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CLASSIFICATION OF SOLUTIONS OF ASYMMETRIC p-LAPLACIAN OSCILLATORS WITH PERIODIC COEFFICIENTS

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ABSTRACT. By exploiting the Denjoy theorem in topological dynamics and the unique ergodic theorem in ergodic theory, we will give a classification of all solutions of asymmetric p-Laplacian oscillators with periodic coefficients.

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1. INTRODUCTION

A basic model for oscillators is the Hill's equation

$$x'' + q(t)x = 0, \qquad x \in \mathbb{R}, \tag{1.1}$$

where $q(t) \in L^1(\mathbb{S}_T) = L^1(\mathbb{S}_T, \mathbb{R})$, called a potential. Here $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$, $T > 0$. Due to the linearity of equation (1.1) and periodicity of potentials, the Floquet theory [11] can be applied to equation (1.1). Consequently, solutions of (1.1) can be classified as follows.

Without loss of generality, we assume that the period of $q(t)$ is 2π and let $\rho = \rho(q)$ be the rotation number of (1.1). See (2.18) for its definition. Note that $\rho(q) \in [0, \infty)$.

Lemma 1.1 (Class 1). *Suppose that $\rho(q) = n/2$, $n \in \mathbb{Z}^+$. Then any non-zero solution $x(t)$ of (1.1) can be decomposed as $x(t) = g(t)h(t)$, where $h(t)$ is either periodic or asymptotical to some periodic function, and $g(t)$ has a linear growth or an exponential growth in t.*

Class 2. *Suppose that $\rho(q)$ is rational and $\rho(q) \neq n/2$, $n \in \mathbb{Z}^+$. Then $x(t)$ is periodic.*

Class 3. *Suppose that $\rho(q)$ is irrational. Then $x(t)$ is quasi-periodic of two frequencies.*

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For the precise statement of these results and its proof based on the Floquet theorem [11], see Section 2.

In this paper we will study the classification of solutions of the so-called scalar asymmetric p-Laplacian oscillator

$$(\phi_p(x'))' + q^+(t)\phi_p(x_+) + q^-(t)\phi_p(x_-) = 0, \qquad x \in \mathbb{R}, \tag{1.2}$$

with T-periodic coefficients $q^\pm \in L^1(\mathbb{S}_T)$. Here $1 < p < \infty$, $\phi_p : \mathbb{R} \to \mathbb{R}$ is $s \mapsto |s|^{p-2}s$, and $y_+ := \max(y, 0)$, $y_- = \min(y, 0)$ for $y \in \mathbb{R}$. Note that (1.2) is linear when and only when $p = 2$ and $q^+ = q^-$. In this case, equation (1.2) is reduced to the Hill's equation (1.1). Equation (1.2) is an interesting model of non-smooth dynamical systems. In recent years, it has received a considerable study, including the Fučik spectrum, existence of periodic solutions of inhomogeneous and nonlinear perturbation, and the Lagrangian stability, etc. We refer to [9, 14].

To the knowledge of authors, a complete understanding for all solutions, even for the homogeneous oscillators (1.2) themselves, is not available. Generally speaking, the linear structure of systems is important in the Floquet theory, though some extension of Floquet theory to nonlinear systems can be obtained [7, 13]. However, it seems that the results in [7, 13] cannot be applied to (1.2).

In this paper, we find that the Denjoy theorem [6, Theorem 12.1.1] in topological dynamics and the unique ergodic theorem in ergodic theory [12, Theorem 6.19] can yield a classification of all solutions of the oscillators (1.2). The results are as follows. Let us use $\rho(q^+, q^-) \in [0, \infty)$ to denote that rotation number of the oscillator (1.2).

Theorem 1.2 (Class A). *Suppose that $\rho(q^+, q^-)$ is rational. Then any non-zero solution $x(t)$ of* (1.2) *can be decomposed as $x(t) = g(t)h(t)$, where $h(t)$ is either periodic or asymptotical to some periodic function and $g(t)$ has a linear or an exponential growth rate in t.*

Class B. *Suppose that $\rho(q^+, q^-)$ is irrational. Then $x(t) = g(t)h(t)$, where $h(t)$ is quasi-periodic of two frequencies and $g(t)$ has the exponential growth rate* 0.

Precisely, we say that $h(t)$ is asymptotical to a periodic function, say $\hat{h}(t)$, if $\lim_{t\to+\infty}(h(t) - \hat{h}(t)) = 0$. The exponential growth rate of $g(t)$ is defined as $\lim_{t\to+\infty}(1/t)\log g(t)$.

Compared with the classification of solutions of linear oscillators (1.1), for the p-Laplacian asymmetric oscillators (1.2), Classes 1 and 2 in Lemma 1.1 are now written as Class A in Theorem 1.2 in an unified way. A crucial difference between (1.1) and (1.2) is as follows. For linear oscillators (1.1), when $\rho(q)$ is a rational number which is not half-integers, the exponential growth rate, or Lyapunov exponent, of all solutions must be zero. However, for asymmetric oscillators, even when $\rho(q^+, q^-)$ is a rational number which is not half-integers, some solutions of (1.2) may have a positive exponential growth rate (or Lyapunov exponent [5, 15]). See Example 4.3.

The paper is organized as follows. In Section 2, we will first give a complete proof of Lemma 1.1 based on the Floquet theorem [11, p. 4]. Note that this approach cannot be applied to oscillators (1.2) due to the nonlinearity. In order to prove Theorem 1.2, we will use the Prüfer transformations to yield a circle diffeomorphism $\mathcal{H}_{q^+,q^-}$ induced from (1.2). It will be proved that $\mathcal{H}_{q^+,q^-}$ just fulfills the minimal regularity requirements in the Denjoy theorem for circle homeomorphisms, i.e., $\mathcal{H}_{q^+,q^-}$ is C^1 and $\log \mathcal{H}'_{q^+,q^-}$ has bounded variation. See Proposition 2.5. In Section 3, we will give a complete description on the dynamics of $\mathcal{H}_{q^+,q^-}$, based on the Denjoy theorem and the unique ergodic theorem. In Section 4, we will give the complete proof of Theorem 1.2. An example to illustrate the difference between Hill's equations and asymmetric oscillators will be given.

2. THE p-POLAR COORDINATES AND REDUCTION OF THE OSCILLATORS

For completeness, we give the proof of Lemma 1.1. Let $y = -x'$. Then equation (1.1) can be written as the planar system

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := A(t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{2.1}$$

Let $M(t)$ be the fundamental matrix solution of (2.1) such that $M(0) = I_2$. Since $A(t)$ is 2π-periodic, one has

$$M(t + 2\pi) = M(t)D,$$

where $D := M(2\pi)$ is the so-called Poincaré matrix of (2.1). Note that $\det D = +1$. The eigenvalues $\lambda_{1,2}$ of matrix D are called the Floquet multipliers of (1.1). Then $\lambda_1 \cdot \lambda_2 = +1$. Let us take λ_i so that $|\lambda_1| = 1/|\lambda_2| \geq 1$.

The classification of (1.1) and its solutions are as follows.

Case (i). Equation (1.1) is hyperbolic. That is, $\lambda_{1,2} \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. In this case, $|\lambda_1| > 1$ and D is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$, and by [11, Floquet's theorem, p. 4], there exist functions $p_i(t)$, $i = 1,\ 2$, such that

- $p_i(t)$ are 2π-periodic if $\lambda_1 > 0$, and $p_i(t)$ are 4π-periodic if $\lambda_1 < 0$, and
- any solution $x(t)$ of (1.1) can be written as

$$x(t) = c_1|\lambda_1|^{t/(2\pi)}p_1(t) + c_2|\lambda_1|^{-t/(2\pi)}p_2(t) = |\lambda_1|^{t/(2\pi)}(c_1p_1(t) + c_2|\lambda_1|^{-t/\pi}p_2(t)),$$

where $c_i \in \mathbb{R}$ are constants.

By setting $g(t) := |\lambda_1|^{t/(2\pi)}$ and $h(t) := c_1 p_1(t) + c_2 |\lambda_1|^{-t/\pi} p_2(t)$, one sees that $g(t)$ has the exponential growth rate $(\log|\lambda_1|)/(2\pi) > 0$, and $h(t)$ is periodic if $c_2 = 0$ and $h(t)$ is asymptotical to the periodic function $c_1 p_1(t)$ if $c_2 \neq 0$. Thus $x(t)$ has the form in Class 1.

Case (ii). Equation (1.1) is parabolic. That is, $\lambda_1 = \lambda_2 = \pm 1$. In this case, the matrix D is similar to $\begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_1 \end{pmatrix}$, where $\beta = 0$ or 1.

If $\lambda_1 = 1$, by the Floquet theorem, (1.1) has two linearly independent solutions $p_1(t)$ and $\hat{p}(t)$ such that $p_1(t)$ is 2π-periodic and $\hat{p}(t)$ satisfies

$$\hat{p}(t + 2\pi) \equiv \hat{p}(t) + \gamma p_1(t),$$

where $\gamma = 0$ if $\beta = 0$ and $\gamma \neq 0$ if $\beta \neq 0$. Then $p_2(t) := \hat{p}(t) - \frac{\gamma}{2\pi} t p_1(t)$ is 2π-periodic. Now any non-zero solution $x(t)$ of (1.1) has the form

$$\begin{aligned} x(t) &= c_1 p_1(t) + c_2 \left(p_2(t) + \frac{\gamma}{2\pi} t p_1(t) \right) \\ &= \left(\frac{c_2 \gamma p_1(t)}{2\pi \sqrt{p_1(t)^2 + p_2(t)^2 + 1}} t + \frac{c_1 p_1(t) + c_2 p_2(t)}{\sqrt{p_1(t)^2 + p_2(t)^2 + 1}} \right) \sqrt{p_1(t)^2 + p_2(t)^2 + 1} \\ &=: g(t) h(t). \end{aligned} \tag{2.2}$$

One sees that $h(t)$ is 2π-periodic, while $g(t)$ has at most a linear growth

$$\limsup_{t \to \infty} \frac{|g(t)|}{t} \in [0, \infty).$$

Thus $x(t)$ has also the form in Class 1.

If $\lambda_1 = -1$, one has also a decomposition like (2.2) where $h(t)$ is 4π-periodic.

Case (iii). Equation (1.1) is elliptic. That is, $\lambda_i \in \mathbb{C} \backslash \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. Since $|\lambda_1| = 1$, we may write λ_1 as $\lambda_1 = e^{i2\alpha\pi}$ where $\alpha \in \mathbb{R}^+ := (0, \infty)$ and $\alpha \neq n/2$ for all $n \in \mathbb{N}$. In this case, any non-zero solution $x(t)$ of (1.1) has the form

$$x(t) = c_1 p_1(t) \cos(\alpha t) + c_2 p_2(t) \sin(\alpha t), \tag{2.3}$$

where $p_i(t)$ are 2π-periodic. We distinguish two cases.

- Subcase (iii)(a): $\alpha = m/n$ is rational, where $m,\ n \in \mathbb{N}$ are co-prime. Then $x(t)$ is $2n\pi$-periodic and is as in Class 2.

- Subcase (iii)(b): α is irrational. It shows form (2.3) that $x(t)$ is quasi-periodic with frequencies α and 1. Thus $x(t)$ belongs to Class 3.

Finally, due to the rotation number approach [14], the rotation number $\rho(q)$ of (1.1) is $n/2$, $n \in \mathbb{Z}^+$, for cases (i) and (ii). For case (iii), one has $\rho(q) = \alpha + n/2$ for some $n \in \mathbb{Z}^+$. Now the proof of Lemma 1.1 is complete. □

Now we are going to study oscillator (1.2). Note that $\phi_p(s) = |s|^{p-2}s$ is $(p-1)$-homogeneous

$$\phi_p(ks) = k^{p-1}\phi_p(s), \qquad k \geq 0,\ s \in \mathbb{R},$$

with the inverse $\phi_p^{-1} = \phi_{p^*}$. Here $p^* := p/(p-1) \in (1, \infty)$. For simplicity, from now on, we always take the period T of $q^{\pm}(t)$ as

$$T = 2\pi_p, \qquad \pi_p := \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}. \tag{2.4}$$

Now we introduce the p-polar coordinates [10] and give some properties. Consider the auxiliary differential equation

$$(\phi_p(x'))' + \phi_p(x) = 0.$$

Set $y = -\phi_p(x')$. Then the equation becomes equivalent to the following planar system

$$x' = -\phi_{p^*}(y), \qquad y' = \phi_p(x). \tag{2.5}$$

This is an integrable Hamiltonian system with the Hamiltonian function

$$H(x, y) = |x|^p/p + |y|^{p^*}/p^*.$$

Let $(C_p(t), S_p(t))$ be the unique solution of (2.5) with the initial value $(x(0), y(0)) = (1, 0)$. Then $C_p(t)$ and $S_p(t)$ are well-defined on the whole real line $\mathbb{R}$. These functions $C_p(t)$ and $S_p(t)$ are called the p-cosine and the p-sine respectively, because they possess many properties similar to the cosine and sine functions. Some properties on $C_p(t)$ and $S_p(t)$ are summarized as follows [10].

Lemma 2.1. (i) *Both $C_p(t)$ and $S_p(t)$ are $2\pi_p$-periodic continuous functions, where π_p is defined by* (2.4)*.*

(ii) *$C_p(t)$ is even in t and $S_p(t)$ is odd in t.*

(iii) *$C_p(t + \pi_p) \equiv -C_p(t)$, and $S_p(t + \pi_p) \equiv -S_p(t)$.*

(iv) *$C_p(t) = 0$ if and only if $t = \pi_p/2 + n\pi_p$, $n \in \mathbb{Z}$, and $S_p(t) = 0$ if and only if $t = n\pi_p,\ n \in \mathbb{Z}$.*

(v) *$C_p(t)$ is strictly decreasing on $[0, \pi_p]$, and $S_p(t)$ is strictly increasing on $[-\pi_p/2, \pi_p/2]$.*

(vi) *$C_p'(t) = -\phi_{p^*}(S_p(t))$, and $S_p'(t) = \phi_p(C_p(t))$.*

(vii) *$|C_p(t)|^p + (p-1)|S_p(t)|^{p^*} \equiv 1$.*

Using these functions C_p and S_p, one can introduce the following p-polar coordinates

$$x = rC_p(\theta), \qquad y = (p-1)^{1/p^*}r^{p-1}S_p(\theta), \qquad r > 0,\ \theta \in \mathbb{R}. \tag{2.6}$$

When $p = 2$, it is the usual polar coordinates. For oscillator (1.2), let $-\phi_p(x') = y$. Then it is transformed into the following Hamiltonian system

$$x' = -\phi_{p^*}(y), \qquad y' = q^+(t)\phi_p(x_+) + q^-(t)\phi_p(x_-).$$

In the p-polar coordinates (2.6), it follows from Lemma 2.1 that the equations for θ and r are respectively

$$\theta' = \Xi(t,\theta), \tag{2.7}$$

$$r' = r\Psi(t,\theta), \tag{2.8}$$

where

$$\Xi(t,\theta) = \begin{cases} (p-1)^{1/p} + Q^+(t)|C_p(\theta)|^p & \text{when } C_p(\theta) \ge 0, \\ (p-1)^{1/p} + Q^-(t)|C_p(\theta)|^p & \text{when } C_p(\theta) \le 0, \end{cases} \tag{2.9}$$

$$\Psi(t,\theta) = \begin{cases} Q^+(t)\phi_p(C_p(\theta))\phi_{p^*}(S_p(\theta)) & \text{when } C_p(\theta) \ge 0, \\ Q^-(t)\phi_p(C_p(\theta))\phi_{p^*}(S_p(\theta)) & \text{when } C_p(\theta) \le 0. \end{cases} \tag{2.10}$$

Here the functions $Q^\pm(t)$ are

$$Q^\pm(t) := (p-1)^{1/p}\left((p-1)^{-1}q^\pm(t) - 1\right) \in L^1(\mathbb{S}_{2\pi_p}). \tag{2.11}$$

We list some properties on vector fields $\Xi(t,\theta)$ and $\Psi(t,\theta)$ in the following lemma. These can be deduced simply from (2.9) and (2.10).

Lemma 2.2. (i) *The functions $\Xi(t,\theta)$ and $\Psi(t,\theta)$ are $2\pi_p$-periodic in both t and θ, and $\Xi(t,\theta)$ is continuously differentiable in θ. In fact, one has*

$$\frac{\partial \Xi(t,\theta)}{\partial \theta} \equiv -p\Psi(t,\theta). \tag{2.12}$$

(ii) *The function $\Psi(t,\theta)$ can be rewritten as*

$$\Psi(t,\theta) \equiv Q^+(t)f_+(\theta) + Q^-(t)f_-(\theta), \tag{2.13}$$

where $Q^\pm(t)$ are defined by (2.11) and $f_\pm(\theta)$ are

$$f_\pm(\theta) := \phi_p\left((C_p(\theta))_\pm\right)\phi_{p^*}(S_p(\theta)) = \left(\phi_p\left(C_p(\theta)\right)\right)_\pm \phi_{p^*}(S_p(\theta)). \tag{2.14}$$

Here both $f_\pm(\theta)$ are continuous and $2\pi_p$-periodic in $\theta \in \mathbb{R}$.

Remark 2.3. Due to the positive homogeneity of (1.2) in x, equation (2.7) is independent of r and equation (2.8) is positively homogeneous in r. Moreover, equality (2.12) is the same as that system (2.7)–(2.8) preserves the area $r^{p-1}dr \wedge d\theta$.

Given $\vartheta \in \mathbb{R}$. We use $(\theta, r) = (\Theta(t, \vartheta), R(t, \vartheta))$ to denote the solution of (2.7)–(2.8) with the initial value $(\Theta(0, \vartheta), R(0, \vartheta)) = (\vartheta, 1)$. From equation (2.8), one has

$$R(t, \vartheta) \equiv \exp\left(\int_0^t \Psi(s, \Theta(s, \vartheta)) ds\right). \tag{2.15}$$

Since the vector field $\Xi(t, \theta)$ is $2\pi_p$-periodic in both t and θ, we have the following periodicity equalities

$$\Theta(t, \vartheta + 2n\pi_p) \equiv \Theta(t, \vartheta) + 2n\pi_p, \qquad n \in \mathbb{Z}, \tag{2.16}$$

$$\Theta(t + 2n\pi_p, \vartheta) \equiv \Theta(t, \Theta(2n\pi_p, \vartheta)), \qquad n \in \mathbb{Z}. \tag{2.17}$$

Now we introduce $\mathcal{H} = \mathcal{H}_{q^+,q^-} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{H}(\vartheta) := \Theta(2\pi_p, \vartheta), \qquad \vartheta \in \mathbb{R}.$$

As $\Xi(t, \theta)$ is continuously differentiable in θ, $\mathcal{H}_{q^+,q^-} : \mathbb{R} \to \mathbb{R}$ is an increasing diffeomorphism satisfying

$$\mathcal{H}(\vartheta + 2\pi_p) \equiv \mathcal{H}(\vartheta) + 2\pi_p.$$

The rotation number $\rho(q^+, q^-)$ of (1.2), or that of (2.7), is defined by

$$\rho = \rho(q^+, q^-) := \lim_{t \to +\infty} \frac{\Theta(t, \vartheta) - \vartheta}{t}, \tag{2.18}$$

which is well-defined and is independent of ϑ. In fact, following Johnson and Moser [4] and Feng and Zhang [2], the rotation number $\rho(q^+, q^-)$ can be introduced even for (1.2) with almost periodic coefficients $q^\pm(t)$. Further extension of rotation numbers to random dynamical systems can be found in Arnold [1] and Li and Lu [8]. Note that the projection of $\mathcal{H}_{q^+,q^-}$ onto $\mathbb{S}_{2\pi_p} := \mathbb{R}/2\pi_p\mathbb{Z}$ yields an orientation-preserving diffeomorphism of $\mathbb{S}_{2\pi_p}$.

By (2.6) and (2.15), non-zero solutions of (1.2) can be written as

$$x(t) = r_0 \exp\left(\int_0^t \Psi(s, \Theta(s, \vartheta)) ds\right) C_p(\Theta(t, \vartheta)) \tag{2.19}$$

for some $r_0 > 0$ and some $\vartheta \in \mathbb{R}$.

Suggested by (2.13) and (2.14), we consider the following $2\pi_p$-periodic continuous functions

$$C(\theta) := \phi_p\left(C_p(\theta)\right), \qquad S(\theta) := \phi_{p^*}(S_p(\theta)), \qquad \theta \in \mathbb{R}.$$

For example, when $p = 2$, $C(\theta) = \cos\theta$ and $S(\theta) = \sin\theta$. However, when $p \neq 2$, one of the numbers $p - 1$ and $p^* - 1$ is less than 1 and one of the functions $C(\theta)$ and $S(\theta)$ is not differentiable on the whole real line $\mathbb{R}$, while another one is continuously differentiable on $\mathbb{R}$. In fact, for the case $p \in (1, 2)$, we have

$$C'(\theta) = \infty \qquad \text{at } \theta = \pi_p/2 + n\pi_p,\ n \in \mathbb{Z}.$$

Then $C(\theta)$ is not Lipschitz continuous. Thus for any $p \neq 2$, neither of the functions $f_\pm(\theta)$ defined by (2.14) is Lipschitz continuous.

Recall that for a function $f(\theta) : I = [a, b] \to \mathbb{R}$, the total variation is defined by

$$\mathrm{Var}(f, I) := \sup \left\{ \sum_{i=0}^{n-1} |f(\theta_{i+1}) - f(\theta_i)| : a = \theta_0 < \theta_1 < \cdots < \theta_n = b,\ n \in \mathbb{N} \right\} \in [0, \infty].$$

We say that f has bounded variation on I if $\mathrm{Var}(f, I) < \infty$. Note that $f_\pm(\theta)$ are $2\pi_p$-periodic continuous functions. Moreover, from Lemma 2.1, it is easy to see that both $f_\pm(\theta)$ are piecewise monotone. Hence we have the following result.

Lemma 2.4. *Both functions $f_\pm(\theta)$ defined by* (2.14) *have bounded variations on any finite closed interval $I \subset \mathbb{R}$.*

We have the following crucial observation on the induced diffeomorphism $\mathcal{H}_{q^+,q^-}$.

Proposition 2.5. The mapping $\mathcal{H} = \mathcal{H}_{q^+,q^-} : \mathbb{R} \to \mathbb{R}$ induced by (1.2) is a C^1 orientation preserving self-diffeomorphism of $\mathbb{R}$. Moreover, the function $\log \mathcal{H}'(\vartheta)$ has bounded variation on $[0, 2\pi_p]$.

Proof. Due to the continuous differentiability in θ of the vector field $\Xi(t, \theta)$ and the continuously differentiable dependence of solutions on the initial value, we know that for any t fixed, $\Theta(t, \cdot)$ is a C^1 increasing diffeomorphism of $\mathbb{R}$. Moreover, by differentiating equation (2.7) with respect to the initial value ϑ, we know that $E(t) := \frac{\partial \Theta(t,\vartheta)}{\partial \vartheta}$ satisfies the variational equation

$$E'(t) = \frac{\partial \Xi(t, \Theta(t, \vartheta))}{\partial \theta} E(t) \equiv -p\Psi(t, \Theta(t, \vartheta))E(t), \qquad E(0) = 1,$$

where (2.12) is used. Integrating this equation, we have

$$E(t) = \exp\left(-p \int_0^t \Psi(s, \Theta(s, \vartheta)) ds \right).$$

Taking $t = 2\pi_p$, we have

$$\log \mathcal{H}'(\vartheta) = \log \frac{\partial \Theta(2\pi_p, \vartheta)}{\partial \vartheta} = \log E(2\pi_p) = -p \int_0^{2\pi_p} \Psi(s, \Theta(s, \vartheta)) ds.$$

Using the expression (2.13) for $\Psi(s, \theta)$, we have

$$\log \mathcal{H}'(\vartheta) = -p \int_0^{2\pi_p} Q^+(t) f_+(\Theta(t, \vartheta)) dt - p \int_0^{2\pi_p} Q^-(t) f_-(\Theta(t, \vartheta)) dt. \tag{2.20}$$

Given $t \in [0, 2\pi_p]$. Since $\vartheta \mapsto \Theta(t, \vartheta)$ is strictly increasing, the range

$$\Theta(t, [0, 2\pi_p]) = [\Theta(t, 0),\ \Theta(t, 2\pi_p)] =: I^t$$

is a closed interval. Some estimates on I^t are as follows. Note from Lemma 2.1 (vii) that

$$|C_p(\theta)| \le 1, \qquad |S_p(\theta)| \le c_p := (p-1)^{-1/p^*}, \qquad \theta \in \mathbb{R}.$$

By (2.9), we have

$$|\Xi(t, \theta)| \le (p-1)^{1/p} + \max\left(|Q^+(t)|,\ |Q^-(t)|\right) =: \check{Q}(t) \in L^1(\mathbb{S}_{2\pi_p}).$$

Let $t \in [0, 2\pi_p]$. Combining with (2.7), we have

$$|\Theta(t,\vartheta) - \vartheta| = \left| \int_0^t \Xi(s, \Theta(s,\vartheta)) ds \right| \le \int_0^{2\pi_p} \check{Q}(s) ds = \|\check{Q}\|_{L^1},$$

i.e., $\vartheta - \|\check{Q}\|_{L^1} \le \Theta(t,\vartheta) \le \vartheta + \|\check{Q}\|_{L^1}$. Here $\|q\|_{L^1} = \int_0^{2\pi_p} |q(t)| dt$ is the L^1 norm for $q \in L^1(\mathbb{S}_{2\pi_p})$. Hence one has

$$I^t \subset \left[-\|\check{Q}\|_{L^1},\, 2\pi_p + \|\check{Q}\|_{L^1} \right] =: I_0 \qquad \text{for any } t \in [0, 2\pi_p]. \tag{2.21}$$

Denote

$$F_\pm(\vartheta) := \int_0^{2\pi_p} Q^\pm(t) f_\pm(\Theta(t,\vartheta)) dt.$$

Let $0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_n = 2\pi_p$ be a partition of $[0, 2\pi_p]$. Then

$$\begin{aligned} \sum_{i=0}^{n-1} |F_\pm(\vartheta_{i+1}) - F_\pm(\vartheta_i)| &= \sum_{i=0}^{n-1} \left| \int_0^{2\pi_p} Q^\pm(t) \left(f_\pm(\Theta(t,\vartheta_{i+1})) - f_\pm(\Theta(t,\vartheta_i)) \right) dt \right| \\ &\le \sum_{i=0}^{n-1} \int_0^{2\pi_p} |Q^\pm(t)| \, |f_\pm(\Theta(t,\vartheta_{i+1})) - f_\pm(\Theta(t,\vartheta_i))| \, dt \\ &= \int_0^{2\pi_p} |Q^\pm(t)| \left(\sum_{i=0}^{n-1} \left| f_\pm(\hat{\vartheta}_{i+1}) - f_\pm(\hat{\vartheta}_i) \right| \right) dt, \end{aligned}$$

where $\{\hat{\vartheta}_i = \Theta(t,\vartheta_i)\}$ is a partition of I^t, because $\Theta(t,\cdot)$ is increasing. By Lemma 2.4 and (2.21), we know that

$$\sum_{i=0}^{n-1} \left| f_\pm(\hat{\vartheta}_{i+1}) - f_\pm(\hat{\vartheta}_i) \right| \le \mathrm{Var}(f_\pm, I^t) \le \mathrm{Var}(f_\pm, I_0) < \infty.$$

Note that the upper bound above is independent of $t \in [0, 2\pi_p]$. Hence we have

$$\sum_{i=0}^{n-1} |F_\pm(\vartheta_{i+1}) - F_\pm(\vartheta_i)| \le \|Q^\pm\|_{L^1} \cdot \mathrm{Var}(f_\pm, I_0) < \infty$$

for any partition $\{\vartheta_i\}$ of $[0, 2\pi_p]$. Thus

$$\mathrm{Var}(F_\pm, [0, 2\pi_p]) \le \|Q^\pm\|_{L^1} \cdot \mathrm{Var}(f_\pm, I_0) < \infty.$$

By (2.20), we conclude that $\log \mathcal{H}'(\vartheta)$ has bounded variation on $[0, 2\pi_p]$. □

For the Hill's equation (1.1), $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ is C^∞. When $p = 2$, it is proved in [15] that $\mathcal{H} = \mathcal{H}_{q^+,q^-}$ is C^1 and $\log \mathcal{H}'(\vartheta)$ is globally Lipschitz continuous in $\vartheta \in \mathbb{R}$, because $f_\pm(\theta) = (\cos\theta)_\pm \sin\theta$ are Lipschitz continuous in θ in this case.

3. DYNAMICS OF THE INDUCED CIRCLE DIFFEOMORPHISMS

Suggested by expression (2.19) for solutions of (1.2), in this section we will study solutions $\Theta(t,\vartheta)$ of (2.7).

Recall that $\mathcal{H} = \mathcal{H}_{q^+,q^-}$ induces an orientation-preserving diffeomorphism $h = h_{q^+,q^-}$ of the circle $\mathbb{S}_{2\pi_p}$. Since we have the regularity result in Proposition 2.5 for $\mathcal{H}$, the dynamics of h, or that of $\mathcal{H}$, is completely clear due to the Denjoy theorem.

Case a. Suppose that $\rho(q^+,q^-) = k/\ell$ is rational, where $k \in \mathbb{Z}^+$, $\ell \in \mathbb{N}$ are co-prime. Define

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{q^+,q^-} : \mathbb{R} \to \mathbb{R}, \qquad \hat{\mathcal{H}}(\vartheta) := \mathcal{H}^\ell(\vartheta) - 2k\pi_p,$$

where $\mathcal{H}^\ell$ is the ℓ-th iteration of $\mathcal{H}$. By equalities (2.16) and (2.17), we have

$$\hat{\mathcal{H}}(\vartheta) \equiv \Theta(2\ell\pi_p, \vartheta) - 2k\pi_p.$$

Then $\rho(\hat{\mathcal{H}}) = 0$. Thus

$$\Omega = \Omega_{q^+,q^-} := \left\{\vartheta_0 \in \mathbb{R} : \hat{\mathcal{H}}(\vartheta_0) = \vartheta_0\right\} = \{\vartheta_0 \in \mathbb{R} : \Theta(2\ell\pi_p, \vartheta) = \vartheta_0 + 2k\pi_p\} \neq \emptyset,$$

and, for any $\vartheta \notin \Omega$, there exists some $\vartheta_0 \in \Omega$ such that

$$\lim_{n\to+\infty} \hat{\mathcal{H}}^n(\vartheta) = \lim_{n\to+\infty} (\mathcal{H}^{n\ell}(\vartheta_0) - 2nk\pi_p) = \vartheta_0. \tag{3.1}$$

By equalities (2.16) and (2.17), we conclude that

- Case $\vartheta_0 \in \Omega$. In this case, we have

$$\Theta(2\ell\pi_p, \vartheta_0) = \vartheta_0 + 2k\pi_p, \qquad \Theta(2n\ell\pi_p, \vartheta_0) = \vartheta_0 + 2nk\pi_p \text{ for } n \in \mathbb{Z}. \tag{3.2}$$

These imply that

$$\Theta(t, \vartheta_0) \equiv (k/\ell)t + \Phi_{\vartheta_0}(t), \qquad \Phi_{\vartheta_0}(t + 2\ell\pi_p) \equiv \Phi_{\vartheta_0}(t). \tag{3.3}$$

- Case $\vartheta \notin \Omega$. As

$$\hat{\mathcal{H}}^n(\vartheta) - \vartheta_0 \equiv \Theta(2n\ell\pi_p, \vartheta) - \Theta(2n\ell\pi_p, \vartheta_0) = \mathcal{H}^{n\ell}(\vartheta) - \mathcal{H}^{n\ell}(\vartheta_0), \tag{3.4}$$

asymptotical result (3.1) gives

$$\lim_{t\to+\infty} (\Theta(t,\vartheta) - \Theta(t,\vartheta_0)) = 0, \tag{3.5}$$

where $\vartheta_0 \in \Omega$ is as in (3.1) and therefore $\Theta(t,\vartheta_0)$ is as in (3.3).

Case b. Suppose that $\rho = \rho(q^+,q^-)$ is irrational. Proposition 2.5 shows that $\mathcal{H} = \mathcal{H}_{q^+,q^-}$ meets with the minimal regularity requirements of the Denjoy theorem [6, Theorem 12.1.1]. We conclude that there exists a homeomorphism $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(\vartheta + 2n\pi_p) = \sigma(\vartheta) + 2n\pi_p, \qquad \vartheta \in \mathbb{R},\ n \in \mathbb{Z}, \tag{3.6}$$

$$\Theta(2\pi_p, \vartheta) = \mathcal{H}(\vartheta) = \sigma^{-1}(\sigma(\vartheta) + 2\pi_p\rho), \qquad \vartheta \in \mathbb{R}. \tag{3.7}$$

That is, $\mathcal{H}$ is topologically conjugate to the translation on $\mathbb{R}$ defined by $\vartheta \mapsto \vartheta + 2\pi_p\rho$. From a fundamental result due to Bohr, see, for example, [3, Theorem 2.6], we know that (3.6) and (3.7) imply the following result. For completeness, a proof will be given.

Proposition 3.1. Suppose that $\rho = \rho(q^+, q^-)$ is irrational. Then there exists a continuous function $\omega(u,v) : \mathbb{R}^2 \to \mathbb{R}$ such that $\omega(u,v)$ is $2\pi_p$-periodic in both u and v, and solutions of (2.7) are given by

$$\Theta(t,\vartheta) \equiv \sigma(\vartheta) + \rho t + \omega(t, \sigma(\vartheta) + \rho t). \tag{3.8}$$

Proof. Let us introduce $\omega(u,v) := \Theta(u, \sigma^{-1}(v - \rho u)) - v$. We will show that $\omega(u,v)$ is $2\pi_p$-periodic in both u and v. Indeed, we have

$$\begin{aligned}
\omega(u+2\pi_p, v) &= \Theta(u+2\pi_p, \sigma^{-1}(v - \rho u - 2\pi_p\rho)) - v \\
&= \Theta(u, \Theta(2\pi_p, \sigma^{-1}(v - \rho u - 2\pi_p\rho))) - v \qquad \text{(by (2.17))} \\
&= \Theta(u, \sigma^{-1}(\sigma(\sigma^{-1}(v - \rho u - 2\pi_p\rho)) + 2\pi_p\rho)) - v \qquad \text{(by (3.7))} \\
&= \Theta(u, \sigma^{-1}(v - \rho u)) - v = \omega(u,v).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\omega(u, v+2\pi_p) &= \Theta(u, \sigma^{-1}(v + 2\pi_p - \rho u)) - v - 2\pi_p \\
&= \Theta(u, \sigma^{-1}(v - \rho u) + 2\pi_p) - v - 2\pi_p \qquad \text{(by (3.6))} \\
&= \Theta(u, \sigma^{-1}(v - \rho u)) + 2\pi_p - v - 2\pi_p = \omega(u,v). \qquad \text{(by (2.16))}
\end{aligned}$$

By setting $u = t$ and $\sigma^{-1}(v - \rho u) = \vartheta$, i.e., $v = \sigma(\vartheta) + \rho t$, we get

$$\Theta(t,\vartheta) = v + \omega(u,v) = \sigma(\vartheta) + \rho t + \omega(t, \sigma(\vartheta) + \rho t),$$

proving equality (3.8). □

The following is a combination of the classification of orientation preserving homeomorphisms on the circle and the unique ergodicity theorem in ergodic theory.

Proposition 3.2. Let $f \in C(\mathbb{S}_{2\pi_p}, \mathbb{R})$ be a continuous function. Then, for any $\vartheta \in \mathbb{S}_{2\pi_p}$, the following ergodic limit exists

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\mathcal{H}^i_{q^+,q^-}(\vartheta)) =: f^*(\vartheta). \tag{3.9}$$

Proof. As mentioned before, we may consider $\mathcal{H} = \mathcal{H}_{q^+,q^-}$ as a diffeomorphism of the circle $\mathbb{S}_{2\pi_p}$.

Case 1. Suppose that $\rho = \rho(\mathcal{H}_{q^+,q^-}) = k/\ell$ is rational. If $\vartheta_0 \in \Omega_{q^+,q^-}$, by (3.2) and the $2\pi_p$-periodicity of $f(\theta)$, then we have

$$f(\mathcal{H}^{i+\ell}(\vartheta_0)) = f(\mathcal{H}^i(\mathcal{H}^\ell(\vartheta_0))) = f(\mathcal{H}^i(\vartheta_0 + 2k\pi_p)) = f(\mathcal{H}^i(\vartheta_0) + 2k\pi_p) = f(\mathcal{H}^i(\vartheta_0))$$

for all $i \in \mathbb{Z}$. That is, the sequence $\{f(\mathcal{H}^i(\vartheta_0))\}_{i\in\mathbb{Z}}$ is ℓ-periodic. Hence the limit of (3.9) is simply

$$f^*(\vartheta_0) := \frac{1}{\ell}\sum_{i=0}^{\ell-1} f(\mathcal{H}^i(\vartheta_0)).$$

If $\vartheta \notin \Omega_{q^+,q^-}$, we take $\vartheta_0 \in \Omega_{q^+,q^-}$ as in (3.1). As $f : \mathbb{S}_{2\pi_p} \to \mathbb{R}$ is uniformly continuous, result (3.1) shows that

$$\lim_{n\to+\infty}(f(\mathcal{H}^n(\vartheta)) - f(\mathcal{H}^n(\vartheta_0))) = 0.$$

See (3.4). That is, the sequence $\{f(\mathcal{H}^i(\vartheta))\}_{i\in\mathbb{Z}}$ is asymptotical to the ℓ-periodic sequence $\{f(\mathcal{H}^i(\vartheta_0))\}_{i\in\mathbb{Z}}$. Hence the limit of (3.9) exists and is just $f^*(\vartheta_0)$.

Case 2. Suppose that $\rho = \rho(\mathcal{H}_{q^+,q^-})$ is irrational. Due to a fundamental result ([12, Theorem 6.18]), we know that the homeomorphism $\mathcal{H} : \mathbb{S}_{2\pi_p} \to \mathbb{S}_{2\pi_p}$ has the unique ergodic Borel probability measure ν. Now the convergence (3.9) follows simply from the unique ergodicity theorem ([12, Theorem 6.19]). Furthermore, in this case, the convergence (3.9) is uniform in $\vartheta \in \mathbb{S}_{2\pi_p}$ and the limiting function f^* is constant which is given by

$$f^*(\vartheta) \equiv \int_{\mathbb{S}_{2\pi_p}} f d\nu, \qquad \vartheta \in \mathbb{S}_{2\pi_p}.$$

Thus the proof is complete. □

4. CLASSIFICATION OF SOLUTIONS

Note that non-zero solutions $x(t)$ of (1.2) are expressed using (2.15). The Lyapunov exponent $\chi(x)$ of $x(t)$ is defined by

$$\chi(x) = \lim_{t\to+\infty}\frac{1}{t}\log\sqrt{(x(t))^2 + (x'(t))^2}$$

when it exists. Using the solutions $\Theta(t,\vartheta)$ and $R(t,\vartheta)$, one has

$$\chi(x) = \lim_{t\to+\infty}\frac{1}{t}\log R(t,\vartheta) = \lim_{t\to+\infty}\frac{1}{t}\int_0^t \Psi(s,\Theta(s,\vartheta))ds =: \chi(\vartheta), \tag{4.1}$$

because the p-polar coordinates lead to

$$r_0 R(t,\vartheta) = \left(|x(t)|^p + |x'(t)|^p\right)^{1/p}.$$

Theorem 4.1. *For any $\vartheta \in \mathbb{R}$, the Lyapunov exponent $\chi(\vartheta)$ of (4.1) does exist.*

Proof. Step 1. We assert that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \Psi(s,\Theta(s,\vartheta))ds = \lim_{n\to+\infty}\frac{1}{2n\pi_p}\int_0^{2n\pi_p}\Psi(s,\Theta(s,\vartheta))ds. \tag{4.2}$$

To see this, one needs only to notice from (2.10) that

$$|\Psi(t,\theta)| \le (p-1)^{-1/p}\max\left(|Q^+(t)|,\ |Q^-(t)|\right) \in L^1(\mathbb{S}_{2\pi_p}).$$

Step 2. We observe that

$$\begin{aligned}\int_0^{2n\pi_p} \Psi(s,\Theta(s,\vartheta))ds &= \sum_{i=0}^{n-1}\int_0^{2\pi_p} \Psi(s+2i\pi_p,\Theta(s+2i\pi_p,\vartheta))ds\\ &= \sum_{i=0}^{n-1}\int_0^{2\pi_p} \Psi(s,\Theta(s+2i\pi_p,\vartheta))ds \qquad \text{(by (2.10))}\\ &= \sum_{i=0}^{n-1}\int_0^{2\pi_p} \Psi(s,\Theta(s,\Theta(2i\pi_p,\vartheta)))ds \qquad \text{(by (2.17))}\\ &= \sum_{i=0}^{n-1}\int_0^{2\pi_p} \Psi(s,\Theta(s,\mathcal{H}^i(\vartheta)))ds\\ &= 2\pi_p\sum_{i=0}^{n-1}\hat{\Psi}(\mathcal{H}^i(\vartheta)), \end{aligned} \tag{4.3}$$

where $\hat{\Psi}$ is

$$\hat{\Psi}(\vartheta) := \frac{1}{2\pi_p}\int_0^{2\pi_p} \Psi(s,\Theta(s,\vartheta))ds, \qquad \vartheta\in\mathbb{R}. \tag{4.4}$$

Therefore, (4.1), (4.2) and (4.3) imply that

$$\chi(\vartheta) = \lim_{n\to+\infty}\frac{1}{2n\pi_p}\int_0^{2n\pi_p}\Psi(s,\Theta(s,\vartheta))ds = \lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\hat{\Psi}(\mathcal{H}^i(\vartheta)). \tag{4.5}$$

Step 3. We assert that the function $\hat{\Psi}(\vartheta)$ defined by (4.4) is $2\pi_p$-periodic and continuous. Indeed, since $\Psi(t,\theta)$ is $2\pi_p$-periodic in both t and θ, the $2\pi_p$-periodicity of $\hat{\Psi}(\vartheta)$ follows simply from (2.16) and Lemma 2.2 (i). Then $\hat{\Psi}(\vartheta)$ may be considered as a function on $\mathbb{S}_{2\pi_p}$. As for the continuity, we have from (2.13) that

$$2\pi_p\hat{\Psi}(\vartheta) = \int_0^{2\pi_p} Q^+(t)f_+(\Theta(t,\vartheta))dt + \int_0^{2\pi_p} Q^-(t)f_-(\Theta(t,\vartheta))dt.$$

Note that $\Theta(t,\vartheta)$ is continuous in (t,ϑ) and $f_\pm(\theta)$ are continuous in θ. By the uniform continuity of the functions $f_\pm(\Theta(t,\vartheta))$ in $(t,\vartheta)\in[0,2\pi_p]^2$, we know that for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\vartheta_i\in[0,2\pi_p],\ |\vartheta_2-\vartheta_1|<\delta \Longrightarrow |f_\pm(\Theta(t,\vartheta_2))-f_\pm(\Theta(t,\vartheta_1))|<\varepsilon \text{ for all } t\in[0,2\pi_p],$$

which, in turn, implies that

$$|\hat{\Psi}(\vartheta_2)-\hat{\Psi}(\vartheta_1)| \le \frac{\varepsilon}{2\pi_p}(\|Q^+\|_{L^1}+\|Q^-\|_{L^1}).$$

Step 4. Applying Proposition 3.2 to $f=\hat{\Psi}$, we know from (4.5) that $\chi(\vartheta)$ exists for any ϑ. □

By Proposition 3.2, if $\rho(q^+, q^-)$ is irrational, then $\chi(\vartheta) = \gamma$ is independent of ϑ and

$$\lim_{n \to +\infty} \frac{\log R(2n\pi_p, \vartheta)}{2n\pi_p} = \gamma \qquad \text{uniformly for all } \vartheta \in \mathbb{S}_{2\pi_p}.$$

As shown in [15], if the constant γ is non-zero, then, as $n \to +\infty$, $R(2n\pi_p, \vartheta)$ will grow or decay exponentially uniformly in $\vartheta \in \mathbb{S}_{2\pi_p}$. It is a contradiction to the area-preserving property of system (2.7)–(2.8). Hence we have in this case the following result.

Theorem 4.2. *Suppose that $\rho(q^+, q^-)$ is irrational. Then $\chi(\vartheta) = 0$ for all ϑ.*

Now we give the proof of Theorem 1.2.

Case (i). Suppose that $\rho = \rho(q^+, q^-) = k/\ell$ is rational. When $\vartheta_0 \in \Omega_{q^+,q^-}$, by (3.3), $\Theta(t, \vartheta_0) \equiv (k/\ell)t + \Phi_{\vartheta_0}(t)$ where $\Phi_{\vartheta_0}(t)$ is $2\ell\pi_p$-periodic. Thus

$$\hat{h}(t) := C_p(\Theta(t, \vartheta_0)) \equiv C_p((k/\ell)t + \Phi_{\vartheta_0}(t))$$

is also $2\ell\pi_p$-periodic. Moreover, as $\Psi(t, \theta)$ is $2\pi_p$-periodic in both t and θ, we know that

$$\hat{g}(t) := \Psi(t, \Theta(t, \vartheta_0))$$

is also $2\ell\pi_p$-periodic. Thus

$$\hat{g}(t) = \chi(\vartheta_0) + \tilde{g}(t),$$

where $\tilde{g}(t)$ is $2\ell\pi_p$-periodic and has mean value 0. Thus

$$R(t, \vartheta_0) = e^{\chi(\vartheta_0)t} \exp\left(\int_0^t \tilde{g}(s)ds\right).$$

By (2.19), we have

$$\begin{aligned} x(t) &= r_0 R(t, \vartheta_0) C_p(\Theta(t, \vartheta_0)) \\ &= e^{\chi(\vartheta_0)t} \cdot \left(r_0 \exp\left(\int_0^t \tilde{g}(s)ds\right) \cdot \hat{h}(t)\right) =: g(t) \cdot h(t). \end{aligned}$$

Notice that

$$\lim_{t \to +\infty} \frac{1}{t} \log g(t) = \chi(\vartheta_0).$$

Since $\hat{g}(t)$ has mean value 0, $h(t)$ is $2\ell\pi_p$-periodic. Hence $x(t)$ is as in Class A.

When $\vartheta \notin \Omega_{q^+,q^-}$, one has from (3.5) that

$$C_p(\Theta(t, \vartheta)) = C_p(\Theta(t, \vartheta_0)) + o(1) \qquad \text{as } t \to +\infty,$$

because $C_p(\theta)$ is uniformly continuous in $\theta \in \mathbb{R}$. By Proposition 3.2 and the proof of Theorem 4.1, we have $\chi(\vartheta) = \chi(\vartheta_0)$. By (4.1), we know that

$$\int_0^t \Psi(s, \Theta(s, \vartheta))ds = (\chi(\vartheta_0) + o(1))t \qquad \text{as } t \to +\infty.$$

Thus

$$\begin{aligned} x(t) &= r_0 R(t,\vartheta) C_p(\Theta(t,\vartheta)) \\ &= r_0 \exp\left(\int_0^t \Psi(s,\Theta(s,\vartheta))ds\right) C_p(\Theta(t,\vartheta)) \\ &= \exp\left((\chi(\vartheta_0)+o(1))t\right)\cdot\left(r_0 C_p(\Theta(t,\vartheta_0))+o(1)\right) \\ &=: g(t)h(t). \end{aligned}$$

Note that $g(t)$ has the Lyapunov exponent $\chi(\vartheta_0)$ and $h(t)$ is asymptotical to the $2\ell\pi_p$-periodic function $r_0 C_p(\Theta(t,\vartheta_0))$. Hence $x(t)$ belongs to Class A as well.

Case (ii). Suppose that $\rho = \rho(q^+, q^-)$ is irrational. By (3.8),

$$C_p(\Theta(t,\vartheta)) = C_p(\sigma(\vartheta_0)+\rho t+\omega(t,\sigma(\vartheta_0)+\rho t)) := C(t,\rho t),$$

which is a quasi-periodic function of two frequencies 1 and ρ. It follows from Theorem 4.2 that the exponential growth rate of $R(t,\vartheta)$ is 0. Thus

$$x(t) = r_0 R(t,\vartheta) C_p(\Theta(t,\vartheta)) = R(t,\vartheta)\cdot(r_0 C(t,\rho t))$$

is as in Class B.

The proof of Theorem 1.2 is complete. □

A simple comparison between Lemma 1.1 for Hill's equations (1.1) and Theorem 1.2 for asymmetric oscillators (1.2) is as follows.

• Classes 1 and 2 in Lemma 1.1 correspond to Class A in Theorem 1.2. However, when rotation number is a rational number which is not half-integers, the result in Theorem 1.2 is weaker than that in Lemma 1.1. In fact, one cannot expect the result in Class 2 for (1.2). See the last example.

• Class 3 in Lemma 1.1 corresponds to Class B in Theorem 1.2. We are not clear if the result in Class B can be improved so that the result is the same as that in Class 3.

Finally we give an example to illustrate a crucial difference between the classification of solutions of (1.1) and (1.2) when $\rho \in (0,\infty)\backslash \frac{1}{2}\mathbb{N}$.

Example 4.3. Let $p = 2$. For each $\ell \in \mathbb{N}$, we will construct $q^{\pm}(t) \in L^1(\mathbb{S}_{2\pi})$ so that (1.2) has rotation number $\rho(q^+, q^-) = 1/\ell$. Moreover, (1.2) has solutions with positive Lyapunov exponents.

At first, let us take a potential $q^-(t) \in L^1(\mathbb{S}_{2\pi})$ so that the Hill's equation

$$x'' + q^-(t)x = 0 \tag{4.6}$$

has the following properties. Denote by $\varphi(t)$ the solution of (4.6) satisfying $(x(0), x'(0)) = (0, -1)$. Then $\varphi(t) < 0$ for $t > 0$ is small. Suppose that $\varphi(t)$ has positive zeros and the first positive zero t_* satisfies

$$t_* \in (0, 2\pi), \qquad \alpha_* := \varphi'(t_*) > 1. \tag{4.7}$$

For example, $q^-(t) = 6\sin t$ fulfills these requirements and

$$t_* \approx 0.4919\pi, \qquad \alpha_* \approx 1.4632.$$

Thus we have

$$(\varphi(0), \varphi'(0)) = (0, -1), \qquad (\varphi(t_*), \varphi'(t_*)) = (0, \alpha_*), \qquad \varphi(t) < 0 \text{ for } t \in (0, t_*). \tag{4.8}$$

Given $\ell \in \mathbb{N}$. Let

$$a_\ell := \left(\pi/(2\ell\pi - t_*)\right)^2.$$

Then we consider the solution $\psi(t)$ of

$$x'' + a_\ell x = 0, \qquad (x(t_*), x'(t_*)) = (0, \alpha_*). \tag{4.9}$$

Here the initial condition comes from the second result of (4.8). Explicitly,

$$\psi(t) = \frac{\alpha_*}{\sqrt{a_\ell}} \sin \sqrt{a_\ell}(t - t_*).$$

The first zero of $\psi(t)$ after t_* is $t_* + \pi/\sqrt{a_\ell} = 2\ell\pi$. Moreover, one has

$$(\psi(2\ell\pi), \psi'(2\ell\pi)) = (0, -\alpha_*), \qquad \psi(t) > 0 \text{ for } t \in (t_*, 2\ell\pi). \tag{4.10}$$

Let us define

$$\eta(t) := \begin{cases} \varphi(t) & \text{for } t \in [0, t_*], \\ \psi(t) & \text{for } t \in [t_*, 2\ell\pi]. \end{cases}$$

From properties (4.7)-(4.8)-(4.10) and equations (4.6)-(4.9), we know that $x = \eta(t)$, $t \in [0, 2\ell\pi]$, satisfies

$$x''(t) + q^+(t)x_+(t) + q^-(t)x_-(t) = 0. \tag{4.11}$$

Here $q^+(t) \equiv a_\ell$. The function $\eta(t)$ can be extended to $\mathbb{R}$ by

$$\eta_*(t) := \alpha_*^n \eta(t - 2n\ell\pi), \qquad t \in [2n\ell\pi, 2(n+1)\ell\pi],\ n \in \mathbb{Z}.$$

For these constructions, see Figure 1.

Since $q^\pm(t)$ are 2π-periodic, from the constructions above, it is easy to see that $\eta_*(t)$ is a solution of (4.11) on $t \in \mathbb{R}$. Furthermore, one has $\rho(q^+, q^-) = 1/\ell$, while equation (4.11) has the solution $\eta_*(t)$ which has the Lyapunov exponent

$$\chi(\eta_*) = \frac{\log \alpha_*}{2\ell\pi} > 0.$$

Thus (4.11) is a desired asymmetric oscillator.

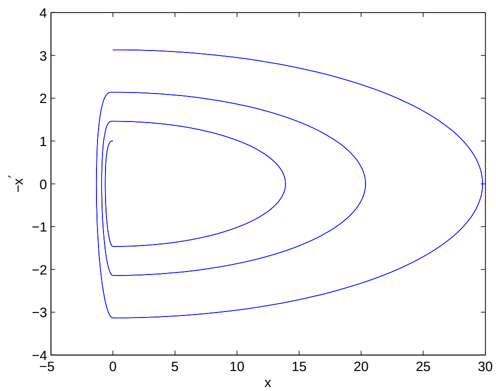

FIGURE 1. The solution $\eta_*(t)$ of (4.11) with $\ell = 5$ in the $(x, -x')$-plane.

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