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TRAVELING WAVEFRONTS FOR A SIS EPIDEMIC MODEL WITH STAGE STRUCTURE

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ABSTRACT. A SIS epidemic model with stage structure is studied. The existence of traveling wavefronts is shown by using the technique of weak upper and lower solutions and Schauder fixed point theorem.

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1. INTRODUCTION

Epidemics always happen in the human history. Some of them, especially the plague epidemics, were threatening scourges not only because of the economic and demographic aspects caused, but also because of the virulence, frequency or endemic persistence. In many diseases, an individual can recover and return to the susceptible state and later become reinfected. The common cold is one of the diseases that most of us get repeatedly. Different epidemics may have different ability to transmit in the different stages. For example, typhus, diphtheria and sexual diseases always transmit only in adult population. Furthermore, since the environment is not often spatially heterogenous and the infectious agents can move around, the effects of spatial diffusion cannot be ignored. As a result, a SIS model with age-structure and diffusion is of significance in describing the dynamic properties of some kinds of epidemics.

There are many phenomena in epidemics where a key element to the developmental process seems to be the appearance of a traveling wave of infectious concentration or mechanical deformation. The study of traveling waves is of current research interest and has developed during these 30 years, for example see [3]-[22] and the references therein. There are also some wonderful research works focus on the traveling waves and spreading speed for SIS models. In [13], Rass and Radcliffe used an approximate saddle-point method to obtain the speed of first spread $\bar{c}$ for some kinds of multi-type SIS models. Weng and Zhao [18] made use of the theory of monotone semiflow (see Liang and Zhao [9]) to obtain the minimal wave speed c_* and spreading speed c^* for

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a multi-type SIS epidemic model and they showed that $c_* = c^* = \bar{c}$ in their situation. The models in [13] and [18] are in the form of ordinary differential equations or integro-differential equations without age-structure.

Gourley and Wu in their recent survey [8] indicated that to model disease spread represented a very natural attempt to bring the model to the biological reality as the age involved. Recall that the following model of a single species with age-structure was first introduced by Aiello and Freedman [2]

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = \alpha u_2 - \gamma u_1 - \alpha \mathrm{e}^{-\gamma\tau} u_2(t-\tau), \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = \alpha \mathrm{e}^{-\gamma\tau} u_2(t-\tau) - \beta u_2^2, \end{cases} \tag{1.1}$$

where α, β, γ and delay τ are positive constants; u_1 and u_2 denote respectively the numbers of immature and mature members of the population. The αu_2 term is the birth function and γu_1, βu_2^2 represent correspondingly the death function of immature and mature individuals. τ is the time for a new-born to become mature and hence the $\alpha \mathrm{e}^{-\gamma\tau} u_2(t-\tau)$ term is the adult recruitment.

More recently, Gourley and Kuang [7] presented a more realistic model by incorporating spatial diffusion and non-local delay into system (1.1):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + \alpha u_2 - \gamma u_1 - \alpha \mathrm{e}^{-\gamma\tau} \int_{-\infty}^{\infty} \dfrac{1}{\sqrt{4\pi D_1 \tau}} \mathrm{e}^{-\frac{(x-y)^2}{4D_1\tau}} u_2(t-\tau, y)\mathrm{d}y, \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \alpha \mathrm{e}^{-\gamma\tau} \int_{-\infty}^{\infty} \dfrac{1}{\sqrt{4\pi D_1 \tau}} \mathrm{e}^{-\frac{(x-y)^2}{4D_1\tau}} u_2(t-\tau, y)\mathrm{d}y - \beta u_2^2. \end{cases} \tag{1.2}$$

When $D_1 \to 0$, that is when the juveniles become immobile, the system (1.2) reduces to the local problem

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha u_2 - \gamma u_1 - \alpha \mathrm{e}^{-\gamma\tau} u_2(t-\tau, x), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \alpha \mathrm{e}^{-\gamma\tau} u_2(t-\tau, x) - \beta u_2^2. \end{cases} \tag{1.3}$$

Al-Omari and Gourley [3] established the traveling waves for the second equation of system (1.3). Gourley and Wu in their survey [8] further emphasized that when the mobility of the immature could be ignored, a reaction diffusion equation with local effect was applicable in population biology, spatial biology and disease spread.

In this paper, inspired by the pioneering work of these authors, we incorporate age-structure and spatial diffusion into a SIS model. To the best of our knowledge, it is the first time that we present a diffusive SIS model with stage structure. The model is taken the following form:

(1.4)
$$\begin{cases} \dfrac{\partial u_1}{\partial t} &= B(u_2(t,x)) - \gamma u_1(t,x) - \mathrm{e}^{-\gamma\tau} B(u_2(t-\tau,x)), \\ \dfrac{\partial u_2}{\partial t} &= D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \mathrm{e}^{-\gamma\tau} B(u_2(t-\tau,x)) - \beta u_2^2(t,x) - U(u_2(t,x)) + ry(t,x), \\ \dfrac{\partial y}{\partial t} &= U(u_2(t,x)) - by(t,x) - ry(t,x), \end{cases}$$

for $t \in \mathbb{R}_+ := [0,\infty), x \in \mathbb{R}$, where $u_1(t,x), u_2(t,x)$ and $y(t,x)$ denote the densities of juvenile, susceptible mature and infective individuals at time t and location x respectively. It is assumed that only the mature individuals are susceptible and responsible to the reproduction of the population. $B(\cdot)$ is a birth function. The susceptible individuals, once infected, can act as the infectious agent and then transmit the infections. Let $U(u_2)$ be the force of infection on the mature population due to a concentration of the infectious agent u_2. $b, r > 0$ denote the death rate and recovery rate of infective individuals, respectively. Note that the system (1.4) models a dispersal of susceptible mature individuals while ignoring the small mobility of the juveniles and infective individuals.

The purpose of this paper is to establish the traveling wavefronts for system (1.4). We show the existence of traveling wavefronts of the mature populations (the last two equations of system (1.4)) by using the technique of weak upper and lower solutions, and Schauder fixed point theorem (see [10] and [16]) for the wave speed $c > c^*$, where c^* is a positive constant determined by an eigenvalue-equation. If $c = c^*$, we also obtain the existence of traveling wavefronts by using a limit argument similar to [21]. Then we return to the immature equation (the first equation of system (1.4)) and establish the traveling wavefronts for system (1.4) when $c \geq c^*$.

2. EXISTENCE OF TRAVELING WAVEFRONTS

Assume that $B(0) = 0$ and $U(0) = 0$. Then (1.4) has two spatially uniform equilibria $E_0(0,0,0)$ and $E^*(u_1^*, u_2^*, y^*)$, where $u_1^* > 0, u_2^* > 0, y^* > 0$ satisfies

$$U(u_2^*) = (b+r)y^*, \;\; y^* = \tfrac{g(u_2^*)}{b}, \;\; u_1^* = \tfrac{B(u_2^*)}{\gamma}(1 - \mathrm{e}^{-\gamma\tau}),$$

and

$$g(u) = \mathrm{e}^{-\gamma\tau} B(u) - \beta u^2.$$

We introduce some assumptions.

(H1) $g(u) = 0$ has a root $\bar{u}_2 > u_2^*$ with $g(u) > 0$ when $0 < u < \bar{u}_2$ and $g(u) < 0$ when $u > \bar{u}_2$.

(H2) $B(u)$ is differentiable and $\rho_1 \geq B'(u) > 0$ for $u \in [0, u_2^*]$, some $\rho_1 > 0$; $B(u) \leq B'(0)u$ for $u \in [0, u_2^*)$; $B'(0)u - B(u) \leq \kappa u^2$ for $u \in [0, u_2^*)$, some $\kappa > 0$.

(H3) $U(u)$ is differentiable and $\rho_2 \geq U'(u) > 0$ for $u \in [0, u_2^*]$, some $\rho_2 > 0$; $U(u) \leq U'(0)u$ for $u \in [0, u_2^*)$; $U'(0)u - U(u) \leq \beta u^2$ for $u \in [0, u_2^*)$.

(H4) $\mathrm{e}^{-\gamma\tau} B'(0) - \frac{b}{b+r} U'(0) > 0$.

We note from (H1) that $g(u_2^*) > 0$ holds. Note that the conditions (H1)-(H4) imposed on functions $B(u), U(u), g(u)$ are natural, and they are not more restrictive conditions. For example, if we take birth function $B(u) = \alpha u$, then (H2) holds spontaneously. Let $U(u) = \frac{u}{1+u}$, then $U'(u) = \frac{1}{(1+u)^2}$. Let $\rho_2 = 1$ and $\beta = 1$, then (H3) holds trivially. In this situation, $g(u) = \alpha \mathrm{e}^{-\gamma\tau} u - \beta u^2$ and $g(u) = 0$ has a positive root $\bar{u}_2 = \frac{\alpha \mathrm{e}^{-\gamma\tau}}{\beta}$. Since $g(u_2^*) > 0$, then $\bar{u}_2 > u_2^*$. The assumption (H1) follows immediately. Finally, if we choose $\alpha > \frac{b}{b+r} \mathrm{e}^{\gamma\tau}$, then (H4) holds. The assumption (H4) makes sense in epidemiological respect. The explanation for this is that, no matter how serious of the disease spread, the species will survive. In fact, since the rate of adult recruitment is larger than that of infected dying, the mature individuals will be still capable of producing offspring.

In system (1.4), the last two equations are uncoupled from the first equation and thus it is sufficient to consider the last two equations in their own:

$$
\text{(2.1)} \quad \begin{cases} \dfrac{\partial u}{\partial t} = D_2 \dfrac{\partial^2 u}{\partial x^2} + \mathrm{e}^{-\gamma\tau} B(u(t-\tau, x)) - \beta u^2(t, x) - U(u(t, x)) + r y(t, x), \\ \dfrac{\partial y}{\partial t} = U(u(t, x)) - b y(t, x) - r y(t, x), \end{cases}
$$

where we use u to replace u_2 for simplicity. Similarly, we use u^* to replace u_2^* in the following.

A traveling wave solution of (2.1) is a solution of (2.1) connecting the two equilibria $(0, 0)$ and (u^*, y^*) with the form $u(t, x) = \psi(x + ct) = \psi(z)$ and $y(t, x) = \phi(x + ct) = \phi(z)$, where $z = x + ct$ and $c > 0$ is the wave speed. Thus we consider the wave profile system

$$
\text{(2.2)} \quad \begin{cases} D_2 \psi''(z) - c\psi'(z) + \mathrm{e}^{-\gamma\tau} B(\psi(z - c\tau)) - \beta \psi^2(z) - U(\psi(z)) + r\phi(z) = 0, \\ -c\phi'(z) + U(\psi(z)) - b\phi(z) - r\phi(z) = 0, \end{cases}
$$

subject to the boundary value conditions

$$
\text{(2.3)} \quad \lim_{z \to -\infty} (\psi(z), \phi(z)) = (0, 0), \quad \lim_{z \to \infty} (\psi(z), \phi(z)) = (u^*, y^*).
$$

Furthermore, a traveling wavefront of (2.1) is a traveling wave solution $(\psi(z), \phi(z))$ of (2.1) which is nondecreasing. For ecological realism, we mention here that $\psi(z) \geq 0, \phi(z) \geq 0$ on $z \in \mathbb{R}$.

From the second equation of (2.2), we obtain

$$
\phi(z) = \mathrm{e}^{-\frac{(b+r)}{c}(z - z_0)} \phi(z_0) + \frac{1}{c} \int_{z_0}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z - s)} U(\psi(s)) \mathrm{d}s
$$

for any $z_0 \in \mathbb{R}$, $z \geq z_0$. Since $\phi(z)$ and $U(\psi(z))$ are bounded functions on $\mathbb{R}$, by taking $z_0 \to -\infty$, we have from (2.3) that

$$\phi(z) = \frac{1}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} U(\psi(s))\mathrm{d}s \quad \text{on } \ z \in \mathbb{R}. \tag{2.4}$$

Substituting (2.4) into the first equation of (2.2), we get

$$D_2\psi''(z) - c\psi'(z) + \mathrm{e}^{-\gamma\tau}B(\psi(z-c\tau)) - \beta\psi^2(z) - U(\psi(z)) + \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} U(\psi(s))\mathrm{d}s = 0 \tag{2.5}$$

on $z \in \mathbb{R}$. Define

$$F[\psi](z) := \mathrm{e}^{-\gamma\tau}B(\psi(z-c\tau)) - \beta\psi^2(z) - U(\psi(z)) + \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} U(\psi(s))\mathrm{d}s.$$

Then we rewrite (2.5) as

$$D_2\psi''(z) - c\psi'(z) + F[\psi](z) = 0. \tag{2.6}$$

Choose a constant $h \geq 2\beta u^* + \rho_2$, where ρ_2 is defined in (H3), then another equivalent form of (2.5) is

$$-D_2\psi''(z) + c\psi'(z) + h\psi(z) = Q[\psi](z), \tag{2.7}$$

where

$$Q[\psi](z) := h\psi(z) + F[\psi](z).$$

Define

$$C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) := \{\psi : \psi \in C(\mathbb{R}, \mathbb{R}), 0 \leq \psi(z) \leq u^* \text{ on } z \in \mathbb{R}\}.$$

Then we have the following.

Lemma 2.1. *Assume that* (H1)–(H3) *hold. Then*

(i) $Q[\varphi](z) \leq Q[\psi](z)$ *on* $z \in \mathbb{R}$, *if* $\varphi, \psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$ *with* $\varphi(z) \leq \psi(z)$ *on* $z \in \mathbb{R}$;
(ii) $0 \leq Q[\psi](z) \leq hu^*$ *for any* $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$;
(iii) $Q[\psi](z)$ *is nondecreasing on* $z \in \mathbb{R}$, *if* $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$ *is nondecreasing on* $z \in \mathbb{R}$.

Proof. (i) If $\varphi, \psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$ and $\varphi(z) \leq \psi(z)$ on $z \in \mathbb{R}$, then

$$\begin{aligned} & Q[\varphi](z) - Q[\psi](z) \\ &\quad = [h - \beta(\varphi(z) + \psi(z))][\varphi(z) - \psi(z)] + \mathrm{e}^{-\gamma\tau}[B(\varphi(z-c\tau)) - B(\psi(z-c\tau))] \\ &\qquad - [U(\varphi(z)) - U(\psi(z))] + \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)}[U(\varphi(s)) - U(\psi(s))]\mathrm{d}s. \\ &\quad \leq [h - \beta(\varphi(z) + \psi(z)) - U'(\varsigma(z))][\varphi(z) - \psi(z)] \leq 0, \end{aligned}$$

where $\varsigma(z)$ is a mean-value function between $\varphi(z)$ and $\psi(z)$. Thus the conclusion of (i) is true.

(ii) If $\psi \in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$, then according to (i) and (2.5), we get

$$\begin{aligned} 0 \le Q[\psi](z) &\le hu^* + \mathrm{e}^{-\gamma\tau} B(u^*) - \beta(u^*)^2 - U(u^*) \\ &\quad + \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} U(u^*)\mathrm{d}s = hu^*, \end{aligned}$$

which implies the conclusion (ii).

(iii) Let $z \in \mathbb{R}$ and $\theta > 0$ be given. Then

$$\begin{aligned} & Q[\psi](z+\theta) - Q[\psi](z) \\ &\quad = [h - \beta(\psi(z+\theta) + \psi(z)) - U'(\xi(z,\theta))][\psi(z+\theta) - \psi(z)] \\ &\qquad + \mathrm{e}^{-\gamma\tau}[B(\psi(z+\theta - c\tau)) - B(\psi(z - c\tau))] \\ &\qquad + \frac{r}{c}\left[\int_{-\infty}^{z+\theta} \mathrm{e}^{-\frac{(b+r)}{c}(z+\theta-s)} U(\psi(s))\mathrm{d}s - \int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} U(\psi(s))\mathrm{d}s\right]. \\ &\quad \ge \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)}[U(\psi(s+\theta)) - U(\psi(s))]\mathrm{d}s \ge 0, \end{aligned}$$

where $\xi(z,\theta)$ is a mean-value function between $\psi(z+\theta)$ and $\psi(z)$. Therefore, the conclusion (iii) follows. □

Recall that for any $\varphi, \psi \in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$ with $\varphi(z) \le \psi(z)$, if there exists a $\ell > 0$ such that $F[\varphi](0) - F[\psi](0) + \ell[\varphi(0) - \psi(0)] \le 0$, then the reaction term in (2.6) satisfies the quasi-monotonic nondecreasing condition [19]. We have from the first conclusion of Lemma 2.1 that the reaction term in (2.6) is in fact quasi-monotonic nondecreasing, namely, for $\ell \ge h$,

$$F[\varphi](0) - F[\psi](0) + \ell[\varphi(0) - \psi(0)] = Q[\varphi](0) - Q[\psi](0) + (\ell - h)[\varphi(0) - \psi(0)] \le 0$$

if $\varphi(z) \le \psi(z)$. The quasi-monotonicity condition previously used in different context by, for example, Ahmad and Vatsala in [1], Martin and Smith in [11], Wu and Zou in [19], Ma in [10]. Assume that a system has only two equilibria. A monotone iteration scheme could be established and the iterative sequence converges to a traveling wavefront connecting these two equilibria provided that the reaction term satisfies the quasi-monotonicity condition along with the existence of a pair of weak upper and lower solutions. Please see [19] and [10] for details. Based on this idea, we therefore are able to consider the nondecreasing solution of (2.6) satisfying

$$\lim_{z\to-\infty}\psi(z) = 0 \quad \text{and} \quad \lim_{z\to\infty}\psi(z) = u^* \tag{2.8}$$

under some conditions given below. However, if the reaction term does not satisfy the quasi-monotonicity condition, then non-monotone traveling waves can appear. For example, T. Faria and S. Trofimchuk in [6] studied a delayed reaction-diffusion equation with a reaction term loss of quasi-monotonicity property and proved that the traveling waves oscillated infinite about the positive equilibrium. They further proved that for large negative values of wave variable, the traveling wave is asymptotically equivalent to an increasing exponential function.

Note that the algebra equation $-D_2\chi^2 + c\chi + h = 0$ has two roots given by

$$\chi_1 = \frac{c - \sqrt{c^2 + 4D_2h}}{2D_2} < 0, \quad \chi_2 = \frac{c + \sqrt{c^2 + 4D_2h}}{2D_2} > 0.$$

Define an operator $\Lambda : C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) \to C^2(\mathbb{R}, \mathbb{R})$ by

$$\Lambda[\psi](z) := \frac{1}{D_2(\chi_2 - \chi_1)} \left\{ \int_{-\infty}^{z} e^{\chi_1(z-s)} Q[\psi](s)ds + \int_{z}^{\infty} e^{\chi_2(z-s)} Q[\psi](s)ds \right\}.$$

Then $\Lambda : C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) \to C^2(\mathbb{R}, \mathbb{R})$ is a well defined map, and satisfies

$$-D_2(\Lambda[\psi])'' + c(\Lambda[\psi])' + h\Lambda[\psi] = Q[\psi] \tag{2.9}$$

for any $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$. Furthermore, any fixed point of Λ is a solution of (2.7). The following lemma can be derived straightly from Lemma 2.1, and we omit the details of the proof.

Lemma 2.2. *Assume that* (H1)–(H3) *hold. Then*

(i) $\Lambda[\varphi](z) \le \Lambda[\psi](z)$ *on* $z \in \mathbb{R}$, *if* $\varphi, \psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$ *with* $\varphi(z) \le \psi(z)$ *on* $z \in \mathbb{R}$;
(ii) $0 \le \Lambda[\psi](z) \le u^*$ *for any* $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$;
(iii) $\Lambda[\psi](z)$ *is nondecreasing on* $z \in \mathbb{R}$, *if* $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$ *is nondecreasing on* $z \in \mathbb{R}$.

Let $\sigma > 0$ be such that $\sigma < \min\{-\chi_1, \chi_2, \frac{b+r}{c}\}$. Denote

$$B_\sigma(\mathbb{R}, \mathbb{R}) = \left\{ \psi \in C(\mathbb{R}, \mathbb{R}) : \sup_{z\in\mathbb{R}} |\psi(z)| \mathrm{e}^{-\sigma|z|} < \infty \right\}$$

with the norm $|\psi|_\sigma = \sup_{z\in\mathbb{R}} |\psi(z)|\mathrm{e}^{-\sigma|z|}$. Then it is easy to verify that $(B_\sigma(\mathbb{R}, \mathbb{R}), |\cdot|_\sigma)$ is a Banach space.

Lemma 2.3. *Assume that* (H1)–(H3) *hold. Then* $\Lambda : B_\sigma(\mathbb{R}, \mathbb{R}) \to B_\sigma(\mathbb{R}, \mathbb{R})$ *is continuous with respect to the norm* $|\cdot|_\sigma$ *in* $B_\sigma(\mathbb{R}, \mathbb{R})$.

Proof. To start with, we show that $Q : C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) \to B_\sigma(\mathbb{R}, \mathbb{R})$ is a well defined map. For any $\psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$, by the conclusion (ii) of Lemma 2.1, we have $|Q[\psi](z)|\mathrm{e}^{-\sigma|z|} \le hu^*$. Then $|Q[\psi]|_\sigma = \sup_{z\in\mathbb{R}} \big|Q[\psi](z)\big|\mathrm{e}^{-\sigma|z|} \le hu^* < \infty$ and thus $Q[\psi] \in B_\sigma(\mathbb{R}, \mathbb{R})$. We now check that $Q : C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) \to B_\sigma(\mathbb{R}, \mathbb{R})$ is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R})$. Indeed, for any $\varphi, \psi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R})$, we have

$$\begin{aligned} |Q[\varphi](z) - Q[\psi](z)|\mathrm{e}^{-\sigma|z|} \le{}& |h - \beta(\varphi(z) + \psi(z)) - U'(\varsigma(z))||\varphi(z) - \ psi(z)|\mathrm{e}^{-\sigma|z|} \\ &+ \mathrm{e}^{-\gamma\tau}|B(\varphi(z - c\tau)) - B(\psi(z - c\tau))|\mathrm{e}^{-\sigma|z|} \\ &+ \frac{r}{c}\mathrm{e}^{-\sigma|z|} \int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)} |U(\varphi(s)) - U(\psi(s))| \mathrm{d}s \end{aligned} \tag{2.10}$$

$$\leq h|\varphi-\psi|_\sigma + \mathrm{e}^{-\gamma\tau}\rho_1|\varphi(z-c\tau)-\psi(z-c\tau)|\mathrm{e}^{-\sigma|z|} + \frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\int_{-\infty}^{z}\mathrm{e}^{\frac{(b+r)}{c}s}|\varphi(s)-\psi(s)|\mathrm{d}s.$$

For any $\varepsilon > 0$, choose $\delta > 0$ such that

$$\delta < \min\left\{\frac{\varepsilon}{3h}, \frac{\varepsilon}{3\rho_1}\mathrm{e}^{(\gamma-\sigma c)\tau}, \frac{\varepsilon(b+r-c\sigma)}{3r\rho_2}\right\}. \tag{2.11}$$

If $|\varphi-\psi|_\sigma < \delta$, then

$$|\varphi(z-c\tau)-\psi(z-c\tau)| \leq \delta\mathrm{e}^{\sigma|z|}\mathrm{e}^{\sigma c\tau} \text{ on } z\in\mathbb{R}. \tag{2.12}$$

Similarly, $|\varphi(z)-\psi(z)| \leq \delta\mathrm{e}^{\sigma|z|}$ on $z\in\mathbb{R}$. Then, for any $z \geq 0$, we have

$$\begin{aligned}
&\frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\int_{-\infty}^{z}\mathrm{e}^{\frac{(b+r)}{c}s}|\varphi(s)-\psi(s)|\mathrm{d}s \leq \frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\delta\int_{-\infty}^{z}\mathrm{e}^{\frac{(b+r)}{c}s}\mathrm{e}^{\sigma|s|}\mathrm{d}s\\
&\quad = \frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\delta\left[\int_{-\infty}^{0}\mathrm{e}^{\frac{(b+r)}{c}s}\mathrm{e}^{-\sigma s}\mathrm{d}s + \int_0^z\mathrm{e}^{\frac{(b+r)}{c}s}\mathrm{e}^{\sigma s}\mathrm{d}s\right]\\
&\quad = \frac{2c\sigma r\rho_2\delta}{(b+r-c\sigma)(b+r+c\sigma)}\mathrm{e}^{-(\sigma+\frac{b+r}{c})z} + \frac{r\rho_2\delta}{b+r+c\sigma}\\
&\quad \leq \frac{2c\sigma r\rho_2\delta}{(b+r-c\sigma)(b+r+c\sigma)} + \frac{r\rho_2\delta}{b+r+c\sigma} = \frac{r\rho_2\delta}{b+r-c\sigma}.
\end{aligned}$$

If $z < 0$, then

$$\begin{aligned}
&\frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\int_{-\infty}^{z}\mathrm{e}^{\frac{(b+r)}{c}s}|\varphi(s)-\psi(s)|\mathrm{d}s \leq \frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\delta\int_{-\infty}^{z}\mathrm{e}^{(\frac{b+r}{c}-\sigma)s}\mathrm{d}s\\
&\quad = \frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\frac{c\delta}{b+r-c\sigma}\mathrm{e}^{(\frac{b+r}{c}-\sigma)z} = \frac{r\rho_2\delta}{b+r-c\sigma}.
\end{aligned}$$

Hence, for any $z\in\mathbb{R}$, we obtain

$$\frac{r}{c}\mathrm{e}^{-\sigma|z|}\mathrm{e}^{-\frac{(b+r)}{c}z}\rho_2\int_{-\infty}^{z}\mathrm{e}^{\frac{(b+r)}{c}s}|\varphi(s)-\psi(s)|\mathrm{d}s \leq \frac{r\rho_2\delta}{b+r-c\sigma}. \tag{2.13}$$

In view of (2.10)–(2.13), we have

$$|Q[\varphi](z)-Q[\psi](z)|\mathrm{e}^{-\sigma|z|} \leq h|\varphi-\psi|_\sigma + \rho_1\delta\mathrm{e}^{(\sigma c-\gamma)\tau} + \frac{r\rho_2\delta}{b+r-c\sigma} < \varepsilon \quad \text{if} \quad |\varphi_1-\varphi_2|_\sigma < \delta.$$

Hence, Q is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R})$.

We now show that Λ is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R})$. For any $z\geq 0$, we get

$$\begin{aligned}
&|\Lambda[\varphi](z)-\Lambda[\psi](z)|\\
&\leq \frac{1}{D_2(\chi_2-\chi_1)}\left\{\int_{-\infty}^{z}\mathrm{e}^{\chi_1(z-s)}|Q[\varphi](s)-Q[\psi](s)|\mathrm{d}s\right.\\
&\qquad \left. + \int_z^{\infty}\mathrm{e}^{\chi_2(z-s)}|Q[\varphi](s)-Q[\psi](s)|\mathrm{d}s\right\}
\end{aligned}$$

$$\leq \frac{1}{D_2(\chi_2-\chi_1)}\left\{\int_{-\infty}^{z} e^{\chi_1(z-s)+\sigma|s|}ds + \int_{z}^{\infty} e^{\chi_2(z-s)+\sigma|s|}ds\right\}|Q[\varphi]-Q[\psi]|_\sigma$$
$$= \frac{1}{D_2(\chi_2-\chi_1)}\left\{\int_{-\infty}^{0} e^{\chi_1(z-s)-\sigma s}ds\right.$$
$$\left. + \int_{0}^{z} e^{\chi_1(z-s)+\sigma s}ds + \int_{z}^{\infty} e^{\chi_2(z-s)+\sigma s}ds\right\}|Q[\varphi]-Q[\psi]|_\sigma$$
$$= \frac{1}{D_2(\chi_2-\chi_1)}\left[\frac{\chi_2-\chi_1}{(\sigma-\chi_1)(\chi_2-\sigma)}e^{\sigma z} + \frac{2\sigma}{\chi_1^2-\sigma^2}e^{\chi_1 z}\right]|Q[\varphi]-Q[\psi]|_\sigma.$$

Therefore, for $z \geq 0$, it follows

(2.14)
$$|\Lambda[\varphi](z)-\Lambda[\psi](z)|e^{-\sigma|z|} \leq \frac{1}{D_2(\chi_2-\chi_1)}\left[\frac{\chi_2-\chi_1}{(\sigma-\chi_1)(\chi_2-\sigma)} + \frac{2\sigma}{\chi_1^2-\sigma^2}\right]|Q[\varphi]-Q[\psi]|_\sigma.$$

If $z < 0$, then

$$|\Lambda[\varphi](z)-\Lambda[\psi](z)|$$
$$\leq \frac{1}{D_2(\chi_2-\chi_1)}\left\{\int_{-\infty}^{z} e^{\chi_1(z-s)-\sigma s}ds + \int_{z}^{0} e^{\chi_2(z-s)-\sigma s}ds\right.$$
$$\left. + \int_{0}^{\infty} e^{\chi_2(z-s)+\sigma s}\right\}|Q[\varphi]-Q[\psi]|_\sigma$$
$$= \frac{1}{D_2(\chi_2-\chi_1)}\left[\frac{\chi_2-\chi_1}{-(\sigma+\chi_1)(\chi_2+\sigma)}e^{-\sigma z} + \frac{2\sigma}{\chi_2^2-\sigma^2}e^{\chi_2 z}\right]|Q[\varphi]-Q[\psi]|_\sigma.$$

Hence, for $z < 0$, it follows

(2.15)
$$|\Lambda[\varphi](z)-\Lambda[\psi](z)|e^{-\sigma|z|} \leq \frac{1}{D_2(\chi_2-\chi_1)}\left[\frac{\chi_2-\chi_1}{-(\sigma+\chi_1)(\chi_2+\sigma)} + \frac{2\sigma}{\chi_2^2-\sigma^2}\right]|Q[\varphi]-Q[\psi]|_\sigma.$$

(2.14)–(2.15) lead to that Λ is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R})$. □

Definition 2.4. A function $\varpi \in C(\mathbb{R},\mathbb{R})$ is called a weak upper-solution of (2.6) if it is twice differentiable on $\mathbb{R}$ except for on $\Sigma := \{z_1, z_2, \ldots, z_m\}$, and satisfies

$$D_2\varpi''(z) - c\varpi'(z) + F[\varpi](z) \leq 0 \text{ on } z \in \mathbb{R}\setminus\Sigma. \tag{2.16}$$

A weak lower-solution of (2.6) is defined in a similar way with a reversing inequality in (2.16).

Definition 2.5. A function $\vartheta \in C^2(\mathbb{R},\mathbb{R})$ is called an upper solution of (2.6) if it satisfies

$$D_2\vartheta''(z) - c\vartheta'(z) + F[\vartheta](z) \leq 0 \text{ on } z \in \mathbb{R}. \tag{2.17}$$

A lower solution of (2.6) is defined in a similar way with a reversing inequality in (2.17).

Lemma 2.6. *If $\varpi \in C(\mathbb{R}, \mathbb{R})$ is a weak upper-solution (weak lower-solution) of (2.6) and $\varpi'(z^+) \leq (\geq) \varpi'(z^-)$ on $z \in \mathbb{R}$, then $\Lambda[\varpi](z) \leq (\geq) \varpi(z)$ on $z \in \mathbb{R}$, and further $\vartheta = \Lambda\varpi$ is an upper (a lower) solution of (2.6).*

Proof. We only verify the conclusion for the upper solution. Assume that $z_0 = -\infty < z_1 < z_2 < \cdots < z_m < z_{m+1} = \infty$. According to Lemma 2.2 and the definition of $\Lambda\varpi$, we see that $\Lambda\varpi \in C_{[0,u^*]}(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$. Since

$$-D_2\varpi''(z) + c\varpi'(z) + h\varpi(z) \geq Q[\varpi](z) \text{ on } z \in \mathbb{R} \setminus \Sigma,$$

for any $z \in (z_k, z_{k+1}), k = 0, 1, \ldots, m$, we obtain

$$\begin{aligned}
\Lambda[\varpi](z) &\leq \frac{1}{D_2(\chi_2 - \chi_1)} \Bigg\{ \int_{-\infty}^{z} \mathrm{e}^{\chi_1(z-s)}[-D_2\varpi''(s) + c\varpi'(s) + h\varpi(s)]\mathrm{d}s \\
&\qquad + \int_{z}^{\infty} \mathrm{e}^{\chi_2(z-s)}[-D_2\varpi''(s) + c\varpi'(s) + h\varpi(s)]\mathrm{d}s \Bigg\} \\
&= \varpi(z) + \frac{1}{\chi_2 - \chi_1} \Bigg\{ \sum_{l=1}^{k} \mathrm{e}^{\chi_1(z-z_l)}[\varpi'(z_l^+) - \varpi'(z_l^-)] \\
&\qquad + \sum_{l=k+1}^{m} \mathrm{e}^{\chi_2(z-z_l)}[\varpi'(z_l^+) - \varpi'(z_l^-)] \Bigg\} \\
&\leq \varpi(z).
\end{aligned} \tag{2.18}$$

Furthermore, since $\Lambda[\varpi]$ and ϖ is continuous on $z \in \mathbb{R}$, letting $z \to z_k^+$ in the inequality (2.18), we have

$$\Lambda[\varpi](z_k) \leq \varpi(z_k)$$

for any $z_k \in \Sigma$.

Therefore, $\Lambda[\varpi](z) \leq \varpi(z)$ on $z \in \mathbb{R}$, which together with Lemma 2.1 and (2.9) implies that

$$\begin{aligned}
&D_2(\Lambda[\varpi])''(z) - c(\Lambda[\varpi])'(z) + F[\Lambda[\varpi]](z) \\
&= D_2(\Lambda[\varpi])''(z) - c(\Lambda[\varpi])'(z) - h\Lambda[\varpi](z) + Q[\Lambda[\varpi]](z) \\
&\leq D_2(\Lambda[\varpi])''(z) - c(\Lambda[\varpi])'(z) - h\Lambda[\varpi](z) + Q[\varpi](z) = 0.
\end{aligned}$$

Thus, $\Lambda\varpi$ is an upper solution of (2.6). □

Remark 2.7. Let

$$\bar{w}(z) := \begin{cases} \mathrm{e}^{-z}, & z > 0, \\ \mathrm{e}^{z}, & z \leq 0. \end{cases} \tag{2.19}$$

Boumenir and Nguyen in Remark 5 [4] thought that the function $\bar{w}$ in (2.19) could serve as a counterexample to the identity (see (2.18) here or the proof of Lemma 2.5

in [10])

$$\frac{1}{D_2(\chi_2-\chi_1)}\left[\int_{-\infty}^{z} \mathrm{e}^{\chi_1(z-s)}\varphi(s)\mathrm{d}s+\int_{z}^{\infty} \mathrm{e}^{\chi_2(z-s)}\varphi(s)\mathrm{d}s\right] \tag{2.20}$$
$$= w(z)+\frac{1}{\chi_2-\chi_1}\left\{\sum_{l=1}^{k} \mathrm{e}^{\chi_1(z-z_l)}[w'(z_l^+)-w'(z_l^-)] + \sum_{l=k+1}^{m} \mathrm{e}^{\chi_2(z-z_l)}[w'(z_l^+)-w'(z_l^-)]\right\},$$

where w is a piecewise C^2 solution of

$$D_2 w''(z)-cw'(z)-hw(z)=-\varphi(z) \qquad \text{on} \quad \mathbb{R}\setminus\{z_1,z_2,\cdots,z_m\}.$$

In the particular case when $D_2=1$, $\chi_1=-1$, $\chi_2=1$, $c=0$, $h=1$, $m=1$, $z_1=0$, then the function $\bar{w}$ in (2.19) defines a solution of $w''(z)-w(z)=0$ for all $z\neq 0$. Boumenir and Nguyen [4] concluded that the left-hand side of (2.20) was zero since $\varphi=0$ while the right-hand side of (2.20) was

$$\bar{w}(z)+\frac{1}{2}\left\{\sum_{l=1}^{k} \mathrm{e}^{-z}[\bar{w}'(0^+)-\bar{w}'(0^-)]+\sum_{l=k+1}^{m} \mathrm{e}^{z}[\bar{w}'(0^+)-\bar{w}'(0^-)]\right\}$$
$$=\bar{w}(z)+\frac{1}{2}\left[\sum_{l=1}^{k} \mathrm{e}^{-z}(-2)+\sum_{l=k+1}^{m} \mathrm{e}^{z}(-2)\right]$$
$$=\bar{w}(z)-\mathrm{e}^{-z}-\mathrm{e}^{z}\neq 0.$$

However, they neglected the behavior of the segmented function $\bar{w}$ in (2.19) and made a mistake in the identity deduced above. In the Appendix of this article, we will show that the function $\bar{w}$ in (2.19) cannot serve as a counterexample to the identity (2.20) here.

Theorem 2.8. *Assume that* (H1)–(H3) *hold. If* (2.6) *has an upper solution* $\bar{\vartheta}\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})\cap C^2(\mathbb{R},\mathbb{R})$ *and a lower solution* $\underline{\vartheta}\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})\cap C^2(\mathbb{R},\mathbb{R})$ *such that* $\bar{\vartheta}\neq\hat{u}^*, \underline{\vartheta}\neq\hat{0}$ *and* $\sup_{s\le z}\underline{\vartheta}(s)\le\bar{\vartheta}(z)$ *on* $z\in\mathbb{R}$. *Then there exists at least one monotone solution of* (2.6) *satisfying* (2.8), *where* $\hat{u}^*$ *and* $\hat{0}$ *denote the constant functions on* $z\in\mathbb{R}$ *with the value* u^* *and* 0, *respectively.*

Proof. Define the following set

$$\Gamma=\left\{\psi\in C_{[0,u^*]}(\mathbb{R},\mathbb{R});\ \begin{array}{l}\text{(i) } \psi \text{ is nondecreasing on } \mathbb{R};\\ \text{(ii) } \underline{\vartheta}(z)\le\psi(z)\le\bar{\vartheta}(z) \text{ on } z\in\mathbb{R};\\ \text{(iii) } |\psi(z_1)-\psi(z_2)|\le\frac{4hu^*}{D_2(\chi_2-\chi_1)}|z_1-z_2| \text{ for } z_1,z_2\in\mathbb{R}.\end{array}\right\}$$

We first show that Γ is a subset of $B_\sigma(\mathbb{R},\mathbb{R})$. For any $\psi\in\Gamma$, we have $|\psi(z)|\le u^*$ on $z\in\mathbb{R}$. It then follows that $|\psi|_\sigma=\sup_{z\in\mathbb{R}}|\psi(z)|\mathrm{e}^{-\sigma|z|}\le u^*<\infty$ and thus $\Gamma\subseteq B_\sigma(\mathbb{R},\mathbb{R})$. Also, Γ is a convex closed subset in $B_\sigma(\mathbb{R},\mathbb{R})$. Moreover, we can verify that Γ is a compact set in $B_\sigma(\mathbb{R},\mathbb{R})$. Indeed, let$\{\psi_n(z)\}\subset\Gamma$ be a sequence. For any given $\varepsilon>0$, choose $T>0$ large enough such that

$$\sup_{|z|\geq T}|\psi_i(z)-\psi_j(z)|\mathrm{e}^{-\sigma|z|}\leq 2u^*\mathrm{e}^{-\sigma T}<\frac{\varepsilon}{2}. \tag{2.21}$$

Since $\{\psi_n(z)\}$ is uniformly bounded and equi-continuous on $[-T,T]$, by Arzerà-Ascoli theorem, $\{\psi_n(z)\}$ has a subsequence which is convergent on $[-T,T]$ with respect to the supremum norm. For convenience, we still denote this subsequence by $\{\psi_n(z)\}$. Then $\{\psi_n(z)\}$ is a Cauchy sequence on $[-T,T]$ with respect to the supremum norm. Therefore, there exists $N>0$ such that

$$\sup_{|z|\leq T}|\psi_i(z)-\psi_j(z)|\mathrm{e}^{-\sigma|z|}\leq\sup_{|z|\leq T}|\psi_i(z)-\psi_j(z)|<\frac{\varepsilon}{2}\ \text{ for } i,j>N.$$

This, together with (2.21), leads to the conclusion that $\{\psi_n(z)\}$ is a Cauchy sequence in $B_\sigma(\mathbb{R},\mathbb{R})$. As $B_\sigma(\mathbb{R},\mathbb{R})$ is a Banach space, hence $\{\psi_n(z)\}$ is convergent in $B_\sigma(\mathbb{R},\mathbb{R})$.

In view of the definition of upper solution and lower solution, we can show that

$$\Lambda[\bar{\vartheta}](z)\leq\bar{\vartheta}(z),\quad \Lambda[\underline{\vartheta}](z)\geq\underline{\vartheta}(z)\text{ on } z\in\mathbb{R}.$$

Let $\varphi(z)=\sup_{s\leq z}\underline{\vartheta}(s)$. Then $\varphi(z)$ is nondecreasing on $\mathbb{R}$ and

$$\underline{\vartheta}(z)\leq\varphi(z)\leq\bar{\vartheta}(z)\text{ on } z\in\mathbb{R}.$$

By Lemma 2.2 and the above inequalities, we obtain

$$\underline{\vartheta}(z)\leq\Lambda[\underline{\vartheta}](z)\leq\Lambda[\varphi](z)\leq\Lambda[\bar{\vartheta}](z)\leq\bar{\vartheta}(z)\text{ on } z\in\mathbb{R}. \tag{2.22}$$

For any $z_1,z_2\in\mathbb{R}$, assuming that $z_1\geq z_2$, recall that $|Q[\varphi](z)|\leq hu^*$ on $z\in\mathbb{R}$ (see Lemma 2.1), $\chi_1<0$ and then

$$\begin{aligned}
&\Big|\int_{-\infty}^{z_1}\mathrm{e}^{\chi_1(z_1-s)}Q[\varphi](s)\mathrm{d}s-\int_{-\infty}^{z_2}\mathrm{e}^{\chi_1(z_2-s)}Q[\varphi](s)\mathrm{d}s\Big|\\
&\leq\Big|\int_{z_2}^{z_1}\mathrm{e}^{\chi_1(z_1-s)}Q[\varphi](s)\mathrm{d}s\Big|+\Big|\int_{-\infty}^{z_2}[\mathrm{e}^{\chi_1(z_1-s)}-\mathrm{e}^{\chi_1(z_2-s)}]Q[\varphi](s)\mathrm{d}s\Big|\\
&\leq hu^*(z_1-z_2)+\int_{-\infty}^{z_2}[\mathrm{e}^{\chi_1(z_2-s)}-\mathrm{e}^{\chi_1(z_1-s)}]Q[\varphi](s)\mathrm{d}s\\
&\leq hu^*(z_1-z_2)+hu^*\int_{-\infty}^{z_2}[\mathrm{e}^{\chi_1(z_2-s)}-\mathrm{e}^{\chi_1(z_1-s)}]\mathrm{d}s\\
&=hu^*(z_1-z_2)+hu^*(\mathrm{e}^{\chi_1z_2}-\mathrm{e}^{\chi_1z_1})\int_{-\infty}^{z_2}\mathrm{e}^{-\chi_1s}\mathrm{d}s\\
&=hu^*(z_1-z_2)+hu^*(\mathrm{e}^{\chi_1z_2}-\mathrm{e}^{\chi_1z_1})\cdot(-\frac{1}{\chi_1})\mathrm{e}^{-\chi_1z_2}\\
&=hu^*(z_1-z_2)+\frac{hu^*}{-\chi_1}\left[1-\mathrm{e}^{\chi_1(z_1-z_2)}\right]\\
&\leq hu^*(z_1-z_2)+hu^*(z_1-z_2)=2hu^*(z_1-z_2),
\end{aligned}$$

where we use the inequality $e^x \geq 1 + x$ for any $x \in \mathbb{R}$. In a similar way, we have

$$\left|\int_{z_1}^{\infty} \mathrm{e}^{\chi_1(z_1-s)}Q[\varphi](s)\mathrm{d}s - \int_{z_2}^{\infty} \mathrm{e}^{\chi_1(z_2-s)}Q[\varphi](s)\mathrm{d}s\right| \leq 2hu^*(z_1 - z_2).$$

Therefore,

$$\begin{aligned}\big|\Lambda[\varphi](z_1) - \Lambda[\varphi](z_2)\big| &= \frac{1}{D_2(\chi_2 - \chi_1)}\Big|\int_{-\infty}^{z_1} \mathrm{e}^{\chi_1(z_1-s)}Q[\varphi](s)\mathrm{d}s - \int_{-\infty}^{z_2} \mathrm{e}^{\chi_1(z_2-s)}Q[\varphi](s)\mathrm{d}s \\ &\quad + \int_{z_1}^{\infty} \mathrm{e}^{\chi_1(z_1-s)}Q[\varphi](s)\mathrm{d}s - \int_{z_2}^{\infty} \mathrm{e}^{\chi_1(z_2-s)}Q[\varphi](s)\mathrm{d}s\Big| \\ &\leq \frac{4hu^*}{D_2(\chi_2 - \chi_1)}|z_1 - z_2| \quad \text{on } z \in \mathbb{R}.\end{aligned}$$

Now we have shown $\Lambda[\varphi] \in \Gamma$, and thus Γ is not empty. On the other hand, we have from (2.22) and Lemma 2.2 that $\Lambda[\Gamma] \subset \Gamma$. Therefore, using Schauder fixed point theorem, we conclude that Λ has a fixed point $\psi \in \Gamma$ which is a solution of (2.6).

In what follows, we show that ψ also satisfies (2.8). Since ψ is nondecreasing and $\Lambda[\psi] = \psi$, we get

$$\begin{aligned}0 \leq \psi_{-\infty} &:= \lim_{z\to-\infty} \psi(z) = \lim_{z\to-\infty} \Lambda[\psi](z) \leq \inf_{z\in\mathbb{R}} \bar{\vartheta}(z), \\ \sup_{z\in\mathbb{R}} \underline{\vartheta}(z) \leq \psi_{\infty} &:= \lim_{z\to\infty} \psi(z) = \lim_{z\to\infty} \Lambda[\psi](z) \leq u^*.\end{aligned}$$

Let $\hat{\psi}_{-\infty}$ and $\hat{\psi}_{\infty}$ be the constant functions on $z \in \mathbb{R}$ with the value $\psi_{-\infty}$ and ψ_{∞}, respectively. We can show

$$\hat{\psi}_{-\infty} = \Lambda[\hat{\psi}_{-\infty}], \quad \hat{\psi}_{\infty} = \Lambda[\hat{\psi}_{\infty}]. \tag{2.23}$$

Indeed, we only need to check

$$\lim_{z\to-\infty} \Lambda[\psi](z) = \Lambda[\hat{\psi}_{-\infty}], \quad \lim_{z\to\infty} \Lambda[\psi](z) = \Lambda[\hat{\psi}_{\infty}]. \tag{2.24}$$

We only verify the first equality in (2.24). Making use of L'Hôspital's rule again, we obtain

$$\begin{aligned}\lim_{z\to-\infty} \Lambda[\psi](z) &= \frac{1}{D_2(\chi_2 - \chi_1)} \lim_{z\to-\infty} \left\{ \frac{\int_{-\infty}^{z} \mathrm{e}^{-\chi_1 s}Q[\psi](s)\mathrm{d}s}{\mathrm{e}^{-\chi_1 z}} + \frac{\int_{z}^{\infty} \mathrm{e}^{-\chi_2 s}Q[\psi](s)\mathrm{d}s}{\mathrm{e}^{-\chi_2 z}} \right\} \\ &= \frac{1}{D_2(\chi_2 - \chi_1)} \left\{ \lim_{z\to-\infty} \frac{Q[\psi](z)}{-\chi_1} + \lim_{z\to-\infty} \frac{-Q[\psi](z)}{-\chi_2} \right\} \\ &= \frac{1}{D_2(\chi_2 - \chi_1)} \cdot \frac{\chi_1 - \chi_2}{\chi_1\chi_2} \lim_{z\to-\infty} Q[\psi](z) \\ &= \frac{-1}{D_2\chi_1\chi_2} \lim_{z\to-\infty} Q[\psi](z).\end{aligned} \tag{2.25}$$

On the other hand, we have

$$\Lambda[\hat{\psi}_{-\infty}] = \frac{Q[\hat{\psi}_{-\infty}]}{D_2(\chi_2 - \chi_1)} \left\{ \int_{-\infty}^{z} \mathrm{e}^{\chi_1(z-s)}\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{\chi_2(z-s)} \right\} \mathrm{d}s \tag{2.26}$$

$$= \frac{Q[\hat{\psi}_{-\infty}]}{D_2(\chi_2-\chi_1)} \cdot \frac{\chi_1-\chi_2}{\chi_1\chi_2} = \frac{-Q[\hat{\psi}_\infty]}{D_2\chi_1\chi_2}.$$

In view of L'Hôspital's rule, we get

$$\lim_{z\to-\infty} \frac{r}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{b+r}{c}(z-s)}U(\psi(s))\mathrm{d}s = \lim_{z\to-\infty}\frac{r}{b+r}U(\psi(z)) = \frac{r}{b+r}U(\psi_{-\infty}).$$

Since

$$Q[\psi](z) = h\psi(z)+\mathrm{e}^{-\gamma\tau}B(\psi(z-c\tau))-\beta\psi^2(z)-U(\psi(z))+\frac{r}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{(b+r)}{c}(z-s)}U(\psi(s))\mathrm{d}s,$$

we obtain

$$\lim_{z\to-\infty} Q[\psi](z) = h\psi_{-\infty}+\mathrm{e}^{-\gamma\tau}B(\psi_{-\infty})-\beta\psi_{-\infty}^2-U(\psi_{-\infty})+\frac{r}{b+r}U(\psi_{-\infty}) = Q[\hat{\psi}_{-\infty}].$$

Now we have from (2.25)–(2.26) that (2.24), and thus (2.23) holds. That is, $\hat{\psi}_{-\infty}$ and $\hat{\psi}_\infty$ are fixed points of Λ. Since Λ has only two constant fixed points $\hat{0}$ and $\hat{u}^*$, we have $\hat{\psi}_{-\infty}=\hat{0}$ and $\hat{\psi}_\infty=\hat{u}^*$. The proof is complete. □

Theorem 2.9. *Assume that* (H1)–(H3) *hold. If* (2.6) *has a weak upper-solution* $\bar{\psi}\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$ *and a weak lower-solution* $\underline{\psi}\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$ *such that*

(i) $\bar{\psi}\neq\hat{u}^*, \underline{\psi}\neq\hat{0}$ *and* $\sup_{s\le z}\underline{\psi}(s)\le\bar{\psi}(z)$ *on* $z\in\mathbb{R}$,

(ii) $\bar{\psi}'(z^+)\le\bar{\psi}'(z^-)$ *and* $\underline{\psi}'(z^+)\ge\underline{\psi}'(z^-)$ *on* $z\in\mathbb{R}$,

then there exists at least one monotone solution of (2.6) *satisfying* (2.8).

Proof. Let $\bar{\vartheta}(z)=\Lambda[\bar{\psi}](z), \underline{\vartheta}(z)=\Lambda[\underline{\psi}](z)$. Then by Lemma 2.6 and (ii) of Theorem 2.9 , we see that $\bar{\vartheta}(z)\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$, $\underline{\vartheta}(z)\in C_{[0,u^*]}(\mathbb{R},\mathbb{R})$ and $\bar{\vartheta},\underline{\vartheta}$ are a pair of upper and lower solutions of (2.6). Furthermore, they satisfy

$$\underline{\psi}(z)\le\underline{\vartheta}(z), \quad \bar{\vartheta}(z)\le\bar{\psi}(z) \quad \text{on } z\in\mathbb{R}.$$

Define $\breve{\psi}(z)=\sup_{s\le z}\underline{\psi}(s)$. Then $\breve{\psi}(z)$ is nondecreasing on $\mathbb{R}$ and $\underline{\psi}(z)\le\breve{\psi}(z)\le\bar{\psi}(z)$ on $z\in\mathbb{R}$. It follows from Lemma 2.2 and (i) of Theorem 2.9 that $\Lambda[\breve{\psi}](z)$ is nondecreasing on $\mathbb{R}$, and

$$\sup_{s\le z}\underline{\vartheta}(s)\le\sup_{s\le z}\Lambda[\breve{\psi}](s)=\Lambda[\breve{\psi}](z)\le\Lambda[\bar{\psi}](z)=\bar{\vartheta}(z) \quad \text{on } z\in\mathbb{R}.$$

According to Theorem 2.8, the proof is complete. □

Corollary 2.10. *Suppose that the conditions in Theorem 2.8 or 2.9 hold, then there exists at least one monotone solution of (2.2) satisfying (2.3).*

Proof. Since (2.6) exists at least a monotone solution $\psi(z)$ satisfying (2.8), we only need to show that ϕ is nondecreasing and satisfies the boundary conditions. Let $\theta>0$ be given. By (2.4) and the nondecreasing property of ψ and U, we see that

$$\phi(z+\theta)-\phi(z)$$

$$= \frac{1}{c}\left[\int_{-\infty}^{z+\theta} \mathrm{e}^{-\frac{(b+r)}{c}(z+\theta-s)}U(\psi(s))\mathrm{d}s - \int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)}U(\psi(s))\mathrm{d}s\right].$$
$$= \frac{1}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)}[U(\psi(s+\theta)) - U(\psi(s))]\mathrm{d}s \geq 0.$$

Substituting $\psi(z)$ into (2.4) and applying L'Hôspital's rule, then $\lim_{z\to-\infty}\phi(z) = 0$ and $\lim_{z\to\infty}\phi(z) = y^*$. This completes the proof. □

In order to construct a pair of appropriate weak upper and lower solutions for (2.6), we linearize (2.6) at $\psi = 0$ to obtain

$$D_2\psi''(z) - c\psi'(z) + \mathrm{e}^{-\gamma\tau}B'(0)\psi(z - c\tau) - U'(0)\psi(z) + \frac{rU'(0)}{c}\int_{-\infty}^{z} \mathrm{e}^{-\frac{(b+r)}{c}(z-s)}\psi(s)\mathrm{d}s = 0.$$

The corresponding characteristic equation is

$$\triangle(\lambda, c) := D_2\lambda^2 - c\lambda + \mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda c\tau} - U'(0) + \frac{rU'(0)}{c\lambda + b + r} = 0.$$

This equation is central to the identification of the speeds $c > 0$ for which wave solutions exist. By direct calculations, we have

$$\triangle(\lambda, 0) = D_2\lambda^2 + \mathrm{e}^{-\gamma\tau}B'(0) - \frac{b}{b+r}U'(0) > 0 \text{ for any } \lambda \geq 0 \text{ if (H4) holds,}$$
$$\frac{\partial^2\triangle(\lambda, c)}{\partial\lambda^2} = 2D_2 + c^2\tau^2\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda c\tau} + \frac{2c^2rU'(0)}{(c\lambda + b + r)^3} > 0 \text{ for any } \lambda > 0,$$
$$\triangle(0, c) = \mathrm{e}^{-\gamma\tau}B'(0) - \frac{b}{b+r}U'(0) > 0 \text{ if (H4) holds,}$$
$$\triangle(\lambda, \infty) = -\infty \text{ for any given } \lambda > 0,$$
$$\triangle(\infty, c) = \infty \text{ for any given } c > 0,$$
$$\frac{\partial\triangle(\lambda, c)}{\partial c} = -\lambda - \lambda\tau\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda c\tau} - \frac{rU'(0)\lambda}{(c\lambda + b + r)^2} < 0 \text{ for } \lambda > 0.$$

Thus, we obtain the following observations.

Lemma 2.11. *Assume that* (H4) *holds, then there exists a pair of* (λ^*, c^*) *such that*

(i) $\triangle(\lambda^*, c^*) = 0, \frac{\partial\triangle(\lambda^*, c^*)}{\partial\lambda} = 0$;
(ii) $\triangle(\lambda, c) > 0$ *for* $0 < c < c^*$ *and* $\lambda > 0$;
(iii) $\triangle(\lambda, c) = 0$ *has two zeros* $0 < \lambda_1 < \lambda_2 < \infty$ *for* $c > c^*$. *Furthermore, there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *with* $0 < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, *we have* $\triangle(\lambda_1 + \varepsilon, c) < 0$ *for* $c > c^*$.

Let $c > c^*$, define $\bar{\psi}(z) = \min\left\{u^*\mathrm{e}^{\lambda_1 z}, u^*\right\}$ and $\underline{\psi}(z) = \max\left\{0, \mu(1 - M\mathrm{e}^{\varepsilon z})\mathrm{e}^{\lambda_1 z}\right\}$ for all $z \in \mathbb{R}$, where $0 < \mu < u^*$ and $M \geq 1$ is to be determined later.

Lemma 2.12. $\bar{\psi}(z)$ *and* $\underline{\psi}(z)$ *are a pair of weak upper-solution and weak lower-solution of* (2.6).

Proof. For $\bar{\psi}$, we have two cases:
(i) If $z>0$, then $\bar{\psi}(z)=u^*, \bar{\psi}(z-c\tau)\le u^*$. It follows that

$$\begin{aligned}D_2\bar{\psi}''(z)-c\bar{\psi}'(z)+F[\bar{\psi}](z) &\le \mathrm{e}^{-\gamma\tau}B(u^*)-\beta(u^*)^2-U(u^*)\\ &\quad+\frac{r}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{(b+r)}{c}(z-s)}U(u^*)\mathrm{d}s\\ &= by^*-(b+r)y^*+\frac{rU(u^*)}{b+r}=-ry^*+ry^*=0.\end{aligned}$$

(ii) If $z<0$, then $\bar{\psi}(z)=u^*\mathrm{e}^{\lambda_1 z}, \bar{\psi}(z-c\tau)=u^*\mathrm{e}^{\lambda_1(z-c\tau)}$. It follows that

$$\begin{aligned}&D_2\bar{\psi}''(z)-c\bar{\psi}'(z)+F[\bar{\psi}](z)\\ &\le u^*\mathrm{e}^{\lambda_1 z}\left[D_2\lambda_1^2-c\lambda_1+\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda_1 c\tau}\right]-\beta(u^*)^2\mathrm{e}^{2\lambda_1 z}-U'(0)u^*\mathrm{e}^{\lambda_1 z}\\ &\quad+\beta(u^*)^2\mathrm{e}^{2\lambda_1 z}+\frac{r}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{(b+r)}{c}(z-s)}U'(0)u^*\mathrm{e}^{\lambda_1 s}\mathrm{d}s\\ &= u^*\mathrm{e}^{\lambda_1 z}\left[D_2\lambda_1^2-c\lambda_1+\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda_1 c\tau}-U'(0)+\frac{rU'(0)}{c\lambda_1+b+r}\right]\\ &\quad+(\beta-\beta)(u^*)^2\mathrm{e}^{2\lambda_1 z}\\ &= u^*\mathrm{e}^{\lambda_1 z}\triangle(\lambda_1,c)=0.\end{aligned}$$

Hence, $\bar{\psi}(z)$ is a weak upper-solution of (2.6).

Now we verify that $\underline{\psi}(z)$ is a weak lower-solution of (2.6). Let $z_*:=-\frac{1}{\varepsilon}\ln M\le 0$. We have to verify the cases $z>z_*$ and $z<z_*$ separately.

(i) If $z\ge z_*$, then $\underline{\psi}(z)=0$. It follows that

$$\begin{aligned}&D_2\underline{\psi}''(z)-c\underline{\psi}'(z)+F[\underline{\psi}](z)\\ &\quad=\mathrm{e}^{-\gamma\tau}B(\underline{\psi}(z-c\tau))+\frac{r}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{(b+r)}{c}(z-s)}U(\underline{\psi}(s))\mathrm{d}s\ge 0.\end{aligned}$$

(ii) If $z<z_*$, then $\underline{\psi}(z)=\mu(1-M\mathrm{e}^{\varepsilon z})\mathrm{e}^{\lambda_1 z}$. For any $\mu<u^*$, it is easy to see that (see [14, p.444]) $\underline{\psi}^2(z)\le(\mu)^2\mathrm{e}^{(\lambda_1+\varepsilon)z}\le(u^*)^2\mathrm{e}^{(\lambda_1+\varepsilon)z}$ for any $z\in\mathbb{R}$. Thus

$$\begin{aligned}&D_2\underline{\psi}''(z)-c\underline{\psi}'(z)+F[\underline{\psi}](z)\\ &\ge D_2\underline{\psi}''(z)-c\underline{\psi}'(z)+\mathrm{e}^{-\gamma\tau}B'(0)\underline{\psi}(z-c\tau)-\mathrm{e}^{-\gamma\tau}\kappa\underline{\psi}^2(z-c\tau)-\beta\underline{\psi}^2(z)\\ &\quad-U'(0)\underline{\psi}(z)+\frac{r}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{(b+r)}{c}(z-s)}\left[U'(0)\underline{\psi}(s)-\beta\underline{\psi}^2(s)\right]\mathrm{d}s\\ &\ge\mu\mathrm{e}^{\lambda_1 z}\left[D_2\lambda_1^2-c\lambda_1+\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-\lambda_1 c\tau}-U'(0)+\frac{rU'(0)}{c\lambda_1+b+r}\right]\\ &\quad-M\mu\mathrm{e}^{(\lambda_1+\varepsilon)z}\Bigg[D_2(\lambda_1+\varepsilon)^2-c(\lambda_1+\varepsilon)+\mathrm{e}^{-\gamma\tau}B'(0)\mathrm{e}^{-(\lambda_1+\varepsilon)c\tau}\\ &\quad-U'(0)+\frac{rU'(0)}{c(\lambda_1+\varepsilon)+b+r}\Bigg]\end{aligned}$$

$$- \mathrm{e}^{-\gamma\tau}\kappa\mu^2 \left[1 - M\mathrm{e}^{\varepsilon(z-c\tau)}\right]^2 \mathrm{e}^{2\lambda_1(z-c\tau)} - \beta\mu^2(1 - M\mathrm{e}^{\varepsilon z})^2\mathrm{e}^{2\lambda_1 z} - \frac{\beta r(u^*)^2\mathrm{e}^{(\lambda_1+\varepsilon)z}}{c(\lambda_1+\varepsilon)+b+r}$$

$$\geq \mu\mathrm{e}^{\lambda_1 z}\triangle(\lambda_1, c) - M\mu\mathrm{e}^{(\lambda_1+\varepsilon)z}\triangle(\lambda_1+\varepsilon, c) - (\mathrm{e}^{-\gamma\tau}\kappa+\beta)\mu^2\mathrm{e}^{2\lambda_1 z} - \frac{\beta r(u^*)^2\mathrm{e}^{(\lambda_1+\varepsilon)z}}{b+r}.$$

Fix any $0 < \varepsilon \leq \lambda_1$ such that $\lambda_1 < \varepsilon + \lambda_1 < 2\lambda_1$ and $\triangle(\lambda_1+\varepsilon, c) < 0$. Note $z < z^* < 0$, then $\mathrm{e}^{2\lambda_1 z} \leq \mathrm{e}^{(\varepsilon+\lambda_1)z}$. Hence

$$\begin{aligned}
&D_2\underline{\psi}''(z) - c\underline{\psi}'(z) + F[\underline{\psi}](z)\\
&\quad\geq -M\mu\mathrm{e}^{(\lambda_1+\varepsilon)z}\triangle(\lambda_1+\varepsilon, c) - \mu^2\mathrm{e}^{(\varepsilon+\lambda_1)z}(\mathrm{e}^{-\gamma\tau}\kappa+\beta) - \frac{\beta r(u^*)^2\mathrm{e}^{(\lambda_1+\varepsilon)z}}{b+r}\\
&\quad= \mathrm{e}^{(\lambda_1+\varepsilon)z}\left[-\mu M\triangle(\lambda_1+\varepsilon, c) - \mu^2(\mathrm{e}^{-\gamma\tau}\kappa+\beta) - \frac{\beta r(u^*)^2}{b+r}\right] \geq 0,
\end{aligned}$$

provided M is sufficiently large. By cases (i) and (ii), it then follows that there exist positive numbers μ, ε and M such that $\underline{\psi}(z)$ is a weak lower-solution of (2.6). This completes the proof. □

Now we give the following theorem.

Theorem 2.13. *Assume that* (H1) – (H4) *hold, then for any* $c \geq c^*$, *(2.1) has a monotone wavefront connecting* $(0, 0)$ *and* (u^*, y^*).

Proof. The conclusion for $c > c^*$ can be obtained from the above discussions. We only need to establish the existence of wavefronts when $c = c^*$. Let $\{c_k\} \subset (c^*, c^*+1)$ with $c_k \to c^*$ as $k \to \infty$. Since $c_k > c^*$, Eq. (2.6) with $c = c_k$ admits a nondecreasing solution $\psi_k(z)$ such that $\lim_{z\to-\infty}\psi_k(z) = 0$ and $\lim_{z\to\infty}\psi_k(z) = u^*$. By the spatial translation invariance of (2.1) [5], we may assume that $\psi_k(0) = \frac{u^*}{2}$ for any $k \geq 1$. Clearly, $|\psi_k(z)| \leq u^*$ for any $z \in \mathbb{R}$, $k \geq 1$, and $\psi_k(z)$ satisfies

$$\psi_k(z) := \frac{1}{D_2(\chi_2^k - \chi_1^k)}\left[\int_{-\infty}^{z} \mathrm{e}^{\chi_1^k(z-s)}Q[\psi_k](s)\mathrm{d}s + \int_z^{\infty} \mathrm{e}^{\chi_2^k(z-s)}Q[\psi_k](s)\mathrm{d}s\right], \tag{2.27}$$

where

$$\chi_1^k = \frac{c_k - \sqrt{{c_k}^2 + 4D_2h}}{2D_2} < 0,\ \ \chi_2^k = \frac{c_k + \sqrt{{c_k}^2 + 4D_2h}}{2D_2} > 0.$$

Since $\{\psi_k(z)\}$ is uniformly bounded and equi-continuous on $\mathbb{R}$, using Arzerà-Ascoli theorem and the standard diagonal method, we can obtain a subsequence of $\{\psi_k(z)\}$, still denoted by $\{\psi_k(z)\}$, such that $\psi_k(z) \to \psi^*(z)$ as $k \to \infty$ uniformly for z in any bounded subset of $\mathbb{R}$. Clearly, $\psi^*(z)$ is nondecreasing and $\psi^*(0) = \frac{u^*}{2}$. By using the dominated convergence theorem and (2.27), it yields that

$$\psi^*(z) := \frac{1}{D_2(\chi_2^* - \chi_1^*)}\left[\int_{-\infty}^{z} \mathrm{e}^{\chi_1^*(z-s)}Q[\psi^*](s)\mathrm{d}s + \int_z^{\infty} \mathrm{e}^{\chi_2^*(z-s)}Q[\psi^*](s)\mathrm{d}s\right],$$

where

$$\chi_1^* = \frac{c^* - \sqrt{c^{*2} + 4D_2h}}{2D_2} < 0,\ \ \chi_2^* = \frac{c^* + \sqrt{c^{*2} + 4D_2h}}{2D_2} > 0.$$

Since $\lim_{z\to\pm\infty}\psi^*(z)$ exist, L'Hôspital rule implies $\lim_{z\to-\infty}\psi^*(z)=0$ and $\lim_{z\to\infty}\psi^*(z)=u^*$. Substituting $\psi^*(z)$ into (2.4) to obtain $\phi^*(z)$, by using L'H$\hat{o}$spital's rule, it is easy to see that $\lim_{z\to-\infty}\phi^*(z)=0$ and $\lim_{z\to\infty}\phi^*(z)=y^*$. Thus the proof is complete. □

In what follows, we turn to study the immature equation. For convenience, we use v to replace u_1. Then the immature equation reads

$$\frac{\partial v}{\partial t}=B(u(t,x))-\gamma v(t,x)-\mathrm{e}^{-\gamma\tau}B(u(t-\tau,x)). \tag{2.28}$$

Assume that $c\geq c^*$. Let $u(t,x)=\psi(z), v(t,x)=V(z)$ with $z=x+ct$. Then we have the following result.

Theorem 2.14. *For any $c\geq c^*$, Eq.* (2.28) *has a traveling wavefront $V(z)$ with $\lim_{z\to-\infty}V(z)=0$ and $\lim_{z\to\infty}V(z)=v^*$ when $u(t,x)=\psi(z)$ with $z=x+ct$.*

Proof. If $u(t,x)=\psi(z)$ with $z=x+ct$, then the wave profile equation of (2.28) is

$$cV'(z)=-\gamma V(z)+B(\psi(z))-\mathrm{e}^{-\gamma\tau}B(\psi(z-c\tau)). \tag{2.29}$$

Eq. (2.29) has a solution given by

$$V(z)=\mathrm{e}^{-\frac{\gamma}{c}(z-z_0)}V(z_0)+\frac{1}{c}\int_{z_0}^{z}\mathrm{e}^{-\frac{\gamma}{c}(z-s)}[B(\psi(s))-\mathrm{e}^{-\gamma\tau}B(\psi(s-c\tau))]\mathrm{d}s$$

for any $z\geq z_0$. Since $V(z)$ and $B(\psi(z))$ are bounded on $\mathbb{R}$, letting $z_0\to-\infty$, we have

$$V(z)=\frac{1}{c}\int_{-\infty}^{z}\mathrm{e}^{-\frac{\gamma}{c}(z-s)}[B(\psi(s))-\mathrm{e}^{-\gamma\tau}B(\psi(s-c\tau))]\mathrm{d}s.$$

Applying L'Hôspital rule, then $\lim_{z\to-\infty}V(z)=0$ and $\lim_{z\to\infty}V(z)=v^*$. □

Finally, combining the Theorems 2.13 and 2.14, we obtain the main result in this article.

Theorem 2.15. *Assume that* (H1)-(H4) *hold, then for any $c\geq c^*$, system* (1.4) *has a traveling wavefront connecting* $(0,0,0)$ *and* (u_1^*,u_2^*,y^*).

3. CONCLUSIONS AND REMARKS

We show the existence of traveling wavefronts as $c\geq c^*$ for a SIS epidemic model with stage structure. From the characteristic equation, we see that c^* depends on the model parameters D_2,γ,τ,r,b and birth function $B(\cdot)$, infection function $U(\cdot)$ in a complicated way. We believe that an appropriate parameter choice can be an implication for control or eradicate the disease transmission. Since system (2.1) generates a monotone semiflow, we can use the theory developed in [9] to show that c^* is the spreading speed for the solution with initial data having compact supports. This, together with Theorem 2.13, implies that c^* is also the minimum wave speed for traveling wavefronts. We will do this work in a forthcoming paper.

4. APPENDIX

Note that the left-hand side of (2.20) is zero since $\varphi = 0$. We will show that the right-hand side of (2.20) must be also zero. We consider two cases.

(i) If $z > 0$, then according to (2.19), $\bar{w}(z) = \mathrm{e}^{-z}$. Since $\bar{w}(z)$ and $\bar{w}'(z)$ are bounded, it follows that

$$\begin{aligned}
&\frac{1}{D_2(\chi_2-\chi_1)}\left[\int_{-\infty}^{z} \mathrm{e}^{\chi_1(z-s)}\varphi(s)\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{\chi_2(z-s)}\varphi(s)\mathrm{d}s\right] \\
&\quad= \frac{1}{2}[\int_{-\infty}^{z} \mathrm{e}^{-(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s] \\
&\quad= \frac{1}{2}[\int_{-\infty}^{0} \mathrm{e}^{-(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s + \int_{0}^{z} \mathrm{e}^{-(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s] \\
&\qquad+ \frac{1}{2}\left[\int_{z}^{\infty} \mathrm{e}^{(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s\right] \\
&\quad= \frac{1}{2}\left[-\mathrm{e}^{-z}\bar{w}'(0^-) + \mathrm{e}^{-z}\bar{w}(0) - \bar{w}'(z) + \mathrm{e}^{-z}\bar{w}'(0^+) + \bar{w}(z) - \mathrm{e}^{-z}\bar{w}(0) + \bar{w}'(z) + \bar{w}(z)\right] \\
&\quad= \frac{1}{2}\left[\mathrm{e}^{-z}(\bar{w}'(0^+) - \bar{w}'(0^-)) + 2\bar{w}(z)\right] \\
&\quad= \frac{1}{2}\left[\mathrm{e}^{-z}(-2) + 2\mathrm{e}^{-z}\right] = 0.
\end{aligned}$$

(ii) If $z \le 0$, then $\bar{w}(z) = \mathrm{e}^{z}$. Similarly, it follows that

$$\begin{aligned}
&\frac{1}{D_2(\chi_2-\chi_1)}\left[\int_{-\infty}^{z} \mathrm{e}^{\chi_1(z-s)}\varphi(s)\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{\chi_2(z-s)}\varphi(s)\mathrm{d}s\right] \\
&\quad= \frac{1}{2}[\int_{-\infty}^{z} \mathrm{e}^{-(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s] \\
&\quad= \frac{1}{2}[\int_{-\infty}^{z} \mathrm{e}^{-(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s] \\
&\qquad+ \frac{1}{2}[\int_{z}^{0} \mathrm{e}^{(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s + \int_{0}^{\infty} \mathrm{e}^{(z-s)}(-\bar{w}''(s)+\bar{w}(s))\mathrm{d}s] \\
&\quad= \frac{1}{2}\left[-\bar{w}'(z) + \bar{w}(z) - \mathrm{e}^{z}\bar{w}'(0^-) + \bar{w}'(z) - \mathrm{e}^{z}\bar{w}(0) + \bar{w}(z) + \mathrm{e}^{z}\bar{w}'(0^+) + \mathrm{e}^{z}\bar{w}(0)\right] \\
&\quad= \frac{1}{2}\left[\mathrm{e}^{z}(\bar{w}'(0^+) - \bar{w}'(0^-)) + 2\bar{w}(z)\right] \\
&\quad= \frac{1}{2}\left[\mathrm{e}^{z}(-2) + 2\mathrm{e}^{z}\right] = 0.
\end{aligned}$$

Therefore, the right-hand side of (2.20) is zero and this implies that the identity (2.20) is correct. The function $\bar{w}$ in (2.19) introduced in Boumenir and Nguyen [4] cannot serve as a counterexample to the identity (2.18) here and thus Lemma 2.6 can be used to prove our main result.

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REFERENCES

[1] S. Ahmad and A.S. Vatsala, Comparison, results of reaction-diffusion equations with delay in abstract cones, *Rend. Sem. Math. Univ. Padova*, 65:19–34, 1981.

[2] W.G. Aiello and H.I. Freedman, A time-delay of single species growth with stage structure, *Math. Biosi.*, 101:47–68, 1990.

[3] J. Al-Omari and S.A. Gourley, Monotone travelling fronts in an age-stuctured reaction-diffusion model for a single species, *J. Math. Biol.*, 45:294–312, 2002.

[4] A. Boumenir and V.M. Nguyen, Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations, *J. Diff. Eqns.*, 244:1551–1570, 2008.

[5] J. Fang, J. Wei and X.-Q. Zhao, Spatial dynamics of a nonlocal and time-delayed reaction-diffusion system, *J. Diff. Eqns.*, 245:2749–2770, 2008.

[6] T. Faria and S. Trofimchuk, Nonmonotone traveling waves in a single species reaction-diffusion equation with delay, *J. Diff. Eqns.*, 228:357–376, 2006.

[7] S.A. Gourley and Y. Kuang, Wavefront and global stability in a time-delayed population model with stage structure, *Proc. R. Soc. Lond. Ser. A.*, 459:1563–1579, 2003.

[8] S.A. Gourley and J.H. Wu, Delayed non-local diffusive sysrems in biological invasion and disease spread, *Nonlinear dynamics and evolution equations, Fields Inst. Commun.*, 48:137–200, Amer. Math. Soc., Providence, RI, 2006.

[9] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with application, *Commun. Pure Appl. Math.*, 60:1–40, 2007. Erratum: 61:137–138, 2008.

[10] S.W. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Diff. Eqns.*, 171:294–314, 2001.

[11] R.H. Martin and H.L. Smith, Abstract functional differential equations and reaction-diffusion systems , *Trans. Am. Math. Soc.*, 321:1–44, 1990.

[12] J.D. Murray, *Mathematical Biology, I & II*, Springer-Verlag, New York, 2002.

[13] L. Rass and J. Radcliffe, *Spatial Deterministic Epidemics, Mathematical Surveys and Monographs*, AMS, Rhode Island, USA, 2003.

[14] H.R. Thieme and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion model, *J. Diff. Eqns.*, 195:430–470, 2003.

[15] Zh.Ch. Wang, W.T. Li and Sh.G. Ruan, Traveling wave fronts in reaction-diffusion systems with spatio-temporal delays, *J. Diff. Eqns.*, 222:185–232, 2006.

[16] P.X. Weng and J.H. Wu, Wavefronts for a non-local reaction-diffusion population model with general distributive maturity, *IMA J. Appl. Math.*, 73:477–495, 2008.

[17] P.X. Weng, H.X. Huang and J.H. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, *IMA J. Appl. Maths.*, 68:409–439, 2003.

[18] P.X. Weng and X.-Q. Zhao, Spreading speed and traveling waves for a mulyi-type SIS epidemic model, *J. Diff. Eqns.*, 229:270–296, 2006.

[19] J.H. Wu and X.F. Zou, Traveling wave fronts of reaction-diffusion systems with delays, *J. Dyna. Diff. Eqns.*, 13:651–687, 2001. Erratum: 20:531–533, 2008.

[20] D.S. Xu and X.-Q. Zhao, Asymptotic speed of spread and traveling waves for a nonlocal epidemic model, *Discrete Contin. Dyn. Syst., Ser. B*, 5:1043–1056, 2005.

[21] X.-Q. Zhao and W.D. Wang, Fisher waves in an epidemic model, *Discrete Contin. Dyn. Syst., Ser. B*, 4:1117–1128, 2004.

[22] X.-Q. Zhao and D.M. Xiao, The asymptotic speed of spread and traveling waves for a vector disease model, *J. Dyna. Diff. Eqns.*, 18:1001–1019, 2006.