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MINIMAL AND MAXIMAL SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEMS FOR THE SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

WAWRZYNIEC SZATANIK

Gdansk University of Technology, Department of Differential Equations
11/12 G.Narutowicz Str., 80–952 Gdańsk, POLAND
wawrzyniec@mif.pg.gda.pl

ABSTRACT. This paper deals with boundary value problems for the second order differential equations with deviating arguments and integral boundary conditions. Sufficient conditions are formulated under which such a problem has at least one solution. A monotone iterative method is used. Example with numerical result is included to illustrate scheme used in the main theorem.

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1. INTRODUCTION

There are many papers which deal with second order BVP and monotone iterative technique [2], [3], [7]–[8], [10]–[14] and [16]. Monotone iterative method for the first order differential equation with integral initial condition was employed in [9]. Interesting results about fourth order BVP and lower and upper solutions method can be found in [1], [5] and [15]. Extension fourth order BVP to $2m$th with full nonlinear BVP is studied in [4]. This paper expands view of BVP with integral boundary conditions and deviating arguments. Requirements for existence of solution are formulated.

First we need some technical lemmas which will be used in the main theorem. Monotone sequences will be defined as solutions of some linear BVP problems. Limits of that sequences appear to be minimal and maximal solutions of out target problem. Next we introduce a non trivial example with numerical illustration of algorithm used in the main theorem.

Let us define a boundary value problem

$$(1.1)\qquad \begin{cases} x''(t) = f\big(t, x(t), x(\alpha(t))\big) \equiv Fx(t), \quad t \in J = [0, T],\ T < \infty, \\ x(0) = \int_0^{\gamma_1} K_1\big(s, x(s)\big)ds, \qquad x(T) = \int_0^{\gamma_2} K_2\big(s, x(s)\big)ds, \end{cases}$$

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where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, K_1, $K_2 \in C(J \times \mathbb{R}, \mathbb{R})$ and γ_1, $\gamma_2 \in J$.

2. **LEMMAS**

To apply monotone iterative technique to problems of type (1.1), we need some results in differential inequalities and integral equations.

Lemma 1. *Assume that:*

$$\alpha \in C(J, J),\ M,\ N \in C(J,\ [0, \infty]), M(t) > 0,\ t \in (0, T), \tag{2.1}$$

$$\max\Big\{ \int_0^T \Big(\int_s^T [M(t) + N(t)]dt \Big) ds, \int_0^T \Big(\int_0^s [M(t) + N(t)]dt \Big) ds \Big\} < 1. \tag{2.2}$$

Let $p \in C^2(J, \mathbb{R})$ *and*

$$\begin{cases} p''(t) \geq M(t)p(t) + N(t)p\big(\alpha(t)\big), & t \in J, \\ p(0) \leq 0, \quad p(T) \leq 0. \end{cases}$$

Then $p(t) \leq 0$ *on* J.

Proof. Suppose, that $p(t) > 0$ for some $t \in J$. Thus we will consider two possible cases:

1. $p(t) > 0$ for $t \in (0, T)$,
2. there exist $t_*, t^* \in J$, such that $p(t_*) < 0$ and $p(t^*) > 0$.

Case 1. Because $p(t) > 0$ for $t \in (0, T)$ then $p(0) = p(T) = 0$. It follows $p''(t) \geq 0$, means that p is convex, thus in holds

$$p(t) \leq \frac{T - t}{T} p(0) + \frac{t}{T} p(T) = 0,$$

for $t \in (0, T)$, is a contradiction.
Case 2. There exists $t^* \in J$, such that

$$p(t^*) = \max\{p(t), t \in J\}, \quad p'(t^*) = 0.$$

Denote $t_* \in J$, such that $p(t_*) = \min\{p(t), t \in J\}$. Then for $t \in J$, we have

$$p''(t) \geq M(t)p(t) + N(t)p\big(\alpha(t)\big) \geq [M(t) + N(t)]p(t_*). \tag{2.3}$$

Suppose that $t_* < t^*$, by integrating on (t, t^*) (2.3) we get

$$p'(t^*) - p'(t) \geq p(t_*) \int_t^{t^*} [M(s) + N(s)]ds.$$

However $p'(t^*) = 0$ and $p(t_*) < 0$, then

$$-p'(t) \geq p(t_*) \int_t^T [M(s) + N(s)]ds, \qquad \text{for } t \in [0, t^*]. \tag{2.4}$$

Now let us integrate (2.4) on (t_*, t^*), thus

$$\begin{aligned} p(t_*) \geq -p(t^*) + p(t_*) &\geq p(t_*) \int_{t_*}^{t^*} \int_s^T [M(\tau) + N(\tau)] d\tau ds \\ &\geq p(t_*) \int_0^T \int_s^T [M(\tau) + N(\tau)] d\tau ds. \end{aligned} \tag{2.5}$$

Hence,

$$p(t_*)\Big(1 - \int_0^T \int_s^T [M(\tau) + N(\tau)] d\tau ds\Big) \geq 0.$$

It is a contradiction with assumption (2.2). For $t_* > t^*$ we omit that part of proof because it is similar. □

Second Lemma follows from Green function properties.

Lemma 2. *Let*

$$G(t,s) = -\frac{1}{T}\begin{cases} (T-t)s \text{ for } 0 \leq s \leq t \leq T, \\ (T-s)t \text{ for } 0 \leq t \leq s \leq T. \end{cases}$$

Let $h : J \to \mathbb{R}$ be integrable on J. Then the problem

$$\begin{cases} u''(t) = h(t), \\ u(0) = \kappa, \quad u(T) = \beta \end{cases}$$

has the exactly one solution given by

$$u(t) = \int_0^T G(t,s)h(s)ds + \frac{\beta}{T}t + \frac{\kappa}{T}(T-t).$$

Also we need some existence lemma ([16]).

Lemma 3. *Let $\alpha \in C(J,J)$, M, $N \in C(J,[0,\infty))$. Assume that*

$$\max_{t\in[0,T]} \int_0^T 2|G(t,s)||M(s) + N(s)|ds < 1. \tag{2.6}$$

Then the problem

$$\begin{cases} y''(t) = M(t)y(t) + N(t)y\big(\alpha(t)\big) + \sigma(t), \ t \in J \\ y(0) = \beta_1, \quad y(T) = \beta_2, \quad \beta_1, \beta_2 \in \mathbb{R}, \end{cases} \tag{2.7}$$

has the exactly one solution $y \in C(J, \mathbb{R})$.

3. MAIN RESULT

First we introduce definitions related with problem (1.1).

Definition 1. A function $y_0 \in C^2(J, \mathbb{R})$ is said to be the lower solution of (1.1) if

$$\begin{cases} y_0''(t) \geq F y_0(t) \quad \text{for } t \in J, \\ y_0(0) \leq \int_0^{\gamma_1} K_1\big(s, y_0(s)\big) ds, \quad y_0(T) \leq \int_0^{\gamma_2} K_2\big(s, y_0(s)\big) ds. \end{cases}$$

Definition 2. A function $z_0 \in C^2(J, \mathbb{R})$ is said to be the upper solution of (1.1) if

$$\begin{cases} z_0''(t) \leq F z_0(t) \quad \text{for } t \in J, \\ z_0(0) \geq \int_0^{\gamma_1} K_1\big(s, z_0(s)\big) ds, \quad z_0(T) \geq \int_0^{\gamma_2} K_2\big(s, z_0(s)\big) ds. \end{cases}$$

Definition 3. Let $u, v \in C^2(J, \mathbb{R})$ with $u(t) \leq v(t)$ for $t \in J$. Solutions U, V of (1.1) are called *minimal and maximal solutions* in segment $[u, v]$ if $u(t) \leq U(t)$, $V(t) \leq v(t)$ for $t \in J$ and for any else Z solution of (1.1), such as $u(t) \leq Z(t) \leq v(t)$ for $t \in J$ we have $U(t) \leq Z(t) \leq V(t)$, $t \in J$.

Now we can formulate the main theorem.

Theorem 1. *Let $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, K_1, $K_2 \in C(J \times \mathbb{R}, \mathbb{R})$, $\gamma_1, \gamma_2 \in J$ and $\alpha \in C(J, J)$. Let y_0, z_0 are lower and upper solutions of* (1.1) *and $y_0(t) \leq z_0(t)$, $t \in J$. Moreover, assume that*

$$f(t, \bar{u}_1, \bar{v}_1) - f(t, u_1, v_1) \geq -M(t)[u_1 - \bar{u}_1] - N(t)[v_1 - \bar{v}_1], \tag{3.1}$$

$$K_1(t, \bar{u}_1) - K_1(t, u_1) \leq h_1(t)[\bar{u}_1 - u_1] \text{ where } h_1(t) \geq 0 \text{ for } t \in J, \tag{3.2}$$

$$K_2(t, \bar{u}_1) - K_2(t, u_1) \leq h_2(t)[\bar{u}_1 - u_1] \text{ where } h_2(t) \geq 0 \text{ for } t \in J, \tag{3.3}$$

for $y_0(t) \leq \bar{u}_1 \leq u_1 \leq z_0(t)$, $y_0(\alpha(t)) \leq \bar{v}_1 \leq v_1 \leq y_0(\alpha(t))$. Also we assume that functions M, N satisfies (2.1), (2.2) *and* (2.6). *Then problem* (1.1) *has in segment $[y_0, z_0]$ the minimal and maximal solutions.*

Proof. Let us define some linear BVP problems:

$$\begin{cases} y_1''(t) = F y_0(t) + M(t)[y_1(t) - y_0(t)] \\ \qquad\qquad + N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)], \quad t \in J, \\ y_1(0) = \int_0^{\gamma_1} K_1\big(s, y_0(s)\big) ds, \qquad y_1(T) = \int_0^{\gamma_2} K_2\big(s, y_0(s)\big) ds, \end{cases} \tag{3.4}$$

$$\begin{cases} z_1''(t) = F z_0(t) + M(t)[z_1(t) - z_0(t)] \\ \qquad\qquad + N(t)[z_1\big(\alpha(t)\big) - z_0\big(\alpha(t)\big)], \quad t \in J, \\ z_1(0) = \int_0^{\gamma_1} K_1\big(s, z_0(s)\big) ds, \qquad z_1(T) = \int_0^{\gamma_2} K_2\big(s, z_0(s)\big) ds. \end{cases} \tag{3.5}$$

Note that problems (3.4) and (3.5) have unique solutions y_1 and z_1, by Lemma (3).

First we will prove that $y_0(t) \le y_1(t)$ for $t \in J$. Put $p(t) = y_0(t) - y_1(t)$, $t \in J$. From definition of lower solution y_0 and (3.4), we obtain,

$$p(0) = y_0(0) - y_1(0) \le \int_0^{\gamma_1} K_1\big(s, y_0(s)\big)ds - \int_0^{\gamma_1} K_1\big(s, y_0(s)\big)ds = 0,$$

$$p(T) = y_0(T) - y_1(T) \le \int_0^{\gamma_2} K_2\big(s, y_0(s)\big)ds - \int_0^{\gamma_2} K_2\big(s, y_0(s)\big)ds = 0.$$

Now we will compute the second derivative of p to show that all assumption of Lemma 1 are fulfilled by p. So,

$$\begin{aligned} p''(t) = y_0''(t) - y_1''(t) \ge{}& Fy_0(t) - Fy_0(t) \\ & - M(t)[y_1(t) - y_0(t)] - N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] \\ ={}& M(t)[y_0(t) - y_1(t)] + N(t)[y_0\big(\alpha(t)\big) - y_1\big(\alpha(t)\big)] \\ ={}& M(t)p(t) + N(t)p\big(\alpha(t)\big). \end{aligned}$$

Then Lemma 1 implies that $p(t) \le 0$ for $t \in J$. It proves that $y_0(t) \le y_1(t)$. In the same way we can show that $z_0(t) \ge z_1(t)$.

In the next step we will show that $y_1(t) \le z_1(t)$ for $t \in J$. Let $p(t) = y_1(t) - z_1(t)$, by using assumptions (3.2) and (3.3), we get

$$\begin{aligned} p(0) = y_1(0) - z_1(0) &= \int_0^{\gamma_1} K_1\big(s, y_0(s)\big)ds - \int_0^{\gamma_1} K_1\big(s, z_0(s)\big)ds \\ &= \int_0^{\gamma_1} \Big[K_1\big(s, y_0(s)\big) - K_1\big(s, z_0(s)\big)\Big]ds \\ &\le \int_0^{\gamma_1} h_1(s)\big[y_0(s) - z_0(s)\big]ds \le 0, \end{aligned}$$

$$\begin{aligned} p(T) = y_1(T) - z_1(T) &= \int_0^{\gamma_2} K_2\big(s, y_0(s)\big)ds - \int_0^{\gamma_1} K_2\big(s, z_0(s)\big)ds \\ &= \int_0^{\gamma_2} \Big[K_2\big(s, y_0(s)\big) - K_2\big(s, z_0(s)\big)\Big]ds \\ &\le \int_0^{\gamma_2} h_2(s)\big[y_0(s) - z_0(s)\big]ds \le 0. \end{aligned}$$

Moreover, using the assumption (3.1), we get

$$\begin{aligned} p''(t) ={}& Fy_0(t) - Fz_0(t) + M(t)[y_1(t) - y_0(t)] - M(t)[z_1(t) - z_0(t)] \\ & + N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] - N(t)[z_1\big(\alpha(t)\big) - z_0\big(\alpha(t)\big)] \\ \ge{}& -M(t)[z_0(t) - y_0(t)] - N(t)[z_0\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] \\ & + M(t)[y_1(t) - y_0(t)] - M(t)[z_1(t) - z_0(t)] \\ & + N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] - N(t)[z_1\big(\alpha(t)\big) - z_0\big(\alpha(t)\big)] \\ ={}& M(t)p(t) + N(t)p\big(\alpha(t)\big). \end{aligned}$$

Hence $y_1(t) \le z_1(t)$, $t \in J$, by Lemma 1.

Now we will show that y_1 and z_1 are lower and upper solutions of (1.1), respectively. From the definition of y_1 and (3.2), we have

$$\begin{aligned} y_1(0) &= \int_0^{\gamma_1} K_1\big(s, y_0(s)\big)ds - \int_0^{\gamma_1} K_1\big(s, y_1(s)\big)ds + \int_0^{\gamma_1} K_1\big(s, y_1(s)\big)ds \\ &\le \int_0^{\gamma_1} h_1(s)\big[y_0(s) - y_1(s)\big]ds + \int_0^{\gamma_1} K_1\big(s, y_1(s)\big)ds \le \int_0^{\gamma_1} K_1\big(s, y_1(s)\big)ds. \end{aligned}$$

In the same way we can show that $y_1(T) \le \int_0^{\gamma_2} K_2\big(s, y_1(s)\big)ds$.

Next we need to show that $y_1''(t) \ge Fy_1(t)$. To do that, we will use the definition of y_1 and condition (3.1). So,

$$\begin{aligned} y_1''(t) &= Fy_0(t) - Fy_1(t) + Fy_1(t) + M(t)[y_1(t) - y_0(t)] \\ &\quad + N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] \\ &\ge Fy_1(t) - M(t)[y_1(t) - y_0(t)] - N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] \\ &\quad + N(t)[y_1\big(\alpha(t)\big) - y_0\big(\alpha(t)\big)] \\ &= Fy_1(t). \end{aligned}$$

Thus y_1 is a lower solution of (1.1). Using same technique we can show that z_1 is a upper solution of (2.7).

Now we can define sequences lower and upper solutions of (1.1), by

$$\text{(3.6)} \qquad \begin{cases} y_n''(t) = Fy_{n-1}(t) + M(t)[y_n(t) - y_{n-1}(t)] \\ \qquad\qquad\qquad\qquad + N(t)[y_n\big(\alpha(t)\big) - y_{n-1}\big(\alpha(t)\big)], \quad t \in J, \\ y_n(0) = \int_0^{\gamma_1} K_1\big(s, y_{n-1}(s)\big)ds, \qquad y_n(T) = \int_0^{\gamma_2} K_2\big(s, y_{n-1}(s)\big)ds, \end{cases}$$

$$\text{(3.7)} \qquad \begin{cases} z_n''(t) = Fz_{n-1}(t) + M(t)[z_n(t) - z_{n-1}(t)] \\ \qquad\qquad\qquad\qquad + N(t)[z_n\big(\alpha(t)\big) - z_{n-1}\big(\alpha(t)\big)], \quad t \in J, \\ z_n(0) = \int_0^{\gamma_1} K_1\big(s, z_{n-1}(s)\big)ds, \qquad z_n(T) = \int_0^{\gamma_2} K_2\big(s, z_{n-1}(s)\big)ds, \end{cases}$$

for $n = 1, 2, \ldots$. We proved for $n = 1$, that problems (3.6) and (3.7) have solutions which are also lower and upper solutions of (1.1). By induction in n, we can prove the relation:

$$y_0(t) \le \cdots \le y_{n-1}(t) \le y_n(t) \le z_n(t) \le z_{n-1}(t) \le \cdots \le z_0(t)$$

for $t \in J$, $n \in \mathbb{N}$. It is not problematic to show that sequences $\{y_n\}$, $\{z_n\}$ are equicontinuous and bounded on J. The Arzeli-Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}\}$, $\{z_{n_k}\}$ and functions $y, z \in C(J, \mathbb{R})$ with $\{y_{n_k}\}$, $\{z_{n_k}\}$ converging uniformly on J to y, z, respectively when $n_k \to \infty$. However, since the sequences $\{y_n\}$, $\{z_n\}$ are monotonic, we conclude that the whole sequences $\{y_n\}$, $\{z_n\}$ converge uniformly on J to y, z, respectively. If $n \to \infty$ in integral equations for y_n, z_n and we compute first and second derivative of them, we get

$$\begin{cases} y''(t) = Fy(t), \quad t \in J, \\ y(0) = \int_0^{\gamma_1} K_1\big(s, y(s)\big)ds, \qquad y(T) = \int_0^{\gamma_2} K_2\big(s, y(s)\big)ds, \quad t \in J \end{cases}$$

and

$$\begin{cases} z''(t) = Fz(t), \quad t \in J, \\ z(0) = \int_0^{\gamma_1} K_1\big(s, z(s)\big)ds, \qquad y(T) = \int_0^{\gamma_2} K_2\big(s, z(s)\big)ds, \quad t \in J, \end{cases}$$

We have proved that the problem (1.1) has the solutions y and z.

In the last step, we will show that y and z are the minimal and maximal solutions in segment $[y_0, z_0]$. Let $\bar{z}$ be the solution of (1.1) such that

$$y_m(t) \leq \bar{z}(t) \leq z_m(t), \quad t \in J$$

for some $m \in \mathbb{N}$. Put $p(t) = y_{m+1}(t) - \bar{z}(t)$, $t \in J$. Form definition of y_{m+1} and conditions (3.1), (3.2), (3.3), we get $p(0) \leq 0$, $p(T) \leq 0$ and

$$\begin{aligned} p''(t) &= Fy_m(t) - F\bar{z}(t) + M(t)[y_{m+1}(t) - y_m(t)] \\ &\quad + N(t)[y_{m+1}\big(\alpha(t)\big) - y_m\big(\alpha(t)\big)] \\ &\geq -M(t)[\bar{z}(t) - y_m(t)] - N(t)[\bar{z}\big(\alpha(t)\big) - y_m\big(\alpha(t)\big)] \\ &\quad + M(t)[y_{m+1}(t) - y_m(t)] + N(t)[y_{m+1}\big(\alpha(t)\big) - y_m\big(\alpha(t)\big)] \\ &= M(t)[y_{m+1}(t) - \bar{z}(t)] + N(t)[y_{m+1}\big(\alpha(t)\big) - \bar{z}\big(\alpha(t)\big)] \\ &= M(t)p(t) + N(t)p\big(\alpha(t)\big). \end{aligned}$$

In the same way we can show that $\bar{z}(t) \leq z_{m+1}(t)$, $t \in J$. By induction, we obtain

$$y_n(t) \leq \bar{z}(t) \leq z_n(t), \quad \text{for } n \in \mathbb{N}.$$

If $n \to \infty$, it yields

$$y(t) \leq \bar{z}(t) \leq z(t), \quad t \in J.$$

It shows that (y, z) are minimal and maximal solutions of problem (1.1) in segment $[y_0, z_0]$. □

Example 1. Let us consider a problem:

$$\begin{cases} x''(t) = t^2 \sin x(t) + \dfrac{\big(x(\sqrt{t})\big)^3}{e^{t+3}} + 1, \; t \in J = [0,1], \\ x(0) = \int_0^{\frac{1}{2}} [1.5 + \sin(s) 2 \arctan x(s)]ds, \\ x(T) = \int_0^1 e^{s-1}\big(x(s) + 1\big)ds \end{cases} \tag{3.8}$$

Let $y_0(t) = t(t-2) - 1$, $z_0(t) = -t(t-2) + 1$.

By numerical computation we can obtain that y_0, z_0 are lower and upper solutions of (1.1). Because $\max\{z_0(t) - y_0(t), t \in J\} = 4$, we can prove then an assumption

(3.1) held by using the mean value theorem. Let $u, \bar{v}, z, \bar{v}$ and $y_0(t) \le \bar{u} \le u \le z_0(t),\ \ y_0(\alpha(t)) \le \bar{v} \le v \le y_0(\alpha(t))$. Thus

$$f(t,\bar{u},\bar{v}) - f(t,u,v) = t^2(\sin\bar{u} - \sin u) + \frac{1}{\mathrm{e}^{t+3}}(\bar{v}^3 - v^3),$$

so there exist $\theta_1 \in (\bar{u}, u)$ and $\theta_2 \in (\bar{v}, v)$ such as

$$\begin{aligned} t^2(\sin\bar{u} - \sin u) + \frac{1}{\mathrm{e}^{t+3}}(\bar{v}^3 - v^3) &= t^2\cos\theta_1(\bar{u} - u) + \frac{3(\theta_2)^3}{\mathrm{e}^{t+3}}(\bar{v} - v) \\ &\le t^2(\bar{u} - u) + \frac{48}{\mathrm{e}^{t+3}}(\bar{v} - v) = -t^2(u - \bar{u}) - \frac{48}{\mathrm{e}^{t+3}}(v - \bar{v}), \end{aligned}$$

Then $M(t) = t^2$ and $N(t) = \dfrac{48}{\exp(t+3)}$. By numerical calculation we can show that functions M and N satisfy (2.2) and (2.6). Conditions (3.2) and (3.3) also holds, $h_1(t) = \exp(t-1)$ and $h_2(t) = \dfrac{2}{17}\sin t$, it can be shown by same way like condition (3.1) . So all conditions from Theorem 1 hold. Then, in the view of Theorem 3, the problem (3.8) has, in segment $[y_0, z_0]$, the minimal and maximal solutions. On Figure 1, we see four numerical results of iterations algorithm from Theorem 3. On Figure 2 there are shown the first eight iterations of algorithm.

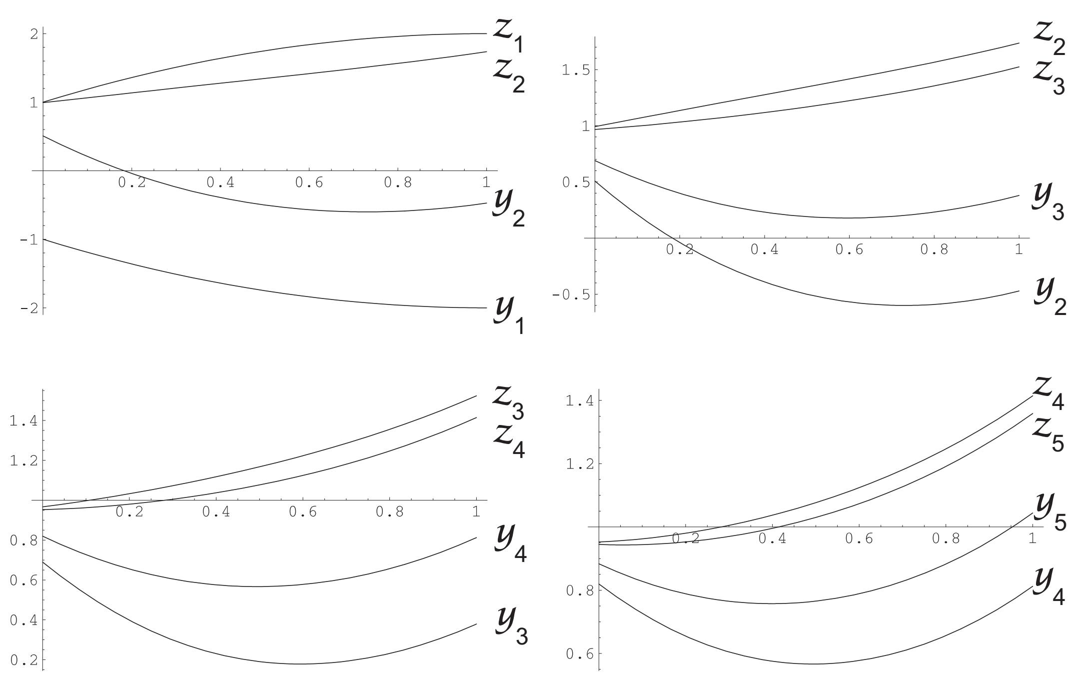

FIGURE 1. First four results of iteration in problem (3.8).

To obtain numerical solutions of problems (3.4) and (3.5), we discreetized it and solved appropriate linear systems by `Mathematica 4.0`. Solutions are interpolated by Lagrange polynomials to obtain values for deviating arguments.

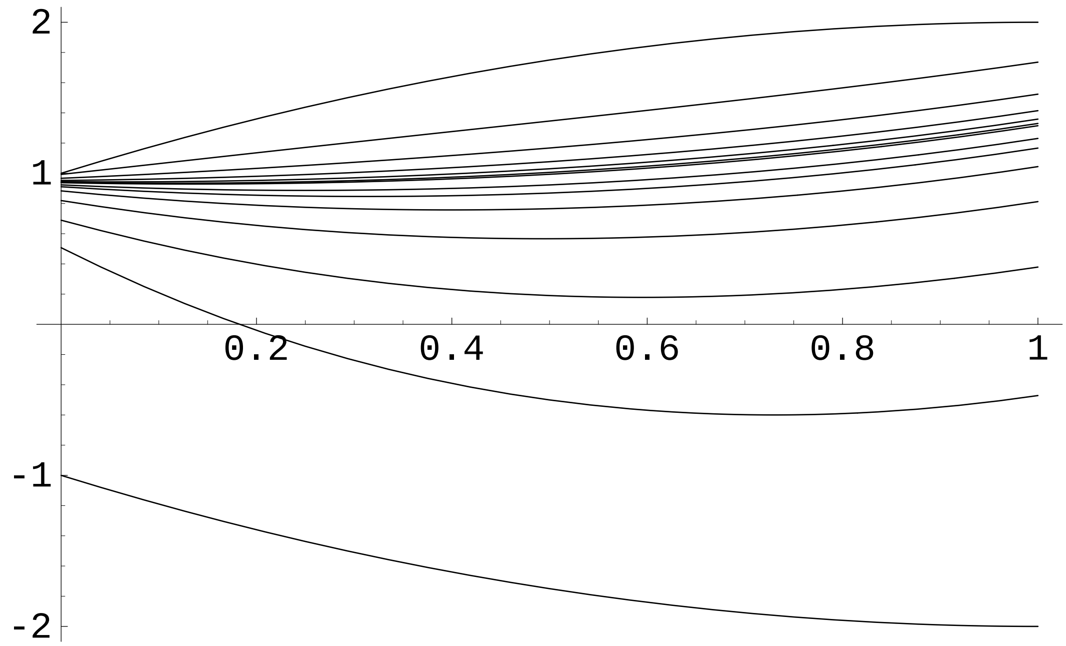

FIGURE 2. First eight results of iteration in problem (3.8).

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