a suitable function space. Finally, depending upon the nature of nonlinearities involved in a differential equation, a fixed point theorem is used to prove the existence of solutions for the so obtained equivalent operator equation which thereby implies the existence results for the functional differential equations in question.

In this article, we characterize the solutions of some nonlinear functional differential equations via applicable classical and hybrid fixed point theorems in abstract spaces. The nonlinearities involved in the equations are not assumed to be continuous and the characterizations of the solutions are obtained under Carathéodory conditions. We claim that our results are new to the theory of nonlinear functional differential equations on unbounded intervals.

2. FUNCTIONAL DIFFERENTIAL EQUATIONS

The differential equations in which the solutions depend upon the past or future states are called functional differential equations. The former are called the functional differential equations of delay type and the later are called the functional differential equations of advanced type. The common nomenclature for both type of functional differential equations is differential equations with deviating arguments. The differential equations in which the solutions depend upon the past velocity or derivatives are called functional differential equations of neutral type. It is needless to say the importance of study of functional differential equations since they arise in several dynamical systems of natural and physical phenomena of the universe. The exhaustive treatment of this topic appear in a monograph of Hale [16]. In this article, we discuss three types of nonlinear functional differential equations on unbounded intervals of real line for existence as well as for some characterizations of the solutions via classical fixed point theorems in Banach spaces.

Let $\mathbb{R}$ be the real line and let $\mathbb{R}_+$ be the set of nonnegative real numbers. Let $I_0 = [-\delta, 0]$ be a closed and bounded interval in $\mathbb{R}$ for some real number $\delta > 0$ and let $J = I_0 \cup \mathbb{R}_+$. Let $\mathcal{C}$ denote the Banach space of continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\phi\|_{\mathcal{C}} = \sup_{t \in I_0} |\phi(t)|.$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm. For a fixed $t \in \mathbb{R}_+$, let x_t denote the element of $\mathcal{C}$ defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\delta, 0].$$

The space $\mathcal{C}$ is called the history space of the past interval I_0 for the functional differential equations to describing the past history of the problems in question.

Let $\mathcal{CRB}(\mathbb{R}_+)$ denote the class of functions $a : \mathbb{R}_+ \to \mathbb{R} - \{0\}$ satisfying the following properties:

(i) a is continuous,
(ii) $\lim_{t\to\infty} a(t) = \pm\infty$, and
(iii) $a(0) = 1$.

There do exist functions satisfying the above conditions. In fact, if $a_1(t) = t + 1$, $a_2(t) = e^t$, then $a_1, a_2 \in \mathcal{CRB}(\mathbb{R}_+)$. Again, the class of continuous and strictly monotone functions $a : \mathbb{R}_+ \to \mathbb{R} - \{0\}$ with $a(0) = 1$ satisfy the above criteria. Note that if $a \in \mathcal{CRB}(\mathbb{R}_+)$, then the reciprocal function $\overline{a} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim_{t\to\infty} \overline{a}(t) = 0$.

Given a function $\phi \in \mathcal{C}$, we consider the following functional differential equation, viz.,

$$\left.\begin{aligned} \frac{d}{dt}\big[a(t)x(t)\big] &= g(t, x(t), x_t) \ \text{ a.e. } t \in \mathbb{R}_+ \\ x_0 &= \phi \end{aligned}\right\} \tag{2.1}$$

where, $a \in \mathcal{CRB}(\mathbb{R}_+)$ and $g : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$. Next, we consider the following perturbed functional differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\big[a(t)x(t) - f(t, x(t))\big] &= g(t, x(t), x_t) \ \text{ a.e. } t \in \mathbb{R}_+ \\ x_0 &= \phi, \end{aligned}\right\} \tag{2.2}$$

where $a \in \mathcal{CRB}(\mathbb{R}_+)$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$.

Finally, we consider the following quadratic functional differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t, x(t))}\right] &= g(t, x(t), x_t) \ \text{ a.e. } t \in \mathbb{R}_+ \\ x_0 &= \phi, \end{aligned}\right\} \tag{2.3}$$

where $a \in \mathcal{CRB}(\mathbb{R}_+)$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$.

It is clear that the functional differential equations (in short FDEs) (2.1), (2.2) and (2.3) are respectively the scalar, linear and quadratic perturbations of second kind for the following nonlinear first order FDE on unbounded interval,

$$\left.\begin{aligned} x'(t) &= g(t, x(t), x_t) \ \text{ a.e. } t \in \mathbb{R}_+ \\ x_0 &= \phi. \end{aligned}\right\} \tag{2.4}$$

A good deal of the discussion on different types of perturbations for the nonlinear differential equations appears in a recent paper of Dhage [13]. Therefore, we claim that the functional differential equations (2.1), (2.2) and (2.3) are new to the theory of nonlinear differential equations and some special cases of these FDEs with $a \equiv 1$

have already been studied in the literature on closed and bounded intervals for various aspects of the solutions. See Hale [16], Ntouyas [19], Dhage *et al.* [14] and the references given therein. However, to the best of author's knowledge, the FDEs (2.1), (2.2) and (2.3) are not discussed so far in the literature on closed but unbounded intervals of real line. In this article, we discuss the above mentioned functional differential equations for existence as well as for different characterizations of the solutions such as attractivity, asymptotic attractivity and ultimate positivity of the solutions. Since the FDEs in question are perturbed, we have to take resort of the hybrid fixed point theory in appropriate functional spaces in order to discuss these problems for different aspects of the solutions. Therefore, the hybrid fixed point theoretic approach is used while formulating the most of our results for FDEs (2.1), (2.2) and (2.3) of this paper. We claim that almost all the results of this article are new to the theory of functional nonlinear ddifferential equations on unbounded intervals of real line..

3. APPLICABLE FIXED POINT THEORY

Let X be a non-empty set and let $T : X \to X$. An invariant point under T in X is called a fixed point of T, that is, the fixed points are the solutions of the functional equation $Tx = x$. Any statement asserting the existence of fixed point of the mapping T is called a fixed point theorem for the mapping T in X. The fixed point theorems are obtained by imposing the conditions on T or on X or on both T and X. By experience, better the mapping T or X, we have better fixed point principles. As we go on adding richer structure to the non-empty set X, we derive richer fixed point theorems useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations. Below we give some fixed point theorems useful in establishing the attractivity and ultimate positivity of the solutions for FDEs (2.1), (2.2) and (2.3) on unbounded intervals. Before stating these results we give some preliminaries.

let X be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $Q : X \to X$ is called **$\mathcal{D}$-Lipschitz** if there is a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Qx - Qy\| \leq \phi(\|x - y\|)$$

for all $x, y \in X$, where $\phi(0) = 0$. If $\phi(r) = k\,r$, $k > 0$, then Q is called **Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **contraction** on X with the contraction constant k. Further, if $\phi(r) < r$ for $r > 0$, then Q is called **nonlinear $\mathcal{D}$-contraction** and the function ϕ is called $\mathcal{D}$-function of Q on X. There do exist $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are $\phi(r) = k\,r$ and $\phi(r) = \dfrac{r}{1+r}$, etc. (see Banas and Dhage [2] and the references therein).

Definition 3.1. An operator Q on a Banach space X into itself is called compact if for any bounded subset S of X, $Q(S)$ is a relatively compact subset of X. If Q is continuous and compact, then it is called completely continuous on X.

Our first fixed point theorem is

Theorem 3.2 (Granas and Dugundji [15]). *Let S be a non-empty, closed, convex and bounded subset of the Banach space X and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation*

$$Qx = x \tag{3.1}$$

has a solution in S.

We employ the following variant of a fixed point theorem of Burton [3] which is a special case of a hybrid fixed point theorem due to the present author [11] in Banach spaces.

Theorem 3.3 (Dhage [7]). *Let S be a closed, convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is nonlinear $\mathcal{D}$-contraction,*
(b) *B is completely continuous, and*
(c) *$x = Ax + By \Longrightarrow x \in S$ for all $y \in S$.*

Then the operator equation

$$Ax + Bx = x \tag{3.2}$$

has a solution in S.

Theorem 3.4 (Dhage [11]). *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function ψ,*
(b) *B is completely continuous,*
(c) *$x = Ax\,By \Longrightarrow x \in S$ for all $y \in S$, and*
(d) *$M\,\psi(r) < r$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.*

Then the operator equation

$$Ax\,Bx = x \tag{3.3}$$

has a solution in S.

A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [15], Deimling [6], Zeidler [20] and the references therein. In the following section we give different types of characterizations of the solutions for nonlinear functional differential equations on unbounded intervals of real line.

4. CHARACTERIZATIONS OF SOLUTIONS

We seek the solutions of the FDEs (2.1), (2.2) and (2.3) in the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ of continuous and bounded real-valued functions defined on $I_0 \cup \mathbb{R}_+$. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ by

$$\|x\| = \sup_{t \in I_0 \cup \mathbb{R}_+} |x(t)| \quad \text{and} \quad (xy)(t) = x(t)y(t), \; t \in \mathbb{R}_+.$$

Clearly, $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ becomes a Banach algebra with respect to the above norm and the multiplication in it. By $L^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ and the norm $\|\cdot\|_{L^1}$ in $L^1(\mathbb{R}_+, \mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|\, ds.$$

In order to introduce further concepts used in this paper, let us assume that $E = BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ and let Ω be a non-empty subset of X. Let $Q : E \to E$ be a operator and consider the following operator equation in E,

$$Qx(t) = x(t) \tag{4.1}$$

for all $t \in I_0 \cup \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator equation (4.1) in the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$.

Definition 4.1. We say that solutions of the operator equation (4.1) are **locally attractive** if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ for some $x_0 \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (4.1) belonging to $\overline{\mathcal{B}}_r(x_0)$ we have that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \tag{4.2}$$

In the case when the limit (4.2) is uniform with respect to the set $\overline{\mathcal{B}}_r(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{4.3}$$

for all $x, y \in \overline{\mathcal{B}}_r(x_0)$ being solutions of (4.1) and for $t \ge T$, we will say that solutions of equation (4.1) are **uniformly locally attractive** on $I_0 \cup \mathbb{R}_+$.

Definition 4.2. A solution $x = x(t)$ of equation (4.1) is said to be **globally attractive** if (4.2) holds for each solution $y = y(t)$ of (4.1) in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$. In other words, we may say that solutions of the equation (4.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (4.1) in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$, the condition (4.2) is satisfied. In the case when the condition (4.2) is satisfied uniformly with respect to the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$, i.e., if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality (4.2) is satisfied for all $x, y \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ being the solutions of (4.1) and for $t \geq T$, we will say that solutions of the equation (4.1) are **uniformly globally attractive** on $I_0 \cup \mathbb{R}_+$.

Remark 4.3. Let us mention that the concept of global attractivity of solutions is recently introduced in Hu and Yan [18] while the concepts of uniform local and global attractivity (in the above sense) were introduced in Banas and Rzepka [1].

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (4.1) in the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$.

Definition 4.4 (Dhage [12])**.** A solution x of the equation (4.1) is called **locally ultimately positive** if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ for some $x_0 \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ such that $x \in \overline{\mathcal{B}}_r(0)$ and

$$\lim_{t \to \infty} \big[|x(t)| - x(t)\big] = 0. \tag{4.4}$$

In the case when the limit (4.4) is uniform with respect to the solution set of the operator equation (4.1) in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$||x(t)| - x(t)| \leq \epsilon \tag{4.5}$$

for all x being solutions of (4.1) in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (4.1) are **uniformly locally ultimately positive** on $\mathbb{R}_+$.

Definition 4.5 (Dhage [12])**.** A solution $x \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ of the equation (4.1) is called **globally ultimately positive** if (4.4) is satisfied. In the case when the limit (4.5) is uniform with respect to the solution set of the operator equation (4.1) in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (4.5) is satisfied for all x being solutions of (4.1) in in $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (4.1) are **uniformly globally ultimately positive** on $I_0 \cup \mathbb{R}_+$.

Remark 4.6. We note that global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (4.1) on $I_0 \cup \mathbb{R}_+$. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (4.1) on unbounded intervals. However, the converse of the above two statements may not be true.

5. ATTRACTIVITY AND POSITIVITY RESULTS

In this section, we prove the global attractivity and positivity results for the FDEs (2.1), (2.2) and (2.3) on $I_0 \cup \mathbb{R}_+$ under some suitable conditions. In what follows, we frequently use the class of almost everywhere differentiable functions to define the solutions for above mentioned functional differential equations. So, before going to the main results, we discuss in brief the class of such functions on closed intervals of real line. Let I be a closed interval in $\mathbb{R}$ and let $AC(I,\mathbb{R})$ be the space of functions which are defined and absolutely continuous on I. As every absolutely continuous functions is continuous on I, we have that $AC(I,\mathbb{R}) \subset C(I,\mathbb{R})$. However, converse implication may not hold. It is also known that if $x \in AC(I,\mathbb{R})$, then it is almost everywhere differentiable on I. In the following, first we prove the global attractivity and ultimate positivity results for the FDE (2.1) on $I_0 \cup \mathbb{R}_+$.

5.1. **Ordinary Functional Differential Equations.** First we discuss the FDE (2.1) for attractivity characterization of the solutions on unbounded interval $I_0 \cup \mathbb{R}_+$. We need the following definitions in the sequel.

Definition 5.1. By a *solution* for the functional differential equation (2.1) we mean a function $x \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R}) \cap AC(\mathbb{R}_+, \mathbb{R})$ such that

(i) The function $t \mapsto a(t)x(t)$ is absolutely continuous on $\mathbb{R}_+$, and
(ii) x satisfies the equations in (2.1),

where $AC(\mathbb{R}_+, \mathbb{R})$ is the space of absolutely continuous real-valued functions on right half real axis $\mathbb{R}_+$.

Definition 5.2. A function $g : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is called *Carathéodory* if

(i) $t \mapsto g(t,x,y)$ is measurable for all $x \in \mathbb{R}$ and $y \in \mathcal{C}$, and
(ii) $(x,y) \mapsto g(t,x,y)$ is continuous for all $t \in \mathbb{R}_+$.

We need the following hypotheses in the sequel.

(H_1) There exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,x,y)| \leq h(t) \text{ a.e. } t \in \mathbb{R}_+$$

for all $x \in \mathbb{R}$ and $y \in \mathcal{C}$. Moreover, we assume that $\displaystyle\lim_{t\to\infty} |\overline{a}(t)| \int_0^t h(s)\,ds = 0$.

(H_2) $\phi(0) \geq 0$.

Remark 5.3. If the hypothesis (H_1) holds and $a \in \mathcal{CRB}(\mathbb{R}_+)$, then $\overline{a} \in BC(\mathbb{R}_+, \mathbb{R})$ and the function $w : \mathbb{R}_+ \to \mathbb{R}$ defined by the expression $w(t) = |\overline{a}(t)| \displaystyle\int_0^t h(s)\,ds$ is continuous on $\mathbb{R}_+$. Therefore, the number $W = \sup_{t\geq 0} w(t)$ exists. Note that $\lim_{t\to\infty} w(t) = 0$ may not always hold. There do exist functions h involved in w such that $\lim_{t\to\infty} w(t) \neq 0$. Indeed, if $h(t) = e^t$ on $\mathbb{R}_+$, then $\lim_{t\to\infty} \frac{1}{t+1} \displaystyle\int_0^t e^s\,ds \neq 0$, even though the function $a(t) = t+1$, $t \in \mathbb{R}_+$, is a member of $\mathcal{CRB}(\mathbb{R}_+)$.

Theorem 5.4. *Assume that the hypotheses (H_1) holds. Then the FDE (2.1) has a solution and solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$.*

Proof. Set $X = BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$. Define an operator Q on X by

$$Qx(t) = \begin{cases} \phi(0)\overline{a}(t) + \overline{a}(t)\displaystyle\int_0^t g(s, x(s), x_s)\, ds, & \text{if } t \in \mathbb{R}_+, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{5.1}$$

We show that Q defines a mapping $Q : X \to X$. Let $x \in X$ be arbitrary. Obviously, Qx is a continuous function on $I_0 \cup \mathbb{R}_+$. We show that Qx is bounded on $I_0 \cup \mathbb{R}_+$. Thus, if $t \in \mathbb{R}_+$, then we obtain:

$$|Qx(t)| \leq |\phi(0)|\,|\overline{a}(t)| + |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds \leq |\phi(0)|\,\|\overline{a}\| + |\overline{a}(t)| \int_0^t h(s)\, ds.$$

Since $\lim_{t\to\infty} |\overline{a}(t)| \displaystyle\int_0^t h(s)\, ds = 0$, and the function $w : \mathbb{R}_+ \to \mathbb{R}$ defined by $w(t) = |\overline{a}(t)| \displaystyle\int_0^t h(s)\, ds$ is continuous, there is a constant $W > 0$ such that

$$\sup_{t\geq 0} w(t) = \sup_{t\geq 0} |\overline{a}(t)| \int_0^t h(s)\, ds \leq W.$$

Therefore,

$$|Qx(t)| \leq |\phi(0)|\|\overline{a}\| + W \leq \|\overline{a}\|\,\|\phi\| + W$$

for all $t \in \mathbb{R}_+$. Similarly, if $t \in I_0$, then $|Qx(t)| \leq \|\phi\|$. As a result, we have that

$$\|Qx\| \leq (\|\overline{a}\| + 1)\|\phi\| + W \tag{5.2}$$

for all $x \in X$ and therefore, Q maps X into X itself. Define a closed ball $\overline{\mathcal{B}}_r(0)$ centered at origin of radius r, where $r = (\|\overline{a}\| + 1)\|\phi\| + W$. Clearly Q defines a mapping $Q : X \to \overline{\mathcal{B}}_r(0)$ and in particular $Q : \overline{\mathcal{B}}_r(0) \to \overline{\mathcal{B}}_r(0)$. We show that Q satisfies all the conditions of Theorem 3.2. First, we show that Q is continuous on $\overline{\mathcal{B}}_r(0)$. To do this, let us fix arbitrarily $\epsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{\mathcal{B}}_r(0)$ converging to a point $x \in \overline{\mathcal{B}}_r(0)$. Then we get:

$$\begin{aligned} |(Qx_n)(t) - (Qx)(t)| &\leq |\overline{a}(t)| \int_0^t |g(s, x_n(s), x_n(\theta + s)) - g(s, x(s), x(\theta + s))| ds \\ &\leq |\overline{a}(t)| \int_0^t [|g(s, x_n(s), x_n(\theta + s))| + |g(s, x(s), x(\theta + s))|] ds \\ &\leq 2|\overline{a}(t)| \int_0^t h(s)\, ds \\ &\leq 2w(t) \end{aligned} \tag{5.3}$$

Hence, by virtue of hypothesis (H_1), we infer that there exists a $T > 0$ such that $w(t) \leq \epsilon$ for $t \geq T$. Thus, for $t \geq T$ from the estimate (5.2) we derive that

$$|(Qx_n)(t) - (Qx)(t)| \leq 2\epsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [0,T]$. Then, following arguments similar to those given in Dhage [7] and Ntouyas [19], by Lebesgue dominated convergence theorem, we obtain the estimate:

$$\begin{aligned} \lim_{n\to\infty} Qx_n(t) &= \lim_{n\to\infty}\left[\phi(0)\overline{a}(t)+\overline{a}(t)\int_0^t g(s,x_n(s),x_n(\theta+s))\,ds\right] \\ &= \phi(0)\overline{a}(t)+\overline{a}(t)\int_0^t\left[\lim_{n\to\infty} g(s,x_n(s),x_n(\theta+s))\right]ds \\ &= Qx(t) \end{aligned} \tag{5.4}$$

for all $t \in [0,T]$. Similarly, if $t \in I_0$, then

$$\lim_{n\to\infty} Qx_n(t) = \phi(t) = Qx(t).$$

Thus, $Qx_n \to Qx$ as $n \to \infty$ uniformly on $\mathbb{R}_+$ and hence Q is a continuous operator on $\overline{\mathcal{B}}_r(0)$ into $\overline{\mathcal{B}}_r(0)$.

Next, we show that B is compact operator on $\overline{\mathcal{B}}_r(0)$. To finish this, it is enough to show that every sequence $\{Qx_n\}$ in $Q(\overline{\mathcal{B}}_r(0))$ has a Cauchy subsequence. Now, by hypotheses (B_2) and (B_3),

$$\begin{aligned} |Qx_n(t)| &\le |\phi(0)||\overline{a}(t)|+|\overline{a}(t)|\int_0^t |g(s,x_n(s),x_n(\theta+s))|\,ds \\ &\le (\|\overline{a}\|+1)|\phi(0)|+w(t) \\ &\le (\|\overline{a}\|+1)\|\phi\|+w(t) \end{aligned} \tag{5.5}$$

for all $t \in \mathbb{R}_+$. Taking supremum over t, we obtain

$$\|Qx_n\| \le (\|\overline{a}\|+1)\|\phi\|+W$$

for all $n \in \mathbb{N}$. This shows that $\{Qx_n\}$ is a uniformly bounded sequence in $Q(\overline{\mathcal{B}}_r(0))$.

Next, we show that $Q(\overline{\mathcal{B}}_r(0))$ is also an equicontinuous set in X. Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T_1 > 0$ such that $|w(t)| < \frac{\epsilon}{8}$ for all $t \ge T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\epsilon > 0$, there is a real number $T_2 > 0$ such that $|\overline{a}(t)| < \frac{\epsilon}{8|\phi(0)|}$ for all $t \ge T_2$. Thus, if $T = \max\{T_1, T_2\}$, then $|w(t)| < \frac{\epsilon}{8}$ and $|\overline{a}(t)| < \frac{\epsilon}{8|\phi(0)|}$ for all $t \ge T$. Let $t, \tau \in I_0 \cup \mathbb{R}_+$ be arbitrary. If $t, \tau \in I_0$, then by uniform continuity of ϕ on I_0, for above ϵ we have a $\delta_1 > 0$ which is a function of only ϵ such that

$$|t-\tau| < \delta_1 \Longrightarrow |Qx_n(t) - Qx_n(\tau)| = |\phi(t) - \phi(\tau)| < \frac{\epsilon}{4}$$

for all $n \in \mathbb{N}$. If $t, \tau \in [0, T]$, then we have

$$\begin{aligned}
|Qx_n(t) - Qx_n(\tau)| &\leq |\phi(0)|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \Big||\overline{a}(t)| \int_0^t g(s, x_n(s), x_n(\theta + s))\,ds - \overline{a}(\tau) \int_0^\tau g(s, x_n(s), x_n(\theta + s))\,ds\Big| \\
&\leq |\phi(0)|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \Big|\overline{a}(t) \int_0^t g(s, x_n(s), x_n(\theta + s))\,ds - \overline{a}(\tau) \int_0^t g(s, x_n(s), x_n(\theta + s))\,ds\Big| \\
&\quad + \Big|\overline{a}(\tau) \int_0^t g(s, x_n(s), x_n(\theta + s))\,ds - \overline{a}(\tau) \int_0^\tau g(s, x_n(s), x_n(\theta + s))\,ds\Big| \\
&\leq |\phi(0)|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)|\Big|\int_0^t g(s, x_n(s), x_n(\theta + s))\,ds\Big| \\
&\quad + |\overline{a}(\tau)|\Big|\int_0^t g(s, x_n(s), x_n(\theta + s))\,ds\Big| \\
&\leq |\phi(0)|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T h(s)\,ds + \|\overline{a}\|\,\Big|\int_\tau^t h(s)\,ds\Big| \\
&\leq |\phi(0)|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T h(s)\,ds + \|\overline{a}\|\,|p(t) - p(\tau)| \\
&\leq \big[|\phi(0)| + \|h\|_{L^1}\big]\,|\overline{a}(t) - \overline{a}(\tau)| + \|\overline{a}\|\,|p(t) - p(\tau)|
\end{aligned}$$

where, $p(t) = \displaystyle\int_0^t h(s)\,ds$ and $\|h\|_{L^1} = \displaystyle\int_0^\infty h(s)\,ds$.

By the uniform continuity of the functions $\overline{a}$ and p on $[0, T]$, for above ϵ we have the real numbers $\delta_2 > 0$ and $\delta_3 > 0$ which are the functions of only ϵ such that

$$|t - \tau| < \delta_2 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\epsilon}{8\big[|\phi(0)| + \|h\|_{L^1}\big]}$$

and

$$|t - \tau| < \delta_3 \Longrightarrow |p(t) - p(\tau)| < \frac{\epsilon}{8\|\overline{a}\|}.$$

Let $\delta_4 = \min\{\delta_2, \delta_3\}$. Then

$$|t - \tau| < \delta_4 \Longrightarrow |Qx_n(t) - Qx_n(\tau)| < \frac{\epsilon}{4}$$

for all $n \in \mathbb{N}$. Similarly, if $t \in I_0$ and $\tau \in [0, T]$, then

$$|Qx_n(t) - Qx_n(\tau)| \leq |Qx_n(t) - Qx_n(0)| + |Qx_n(0) - Qx_n(\tau)|.$$

Take $\delta_5 = \min\{\delta_1, \delta_4\} > 0$ which is again a function of only ϵ. Hence by above estimated facts it follows that

$$|t - \tau| < \delta_5 \Longrightarrow |Qx_n(t) - Qx_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$.

Again, if $t, \tau > T$, then we have a real number $\delta_6 > 0$ which is a function of only ϵ such that

$$\begin{aligned}|Qx_n(t) - Qx_n(\tau)| &\leq |\phi(0)|\, |a(t) - a(\tau)| \\ &\quad + \left|\overline{a}(t) \int_0^t g(s, x_n(s), x_n(\theta + s))\, ds - \overline{a}(t) \int_0^\tau g(s, x_n(s), x_n(\theta + s))\, ds\right| \\ &\leq |\phi(0)|\, |a(t)| + |\phi(0)|\, |a(\tau)| + w(t) + w(\tau) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4}\end{aligned}$$

for all $n \in \mathbb{N}$, whenever $t - \tau| < \delta_6$. Similarly, if $t, \tau \in I_0 \cup \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|Qx_n(t) - Qx_n(\tau)| \leq |Qx_n(t) - Qx_n(T)| + |Qx_n(T) - Qx_n(\tau)|.$$

Take $\delta = \min\{\delta_5, \delta_6\} > 0$ which is again a function of only ϵ. Therefore, from the above obtained estimates, it follows that

$$|Qx_n(t) - Qx_n(T)| < \frac{\epsilon}{2} \quad \text{and} \quad |Qx_n(T) - Qx_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$, whenever $t - \tau| < \delta$. As a result, $|Qx_n(t) - Qx_n(\tau)| < \epsilon$ for all $t, \tau \in I_0 \cup \mathbb{R}_+$ and for all $n \in \mathbb{N}$, whenever $t - \tau| < \delta$. This shows that $\{Qx_n\}$ is a equicontinuous sequence in X. Now an application of Arzela-Ascoli theorem yields that $\{Qx_n\}$ has a uniformly convergent subsequence on the compact subset $I_0 \cup [0, T]$ of $I_0 \cup \mathbb{R}$. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{Qx_n\}$ is Cauchy in X. Now $|Qx_n(t) - Qx(t)| \to 0$ as $n \to \infty$ for all $t \in I_0 \cup [0, T]$. Then for given $\epsilon > 0$ there exits an $n_0 \in \mathbb{N}$ such that

$$\sup_{-\delta \leq p \leq T} |\overline{a}(p)| \int_0^p |g(s, x_n(s), x_n(\theta + s)) - g(s, x_n(s), x_n(\theta + s))|\, ds < \frac{\epsilon}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$\begin{aligned}\|Qx_m - Qx_n\| &= \sup_{-\delta \leq t < \infty} \left| |\overline{a}(t)| \int_0^t |g(s, x_n(s), x_n(\theta + s)) - g(s, x_n(s), x_n(\theta + s))|\, ds \right| \\ &\leq \sup_{-\delta \leq p \leq T} \left| |\overline{a}(p)| \int_0^p |g(s, x_n(s), x_n(\theta + s)) - g(s, x_n(s), x_n(\theta + s))|\, ds \right| \\ &\quad + \sup_{p \geq T} |\overline{a}(p)| \int_0^p \big[|g(s, x_n(s), x_n(\theta + s))| + |g(s, x_n(s), x_n(\theta + s))|\big]\, ds \\ &< \epsilon.\end{aligned}$$

This shows that $\{Qx_n\} \subset Q(\overline{\mathcal{B}}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{Qx_n\}$ converges to a point in X. As $Q(\overline{\mathcal{B}}_r(0))$ is closed $\{Qx_n\}$ converges to a point in $Q(\overline{\mathcal{B}}_r(0))$. Hence $Q(\overline{\mathcal{B}}_r(0))$ is relatively compact and consequently Q is a continuous and compact operator on $\overline{\mathcal{B}}_r(0)$ into itself. Now an application of Theorem 3.2 to the operator Q on $\overline{\mathcal{B}}_r(0)$ yields that Q has a fixed point in $\overline{\mathcal{B}}_r(0)$ which further implies that the FDE (2.1) has a solution defined on $I_0 \cup \mathbb{R}_+$.

Finally, we show that the solutions are uniformly attractive on $I_0 \cup \mathbb{R}_+$. Let $x, y \in \overline{\mathcal{B}}_r(0)$ be any two solutions the FDE (4.1) defined on $I_0 \cup \mathbb{R}_+$. Then,

$$
\begin{aligned}
|x(t) - y(t)| &\leq \left| \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds - \overline{a}(t) \int_0^t g(s, y(s), y_s)\, ds \right| \\
&\leq |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds + |\overline{a}(t)| \int_0^t |g(s, y(s), y_s)|\, ds \\
&\leq 2w(t) \qquad (5.6)
\end{aligned}
$$

for all $t \in I_0 \cup \mathbb{R}_+$. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T > 0$ such that $w(t) < \frac{\epsilon}{2}$ for all $t \geq T$. Therefore, $|x(t) - y(t)| \leq \epsilon$ for all $t \geq T$, and so all the solutions of the FDE (2.1) are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. □

Theorem 5.5. *Assume that the hypotheses* (H_1)-(H_2) *hold. Then the FDE (2.1) has a solution and solutions are uniformly globally attractive and ultimately positive on* $I_0 \cup \mathbb{R}_+$.

Proof. By Theorem 5.4, the FDE (2.1) has a solution in $\overline{\mathcal{B}}_r(0)$, where $r = \|\phi\| + W$ and the solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. We know that for any $x, y \in \mathbb{R}$, one has the inequality,

$$|x| + |y| \geq |x + y| \geq x + y,$$

and, therefore,

$$\big||x + y| - (x + y)\big| \leq \big|\, |x| + |y| - (x + y)\big| \leq \big||x| - x\big| + \big||y| - y\big| \qquad (5.7)$$

for all $x, y \in \mathbb{R}$. Now for any solution $x \in \overline{\mathcal{B}}_r(0)$, one has

$$
\begin{aligned}
\big||x(t)| - x(t)\big| &= \Bigg| \left| \phi(0)\overline{a}(t) + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds \right| \\
&\qquad - \Big(\phi(0)\overline{a}(t) + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds \Big) \Bigg| \\
&\leq \big||\phi(0)| - \phi(0)\big|\, |\overline{a}(t)| + |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds \\
&\qquad + |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds \\
&\leq 2w(t).
\end{aligned}
$$

Since $\lim\limits_{t\to\infty} w(t) = 0$, there is a real number $T > 0$ such that $|\,|x(t)| - x(t)| \leq \epsilon$ for all $t \geq T$. Hence solutions of the FDE (2.1) are also uniformly globally ultimately positive on $I_0 \cup \mathbb{R}_+$. This completes the proof. □

Example 5.6. Let $I_0 = [-\pi/2, 0]$ be a closed and bounded interval in $\mathbb{R}$ and define a function $\phi : I_0 \to \mathbb{R}$ by $\phi(t) = \cos t$. Consider the following FDE,

$$\left.\begin{aligned} \big(e^t x(t)\big)' &= e^{-t}\frac{(x(t)+x_t)}{|x(t)|+\|x_t\|_{\mathcal{C}}} \ \text{ a.e. } t \in \mathbb{R}_+ \\ x_0 &= \phi, \end{aligned}\right\} \tag{5.8}$$

where, $e^{-t} \in C(\mathbb{R}_+, \mathbb{R}) \subset L^1(\mathbb{R}_+, \mathbb{R})$ and $\lim_{t\to\infty} e^{-t}\int_0^t e^{-s}\,ds = 0$.

Here, $a(t) = e^t$ which is positive and increasing on $\mathbb{R}_+$ and so $a \in \mathcal{CRB}(\mathbb{R}_+)$ and

$$\|\overline{a}\| = \sup_{t\geq 0}\overline{a}(t) = \sup_{t\geq 0} e^{-t} \leq 1.$$

Again, $g(t, x, y) = \dfrac{e^{-t}(x+y)}{|x|+\|y\|_{\mathcal{C}}}$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y \in \mathcal{C}$. Clearly, the function g satisfies the hypothesis (H$_1$) with growth function $h(t) = e^{-t}$ on $\mathbb{R}_+$ so that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-t}\int_0^t e^{-s}\,ds = 0$. Now we apply Theorem 5.4 to FDE (2.1) to conclude that it has a solution and solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. As $\phi(0) = 1 \geq 0$, the hypothesis (H$_2$) of Theorem 5.5 is satisfied. Hence, solutions of the FDE (5.6) are also uniformly globally ultimately positive on $I_0 \cup \mathbb{R}_+$.

5.2. **Perturbed Functional Differential Equations.** Next, we establish the attractivity and positivity results for the FDE (2.2) on unbounded interval $I_0 \cup \mathbb{R}_+$. The following definition is useful in the sequel.

Definition 5.7. By a *solution* for the functional differential equation (2.2) we mean a function $x \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R}) \cap AC(\mathbb{R}_+, \mathbb{R})$ such that

(i) the function $t \mapsto [a(t)x(t) - f(t, x(t))]$ is absolutely continuous on $\mathbb{R}_+$, and
(ii) x satisfies the equations in (2.2),

where $AC(\mathbb{R}_+, \mathbb{R})$ is the space of absolutely continuous real-valued functions on right half real axis $\mathbb{R}_+$.

We need the following hypotheses in the sequel.

(H$_3$) The function $t \to f(t, 0, 0)$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup\{|f(t, 0, 0)| : t \in \mathbb{R}_+\}$.
(H$_4$) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell \in BC(\mathbb{R}_+, \mathbb{R})$ and a real number $K > 0$ such that

$$|f(t, x) - f(t, y)| \leq \ell(t)\frac{|x-y|}{K+|x-y|}$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Moreover, we assume $\sup_{t\geq 0}\ell(t) = L$.
(H$_5$) $\lim_{t\to\infty}\big[\,|f(t, x)| - f(t, x)\big] = 0$ for all $x \in \mathbb{R}$.
(H$_6$) $f(0, \phi(0)) \geq 0$.

Remark 5.8. Hypothesis (H_4) is more general than that existing in the literature. Indeed, if $L < K$, then it reduces to the usual Lipschitz condition of the function f on $\mathbb{R}_+ \times \mathbb{R}$.

Theorem 5.9. *Assume that the hypotheses (H_1), (H_3) and (H_4) hold. Further if $L\|\overline{a}\| \leq K$, then the FDE (2.2) has a solution and solutions are uniformly globally attractive defined on $I_0 \cup \mathbb{R}_+$.*

Proof. Now the FDE (2.2) is equivalent to the functional integral equation,

$$x(t) = \begin{cases} [\phi(0) - f(0, \phi(0))]\overline{a}(t) + \overline{a}(t) f(t, x(t)) \\ \qquad +\overline{a}(t) \displaystyle\int_0^t g(s, x(s), x_s)\, ds, \quad \text{if } t \in \mathbb{R}_+ \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{5.9}$$

Set $X = BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ and define the closed ball $\overline{\mathcal{B}}_r(0)$ in X, where the real number r is defined by $r = \|\overline{a}\| \left[\|\phi\| + |f(0, \phi(0))| + L + F_0 + W\right]$. Define the operators A on X and B on $\overline{\mathcal{B}}_r(0)$ by

$$Ax(t) = \begin{cases} -f(0, \phi(0))\overline{a}(t) + \overline{a}(t) f(t, x(t)), & \text{if } t \in \mathbb{R}_+, \\ 0, \quad \text{if } t \in I_0. \end{cases} \tag{5.10}$$

for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, and

$$Bx(t) = \begin{cases} \phi(0)\overline{a}(t) + \overline{a}(t) \displaystyle\int_0^t g(s, x(s), x_s)\, ds, & \text{if } t \in \mathbb{R}_+, \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{5.11}$$

Then the FIE (5.9) is transformed into the operator equation as

$$Ax(t) + Bx(t) = x(t), \;\; t \in I_0 \cup \mathbb{R}_+. \tag{5.12}$$

We show that A and B satisfy all the conditions of Theorem 3.3 on $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$. First we we show that the operators A and B define the mappings $A : X \to X$ and $B : \overline{\mathcal{B}}_r(0) \to X$. Let $x \in X$ be arbitrary. Obviously, Ax is a continuous function on $I_0 \cup \mathbb{R}_+$. We show that Ax is bounded on $I_0 \cup \mathbb{R}_+$. Thus, if $t \in \mathbb{R}_+$, then we obtain:

$$\begin{aligned} |Ax(t)| &= |f(0, \phi(0))|\|\overline{a}\| + \|\overline{a}\||f(t, x(t))| \\ &\leq \|\overline{a}\|[|f(0, \phi(0))| + |f(t, x(t)) - f(t, 0)| + |f(t, 0)|] \\ &\leq \|\overline{a}\| \left[f(0, \phi(0))| + \ell(t) \frac{|x(t)|}{K + |x(t)|} + F_0 \right] \\ &\leq \|\overline{a}\| \left[|f(0, \phi(0))| + L + F_0\right]. \end{aligned}$$

Taking supremum over t, $\|Ax\| \leq \|\overline{a}\| \left[|f(0, \phi(0))| + L + F_0\right]$. Thus Ax is continuous and bounded on $I_0 \cup \mathbb{R}_+$. As a result $Ax \in X$. Similarly, as in the proof of Theorem 5.4 above , it can be shown that $B : X \to X$ and in particular, $B : \overline{\mathcal{B}}_r(0) \to X$ for all $x \in X$. We show that A is a contraction on X. Let $x, y \in X$ be arbitrary. Then by hypothesis (H_3),

$$\begin{aligned}\|Ax - Ay\| &= \sup_{t \in I_0 \cup \mathbb{R}_+} |Ax(t) - Ay(t)| \\ &\leq \max\left\{\sup_{t \in I_0} |Ax(t) - Ay(t)|, \sup_{t \in \mathbb{R}_+} |Ax(t) - Ay(t)|\right\} \\ &\leq \max\left\{0\,,\ \sup_{t \in \mathbb{R}_+} \overline{a}(t) \frac{\ell(t)|x(t) - y(t)|}{K + |x(t) - y(t)|}\right\} \\ &\leq \frac{L\|\overline{a}\|\|x - y\|}{K + \|x - y\|}\end{aligned}$$

for all $x, y \in X$. This shows that A is a nonlinear $\mathcal{D}$-contraction on X with the $\mathcal{D}$-function ψ defined by $\psi(r) = \dfrac{L\|\overline{a}\|r}{K + r}$. Next, it can be shown as in the proof of Theorem 5.4 that B is a compact and continuous operator on X and in particular on $\overline{\mathcal{B}}_r(0)$.

Next, let $x, y \in X$ be arbitrary. Then,

$$\begin{aligned}|x(t)| &\leq |Ax(t)| + |By(t)| \\ &\leq |\overline{a}(t)||\phi(0)| + |f(0, \phi(0))||\overline{a}(t)| + |\overline{a}(t)||f(t, x(t))| + \overline{a}(t) \int_0^t |g(s, y(s), y_s)|\, ds \\ &\leq \|\overline{a}\| \big[|\phi(0)| + |f(0, \phi(0))| + |f(t, x(t)) - f(t, 0)| + |f(t, 0)|\big] \\ &\qquad + |\overline{a}(t)(t)| \int_0^t h(s)\, ds \\ &\leq \|\overline{a}\| \big[|\phi(0)| + |f(0, \phi(0))|\big] + \|\overline{a}\| \frac{\ell(t)|x(t)|}{K + |x(t)|} + F_0\|\overline{a}\| + w(t) \\ &\leq \|\overline{a}\|\, \big[\|\phi\| + |f(0, \phi(0))| + L + F_0\big] + W \\ &= r\end{aligned}$$

for all $t \in I_0 \cup \mathbb{R}_+$. This shows that $x \in \overline{\mathcal{B}}_r(0)$ and hypothesis (c) of Theorem 3.3 is satisfied. Hence an application of it yields that the operator equation $Ax + Bx = x$ has a global solution in $\overline{\mathcal{B}}_r(0)$.

Finally, let $x, y \in \overline{\mathcal{B}}_r(0)$ be any two solutions of the FDE (2.2) on $I_0 \cup \mathbb{R}_+$. Then,

$$\begin{aligned}|x(t) - y(t)| &\leq |\overline{a}(t)||f(t, x(t)) - f(t, y(t))| \\ &\qquad + \left|\overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds - \overline{a}(t) \int_0^t g(s, y(s), y_s)\, ds\right| \\ &\leq |\overline{a}(t)| \frac{\ell(t)|x(t) - y(t)|}{K + |x(t) - y(t)|} + 2|\overline{a}(t)| \int_0^t h(s)\, ds \\ &\leq \frac{L\|\overline{a}\|\, |x(t) - y(t)|}{K + |x(t) - y(t)|} + 2w(t)\end{aligned}$$

Taking the limit superior as $t \to \infty$ in the above inequality yields,

$$\lim_{t\to\infty} |x(t) - y(t)| = 0.$$

Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the solutions of FDE (2.2) are globally uniformly attractive defined on $I_0 \cup \mathbb{R}_+$. This completes the proof. $\square$

Theorem 5.10. *Assume that the hypotheses (H$_1$)-(H$_6$) hold. Then the FDE (2.2) has a solution and solutions are uniformly globally attractive and ultimately positive on* $I_0 \cup \mathbb{R}_+$.

Proof. By Theorem 5.9, the FDE (2.2) has a solution in $\overline{\mathcal{B}}_r(0)$, where $r = \|\overline{a}\| \, [\|\phi\| + |f(0, \phi(0))| + L + F_0] + W$ and solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. Now for any solution $x \in \overline{\mathcal{B}}_r(0)$, by inequality (5.7), one has

$$\begin{aligned}
\big||x(t)| - x(t)\big| &= \bigg|\Big|\phi(0)\overline{a}(t) + f(0, \phi(0))\overline{a}(t) + \overline{a}(t) f(t, x(t)) \\
&\qquad\qquad + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds\Big| \\
&\qquad - \Big(\phi(0)\overline{a}(t) + f(0, \phi(0))\overline{a}(t) + \overline{a}(t) f(t, x(t)) \\
&\qquad\qquad + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds\Big)\bigg| \\
&\leq \big||\phi(0)| - \phi(0)\big| |\overline{a}(t)| + \big||f(0, \phi(0))| - f(0, \phi(0))\big| \overline{a}(t) \\
&\qquad + |\overline{a}(t)| \big||f(t, x(t))| - f(t, x(t))\big| \\
&\qquad + |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds + |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds \\
&\leq |\overline{a}(t)| \big||f(t, x(t))| - f(t, x(t))\big| + 2|\overline{a}(t)| \int_0^t h(s)\, ds \\
&\leq \|\overline{a}\| \big||f(t, x(t))| - f(t, x(t))\big| + 2w(t).
\end{aligned}$$

Since (H$_3$) holds, taking limit superior as $t \to \infty$ on both sides of above inequality yields that $\lim_{t\to\infty} \big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $\big|\,|x(t)| - x(t)\big| \leq \epsilon$ for all $t \geq T$. Hence solutions of the FDE (2.2) are also uniformly globally ultimately positive defined on $I_0 \cup \mathbb{R}_+$. This completes the proof. $\square$

Example 5.11. Let $I_0 = [-\pi/2, 0]$ and define a function $\phi : I_0 \to \mathbb{R}$ by $\phi(t) = \cos t$. Consider the FDE

$$\left.\begin{aligned}
\frac{d}{dt}\left[(t+1)x(t) - \frac{|x(t)|}{1 + |x(t)|}\right] &= e^{-t} \frac{x(t) + x_t}{|x(t)| + \|x_t\|_{\mathcal{C}}} \quad \text{a.e. } t \in \mathbb{R}_+ \\
x_0 &= \phi.
\end{aligned}\right\} \tag{5.13}$$

Here, $a(t) = t + 1$ which is positive and increasing on $\mathbb{R}_+$ and so $a \in \mathcal{CRB}(\mathbb{R}_+)$ and $\|\overline{a}\| = \sup_{t\geq 0} \overline{a}(t) = \sup_{t\geq 0} \dfrac{1}{t+1} \leq 1$. Again,

$$f(t,x) = \frac{|x|}{1+|x|} \quad \text{and} \quad g(t,x,y) = e^{-t}\frac{x+y}{|x| + \|y\|_{\mathcal{C}}}$$

for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y \in \mathcal{C}$. First, we show that the function f satisfies hypothesis (H_3) on $\mathbb{R}_+ \times \mathbb{R}$. Let $(t,x), (t,y) \in \mathbb{R}_+ \times \mathbb{R}$ be arbitrary. Then,

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leq \left| \frac{|x|}{1+|x|} - \frac{|y|}{1+|y|} \right| \\
&= \frac{\big|\,|x| - |y|\,\big|}{1 + \big|\,|x| - |y|\,\big|} \\
&\leq \frac{|x-y|}{1+|x-y|}.
\end{aligned}$$

Therefore, here $\ell(t) = 1$ for all $t \in \mathbb{R}_+$ and $\|\overline{a}\| = L = 1 = K$ and hence $L\|\overline{a}\| \leq K$. Further g is Carathéodory and $|g(t,x,y)| \leq e^{-t}$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Clearly, $\lim_{t\to\infty} \dfrac{1}{(t+1)} \displaystyle\int_0^t e^{-s}\,ds = 0$. Now we apply Theorem 5.9 to the FDE (2.2) and conclude that it has a solution and solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. Moreover, hypotheses (H_2), (H_5) and (H_6) are also satisfied by the functions ϕ and f. Hence by Theorem 5.10, the solutions of the FDE (2.2) are also uniformly globally ultimately positive defined on $I_0 \cup \mathbb{R}_+$.

5.3. **Quadratic Functional Differential Equations.** Now, finally we discuss the attractivity results for the quadratic perturbations of the first order ordinary differential equation (2.3) on $I_0 \cup \mathbb{R}_+$.

Definition 5.12. By a *solution* for the functional differential equation (2.3) we mean a function $x \in BC(I_0 \cup \mathbb{R}_+, \mathbb{R}) \cap AC(\mathbb{R}_+, \mathbb{R})$ such that

(i) the function $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t))}$ is absolutely continuous on $\mathbb{R}_+$, and
(ii) x satisfies the equations in (2.3) on $I_0 \cup \mathbb{R}_+$,

where $AC(\mathbb{R}_+, \mathbb{R})$ is the space of absolutely continuous real-valued functions on right half real axis $\mathbb{R}_+$.

We need the following hypothesis in the sequel.

(H_7) $f(0, \phi(0)) = 1$.
(H_8) The function $x \mapsto \dfrac{x}{f(0,x)}$ is injective in $\mathbb{R}$.

Theorem 5.13. *Assume that the hypotheses* (H_1), (H_3), (H_4), (H_7) *and* (H_8) *hold. Further, assume that*

$$L \max\big\{\|\phi\|, |\phi(0)|\,\|\overline{a}\| + W\big\} \leq K. \tag{5.14}$$

Then the FDE (2.3) has a solution and solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$.

Proof. Now, using hypotheses (H_7) and (H_8) it can be shown that the FDE (2.3) is equivalent to the functional integral equation

$$x(t) = \begin{cases} \big[f(t, x(t))\big] \left(\phi(0)\overline{a}(t) + \overline{a}(t) \displaystyle\int_0^t g(s, x(s), x_s)\, ds \right), & \text{if } t \in \mathbb{R}_+ \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{5.15}$$

Set $X = BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$ and define a closed ball $\overline{\mathcal{B}}_r(0)$ in X centered at origin of radius r given by

$$r = \max\{1, L + F_0\} \max\big\{\|\phi\|, |\phi(0)|\,\|\overline{a}\| + W\big\}.$$

Define the operators A on X and B on $\overline{\mathcal{B}}_r(0)$ by

$$Ax(t) = \begin{cases} f(t, x(t)), & \text{if } t \in \mathbb{R}_+ \\ 1, \quad \text{if } t \in I_0. \end{cases} \tag{5.16}$$

and

$$Bx(t) = \begin{cases} \phi(0)\overline{a}(t) + \overline{a}(t) \displaystyle\int_0^t g(s, x(s), x_s)\, ds, & \text{if } t \in \mathbb{R}_+ \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{5.17}$$

Then the FIE (5.15) is transformed into the operator equation as

$$Ax(t)\, Bx(t) = x(t),\ t \in I_0 \cup \mathbb{R}_+. \tag{5.18}$$

We show that A and B satisfy all the conditions of Theorem 3.4 on $BC(I_0 \cup \mathbb{R}_+, \mathbb{R})$. First we we show that the operators A and B define the mappings $A : X \to X$ and $B : \overline{\mathcal{B}}_r(0) \to X$. Let $x \in X$ be arbitrary. Obviously, Ax is a continuous function on $I_0 \cup \mathbb{R}_+$. We show that Ax is bounded on $I_0 \cup \mathbb{R}_+$. Thus, if $t \in \mathbb{R}_+$, then we obtain:

$$\begin{aligned} |Ax(t)| = |f(t, x(t))| &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq \ell(t) \frac{|x(t)|}{K + |x(t)|} + F_0 \leq L + F_0. \end{aligned}$$

Similarly, $|Ax(t)| \leq 1$ for all $t \in I_0$. Therefore, taking the supremum over t,

$$\|Ax\| \leq \max\{1, L + F_0\} = N.$$

Thus Ax is continuous and bounded on $I_0 \cup \mathbb{R}_+$. As a result $Ax \in X$. Similarly, as in the proof of Theorem 5.4 above , it can be shown that $Bx \in X$ and in particular,

$A : X \to X$ and $B : \overline{\mathcal{B}}_r(0) \to X$. We show that A is a Lipschitz on X. Let $x, y \in X$ be arbitrary. Then, by hypothesis (H_3),

$$\begin{aligned}
\|Ax - Ay\| &= \sup_{t \in I_0 \cup \mathbb{R}_+} |Ax(t) - Ay(t)| \\
&\leq \max\left\{ \sup_{t \in I_0} |Ax(t) - Ay(t)|, \sup_{t \in \mathbb{R}_+} |Ax(t) - Ay(t)| \right\} \\
&\leq \max\left\{ 0\,,\ \sup_{t \in \mathbb{R}_+} \ell(t) \frac{|x(t) - y(t)|}{K + |x(t) - y(t)|} \right\} \\
&\leq \frac{L\|x - y\|}{K + \|x - y\|}
\end{aligned}$$

for all $x, y \in X$. This shows that A is a $\mathcal{D}$-Lipschitz on X with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{K+r}$. Next, it can be shown as in the proof of Theorem 5.4 that B is a compact and continuous operator on X and in particular on $\overline{\mathcal{B}}_r(0)$.

Next, we estimate the value of the constant M. By definition of M, one has

$$\begin{aligned}
\|B(\overline{\mathcal{B}}_r(0))\| &= \sup\{\|Bx\| \ :\ x \in \overline{\mathcal{B}}_r(0)\} \\
&= \sup\left\{ \sup_{t \in I_0 \cup \mathbb{R}_+} |Bx(t)| \ :\ x \in \overline{\mathcal{B}}_r(0) \right\} \\
&\leq \sup\left\{ \max\left\{ \sup_{t \in I_0} |Bx(t)|, \sup_{t \in \mathbb{R}_+} |Bx(t)| \right\} \ :\ x \in \overline{\mathcal{B}}_r(0) \right\} \\
&\leq \sup_{x \in \overline{\mathcal{B}}_r(0)} \left\{ \max\left\{ \|\phi\|\,,\ |\phi(0)||\overline{a}(t)| \right.\right. \\
&\qquad\qquad \left.\left. + \sup_{t \in \mathbb{R}_+} |\overline{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds \right\}\right\} \\
&\leq \max\big\{ \|\phi\|, |\phi(0)|\, \|\overline{a}\| + W \big\}.
\end{aligned}$$

Thus,

$$\|Bx\| \leq \max\big\{ \|\phi\|, |\phi(0)|\, \|\overline{a}\| + W \big\} = M$$

for all $x \in \overline{\mathcal{B}}_r(0)$. Next, let $x, y \in X$ be arbitrary. Then,

$$\begin{aligned}
|x(t)| &\leq |Ax(t)|\, |By(t)| \\
&\leq \|Ax\|\, \|By\| \\
&\leq \|A(X)\|\, \|B(\overline{\mathcal{B}}_r(0))\| \\
&\leq \max\{1, L + F_0\} M \\
&\leq \max\{1, L + F_0\} \max\big\{ \|\phi\|, |\phi(0)|\, \|\overline{a}\| + W \big\} \\
&= r
\end{aligned}$$

for all $t \in I_0 \cup \mathbb{R}_+$. Therefore, we have:

$$\|x\| \leq \max\{1, L + F_0\} \max\big\{ \|\phi\|, |\phi(0)|\, \|\overline{a}\| + W \big\} = r.$$

This shows that $x \in \overline{\mathcal{B}}_r(0)$ and hypothesis (c) of Theorem 3.4 is satisfied. Again,

$$M\phi(r) \leq \frac{L \max\big\{\|\phi\|, |\phi(0)|\, \|\overline{a}\| + W\big\} r}{K + r} < r$$

for $r > 0$, because

$$L \max\big\{\|\phi\|, |\phi(0)|\, \|\overline{a}\| + W\big\} \leq K.$$

Therefore, hypothesis (d) of Theorem 3.4 is satisfied. Now we apply Theorem 3.4 to the operator equation $Ax\,Bx = x$ to yield that the FDE (2.3) has a solution on $I_0 \cup \mathbb{R}_+$. Moreover, the solutions of the FDE (2.3) are in $\overline{\mathcal{B}}_r(0)$. Hence, solutions are global in nature.

Finally, let $x, y \in \overline{\mathcal{B}}_r(0)$ be any two solutions of the FDE (2.3) on $I_0 \cup \mathbb{R}_+$. Then

$$\begin{aligned}
|x(t) - y(t)| &\leq \bigg| [f(t, x(t))] \left(\phi(0)\overline{a}(t) + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds \right) \\
&\qquad - [f(t, y(t))] \left(\phi(0)\overline{a}(t) + \overline{a}(t) \int_0^t g(s, y(s), y_s)\, ds \right) \bigg| \\
&\leq \bigg| \big[f(t, x(t)) - f(t, y(t)) \big] \left(\phi(0)\overline{a}(t) + \overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds \right) \bigg| \\
&\qquad + \bigg| f(t, y(t)) \Big(\overline{a}(t) \int_0^t g(s, x(s), x_s)\, ds - g(s, y(s), y_s)\, ds \Big) \bigg| \\
&\leq |f(t, x(t)) - f(t, y(t))| \left(|\phi(0)||\overline{a}(t)| + |\overline{a}(t)| \int_0^t h(s)\, ds \right) \\
&\qquad + 2\big[|f(t, x(t)) - f(t, 0)| + |f(t, 0)| \big] w(t) \\
&\leq \ell(t) \frac{|x(t) - y(t)|}{K + |x(t) - y(t)|} \left(|\phi(0)|\, \|\overline{a}\| + W \right) \\
&\qquad + 2 \left[\frac{\ell(t)|y(t)|}{K + |y(t)|} + F_0 \right] w(t) \\
&\leq \frac{L \left(|\phi(0)|\, \|\overline{a}\| + W \right) |x(t) - y(t)|}{K + |x(t) - y(t)|} + 2(L + F_0) w(t)
\end{aligned} \tag{5.19}$$

Taking the limit superior as $t \to \infty$ in the above inequality yields,

$$\lim_{t \to \infty} |x(t) - y(t)| = 0.$$

Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the solutions of FDE (2.3) are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. This completes the proof. □

Theorem 5.14. *Assume that the hypotheses (H_1)-(H_6) hold. Then the FDE (2.1) has a solution and solutions are uniformly globally attractive and ultimately positive defined on $I_0 \cup \mathbb{R}_+$.*

Proof. By Theorem 5.13, the FDE (2.3) has a global solution in the closed ball $\overline{\mathcal{B}}_r(0)$, where the radius r is given as in the proof of Theorem 5.13, and the solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. We know that for any $x, y \in \mathbb{R}$, one has the inequality,

$$|x|\,|y| = |xy| \geq xy,$$

and therefore,

$$\big||xy| - (xy)\big| \leq |x|\big||y| - y\big| + \big||x| - x\big|\,|y| \tag{5.20}$$

for all $x, y \in \mathbb{R}$. Now for any solution $x \in \overline{\mathcal{B}}_r(0)$, one has

$$\begin{aligned}
\big||x(t)| - x(t)\big| &= \Bigg|\,|\big[f(t,x(t))\big]\left(\phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t g(s,x(s),x_s)\,ds\right)| \\
&\qquad -\left(\big[f(t,x(t))\big]\left(\phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t g(s,x(s),x_s)\,ds\right)\right)\Bigg| \\
&\leq |\big[f(t,x(t))\big]|\,(|\phi(0)| - \phi(0)|)\,|\overline{a}(t)| \\
&\quad + |\big[f(t,x(t))\big]\,\left||\overline{a}(t)\int_0^t g(s,x(s),x_s)\,ds| - \overline{a}(t)\int_0^t g(s,x(s),x_s)\,ds\right| \\
&\quad + |\big|f(t,x(t))\big| - f(t,x(t))|\left|\phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t g(s,x(s),x_s)\,ds\right| \\
&\leq |\big|f(t,x(t))\big| - f(t,x(t))|\,(|\phi(0)|\,\|\overline{a}\| + W) + 2[L + F_0]w(t).
\end{aligned} \tag{5.21}$$

Taking the limit superior as $t \to \infty$ in the above inequality (5.21), we obtain $\lim\limits_{t\to\infty} \big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $|\,|x(t)| - x(t)| \leq \epsilon$ for all $t \geq T$. Hence, solutions of the FDE (2.3) are uniformly globally attractive as well as ultimately positive defined on $I_0 \cup \mathbb{R}_+$. This completes the proof. □

Example 5.15. Let $I_0 = [-\pi/2, 0]$ be a closed and bounded interval in $\mathbb{R}$ and define a function $\phi : I_0 \to \mathbb{R}$ by $\phi(t) = \cos t$. Consider the quadratic FDE,

$$\left.\begin{aligned}
\frac{d}{dt}\left[\frac{e^t x(t)}{1 + \frac{(\pi+4)t}{2(\pi+6)(t+1)}\tan^{-1}(|x(t)|)}\right] &= e^{-t}\frac{x(t) + x_t}{|x(t)| + \|x_t\|_{\mathcal{C}}} \quad \text{a.e. } t \in \mathbb{R}_+ \\
x_0 &= \phi.
\end{aligned}\right\} \tag{5.22}$$

Here, $a(t) = e^t$ for $t \in \mathbb{R}_+$. As in example 5.1, $a \in \mathcal{CRB}(\mathbb{R}_+)$ and $\|\overline{a}\| \leq 1$. Again, here, we have:

$$f(t,x) = 1 + \frac{(\pi+4)t}{2(\pi+6)(t+1)}\tan^{-1}(|x|) \quad \text{and} \quad g(t,x,y) = e^{-t}\frac{x + y}{|x| + \|y\|_{\mathcal{C}}}$$

for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y \in \mathcal{C}$. First, we show that the function f satisfies hypothesis (H_3) on $\mathbb{R}_+ \times \mathbb{R}$. Let $(t,x), (t,y) \in \mathbb{R}_+ \times \mathbb{R}$ be arbitrary. Then,

$$\begin{aligned} |f(t,x) - f(t,y)| &\leq \left| \frac{(\pi+4)t}{2(\pi+6)(t+1)} \tan^{-1}(|x|) - \frac{(\pi+4)t}{2(\pi+6)(t+1)} \tan^{-1}(|y|) \right| \\ &\leq \frac{(\pi+4)t}{2(\pi+6)(t+1)} \cdot \frac{\big|\,|x| - |y|\,\big|}{1 + \big|\,|x| - |y|\,\big|} \\ &\leq \frac{(\pi+4)t}{2(\pi+6)(t+1)} \cdot \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Therefore, here $\ell(t) = \dfrac{(\pi+4)t}{2(\pi+6)(t+1)}$ for all $t \in \mathbb{R}_+$ so that $L = \dfrac{1}{2}$. Furthermore, the function g is Carathéodory and $|g(t,x,y)| \leq e^{-t}$ for $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C}$. Clearly, $\displaystyle\lim_{t\to\infty} e^{-t} \int_0^t e^{-s}\, ds = 0$. Finally,

$$L \max\left\{ \|\phi\|, |\phi(0)|\,\|\overline{a}\| + W \right\} \leq 1 = K.$$

Now, we apply Theorem 5.13 to the FDE (5.19) and conclude that it has a solution on $I_0 \cup \mathbb{R}_+$. Moreover, the solutions are uniformly globally attractive on $I_0 \cup \mathbb{R}_+$. Further,

$$|f(t,x)| = 1 + \frac{(\pi+4)t}{2(\pi+6)(t+1)} \tan^{-1}(|x|) = f(t,x)$$

for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ and hence solutions of the quadratic FDE (5.19) are also uniformly globally ultimately positive on $I_0 \cup \mathbb{R}_+$.

6. **THE CONCLUSION**

From foregoing discussion, it is clear that the fixed point theorems are useful for proving the existence theorems as well as for characterizing the solutions of different types of functional differential equations on unbounded intervals of real line. The choice of the fixed point theorems depends upon the situations and the circumstances of the nonlinearities involved in the problems. The clever selection of the fixed point theorems yields very powerful existence results as well as different characterizations of the nonlinear functional differential equations. In this article, we have been able to prove the existence as well as global attractivity and ultimate positivity of the solutions for three types of nonlinear functional differential equations unbounded intervals. However, other nonlinear functional differential equations can be treated in the similar way for these and some other characterizations such as monotonic global attractivity, monotonic asymptotic attractivity and monotonic ultimate positivity of the solutions for such equations on unbounded intervals of real line. In a forthcoming paper, it is planed to discuss the global asymptotic and monotonic attractivity of solutions for nonlinear functional differential equations via classical and hybrid applicable fixed point theory.

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