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ASYMPTOTIC BEHAVIOR OF FUNCTIONAL DYNAMIC EQUATIONS IN TIME SCALE

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ABSTRACT. It is considered a scalar linear functional dynamic equation in time scale with delayed argument of the form

$$y^{\Delta}(t) = b(t)y(\tau(t)),\ t \in \mathbb{T} \cap [0, +\infty[, \tag{0.1}$$

where $\mathbb{T}$, the time scale, is a closed subset of $\mathbb{R}$ without upper bound for this case, $^{\Delta}$ is de Hilger's derivate, which among other things, unifies difference operator for sequences and the derivate.

The functions $b, \tau : \mathbb{T} \to \mathbb{C}$, $\tau > 0$, are "locally integrable" and satisfy integral smallness conditions in a sense to be defined later. Asymptotic formulas of solutions of equation (0.1) are given. They unify and extend asymptotic formulas of difference and differential equations.

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1. INTRODUCTION

For differential equations and difference equations, a sort of results about qualitative description of their solutions have been obtained (see, for example [9, 10, 12, 13, 14]). There are some of these results which are similar but they have been proved with different techniques. Those results can be unified and extended by mean dynamic equations in time scales.

Asymptotic behavior for dynamic equations on time scales is an issue which has been being recently studied (see [1, 2, 7, 17]).

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The theory of time scales was introduced in 1988 by S. Hilger [15] in his Ph.D. Thesis in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this theory, see [4, 8, 16] and the references cited therein.

The Hilger's derivate or Hilger's Δ is an operator $^{\Delta}$ such that if E a complex Banach space and f is a function $f : \mathbb{T} \to E$, f is differentiable in $t \in \mathbb{T}$ if there is a

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real number $f^{\Delta}(t)$ such that given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \tag{1.1}$$

where $s \in]t - \delta, t + \delta[\cap \mathbb{T}$ and $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. σ is called the forward jump function. The backward jump function is given by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, for all $t \in \mathbb{T}$. The length of the forward jump is denoted and defined by $\mu(t) = \sigma(t) - t$.

Consider

$$\mathbb{T} = \dot{\bigcup}_{j=1}^{+\infty} ([a_j, b_j] \cup \{p_{j,0}, p_{j,1}, \ldots, p_{j,m_j}\}), \tag{1.2}$$

where $b_j = p_{j,0} < p_{j,1} < \cdots < p_{j,m_j} = a_{j+1}$. Then, $\mathbb{T}$ is a time scale, $f^{\Delta}(t) = \frac{df}{dt}$ for $t \in [a_n, b_n[$ and $f^{\Delta}(p_{j,k}) = \frac{f(p_{j,k+1}) - f(p_{j,k})}{p_{j,k+1} - p_{j,k}}$.

Particularly, $\mathbb{T} = \dot{\cup}_{j=1}^{+\infty} [a_j, b_j]$ is a time scale with $f^{\Delta}(b_j) = \dfrac{f(a_{j+1}) - f(b_j)}{a_{j+1} - b_j}$, where $b_j < a_{j+1}$ for all $j \in \mathbb{N}$.

When $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ the Hilger's derivate is

$$f^{\Delta}(t) = \frac{1}{h}(f(t+h) - f(t)),$$

which is an approximation to the ordinary derivate when h is small enough. Obviously, the continuous case is the case when the Hilger's derivate is the ordinary derivate $f^{\Delta}(t) = \frac{df}{dt}$ which is obtained for $\mathbb{T} = \mathbb{R}$.

There are examples where time scales can be used. A particular case is *Magicicada Septendecim* which is a specie of insect in the family Cicadidae. It is found in Canada and the United States. It lives as a larva for 17 years and as an adult for around a week. Other similar case, worthy to be considered, is the mayfly *Stenonema Canadense*. It lives as a larva for 1 year and as an adult for less than a day. In the both cases, the population can be expressed as a function of the time in a time scale of the form $\mathbb{T} = \dot{\bigcup}_{j=1}^{+\infty} C_j$, where $C_j = I_j \dot{\cup} \{p_j\}$, I_j is a closed interval which can represent the life as larvas in both species and p_j can represent one unity of time in the adult life of the species. Those examples can be found in the Christiansen-Fenchel's book [11] and M. Bohner and A. Peterson's book [8, pag. 15, 71]. A model with the same kind of time scale for an electric circuit can be found in [8, pag. 16].

This work concerns asymptotic representations for certain solutions of a class of delayed equations, which have been called functional dynamic equations on time scales. These are generalizations of classically-studied delay differential equations or delay difference equations corresponding to the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

The object of study in this work is the asymptotic behavior of the solutions of the dynamic equation in time scale

$$y^{\Delta}(t) = b(t)y(\tau(t)), \quad t \in \mathbb{T} \cap [0, +\infty[, \tag{1.3}$$

where $\mathbb{T}$ is a time scale such that

$$\sup \mathbb{T} = +\infty, \tag{1.4}$$

and it is assumed that:

(H1) $b : \mathbb{T} \cap [0, +\infty[\to \mathbb{C}$ is a locally integrable function.

(H2)

1. $\tau : \mathbb{T} \cap [0, +\infty[\to]0, +\infty[$ is a locally integrable function;
2. $t - \tau(t) \geq 0$, for all $t \in \mathbb{T} \cap [0, +\infty[$;
3. $r_0 := \sup\limits_{t \in \mathbb{T} \cap [0,+\infty[} [t - \tau(t)] < +\infty$;
4. τ is strictly increasing.

Equation (1.3) unifies a class of differential and difference equations with delayed argument and more general cases.

This work is organized as follows: In the next section some previous facts are given. Then, the section with a result about asymptotic constancy like [3, 5], the main result and their respective proofs are presented. Finally, a section with some corollaries is showed.

2. PRELIMINARIES

Basic properties of the operator $^\Delta$ are given in [4, Theorems 3 and 4] and some properties for its respective integral in time scales can be found in [4, Theorems 6 and 7].

Now, it is presented a little extension of the integral for time scales in order to obtain Banach spaces with a norm defined in terms of such an integral.

Let $f : \mathbb{T} \to \mathbb{C}$. A extension for f is denoted as $\mathrm{ext}_1(f)$, where $g := \mathrm{ext}_1(f) : \mathbb{R} \to \mathbb{C}$ is a function such that $g(t) = f(t)$ for all $t \in \mathbb{T}$, $g(s) = g(t)$ for all $s \in [t, \sigma(t)[$ and $t \in \mathbb{T}$ whenever $\mu(t) > 0$ and if there is $t_0 = \min \mathbb{T}$ or there is $t_1 = \max \mathbb{T}$, $g(s) = 0$ if $s < t_0$ or $s > t_1$, respectively.

Let A be a Lebesgue measurable subset of $\mathbb{R}$. $L^{loc}_{\mathbb{T}}(A)$ denotes the functions $f : \mathbb{T} \to \mathbb{C}$ such that $\mathrm{ext}_1(f) : \mathbb{R} \to \mathbb{C}$ is locally integrable in the Lebesgue on A.

When the context is known, $L^{loc}_{\mathbb{T}}(A)$ is denoted as $L^{loc}_{\mathbb{T}}$.

If J is a measurable subset of $\mathbb{R}$ and $f \in L^{loc}_{\mathbb{T}}(J, E)$, it is defined $\int_J f(\zeta)\Delta\zeta = \int_J \mathrm{ext}_1(f)(\zeta)d\zeta$, where the right side member of the recent equality is an integral in the Lebesgue sense. Let $T, t \in \mathbb{R} \cup \{\pm\infty\}$ be such that $T < t$ and $f \in L^{loc}_{\mathbb{T}}(]T, t[, E)$. Then,

$$\int_T^t f(\zeta)\Delta\zeta = \int_T^t \mathrm{ext}_1(f)(\zeta)d\zeta,$$

where the right side member of the recent equality is an integral in the Lebesgue sense.

A function $F : \mathbb{T} \to \mathbb{C}$ is called an *anti-derivate* of f if and only if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.1. It can be proved that if f is a rd-continuous function, i.e.,

1. $\lim_{s\to t^-} f(s)$ exists when $\rho(t) = t$,
2. $\lim_{s\to t} f(s) = f(t)$ when $\sigma(t) = t$,

then f has an anti-derivate. If it is denoted by F, then

$$\int_T^t f(\zeta)\Delta\zeta := F(t) - F(T). \tag{2.1}$$

If $\mathbb{T}$ is given by (1.2) then a discrete-continuous situation is obtained:

$$\begin{aligned} \int_T^t f(\zeta)\Delta\zeta &= \int_T^{b_{n_T}} f(\zeta)d\zeta + \sum_{j=n_T}^{n_t-1} \sum_{k=1}^{m_j} f(p_{j,k-1})(p_{j,k} - p_{j,k-1}) \\ &+ \sum_{j=n_T+1}^{n_t-1} \int_{a_j}^{b_j} f(\zeta)d\zeta + \int_{a_{n_t}}^t f(\zeta)d\zeta, \end{aligned} \tag{2.2}$$

where for $s = T$ or $s = t$ there is $n_s \in \mathbb{N}$ such that $a_{n_s} \leq s \leq b_{n_s}$. Extensions for $t = p_{j,k}$ or $T = p_{j,k}$ can be naturally obtained.

If $\mathbb{T} = \{T_n\}_{n=1}^{+\infty}$ with $T_{n+1} > T_n$ for all $n \in \mathbb{N}$ then

$$\int_T^t f(\zeta)\Delta\zeta = \sum_{T\leq T_j<t} f(T_j)(T_{j+1} - T_j).$$

Obviously, if $\mathbb{T} = \mathbb{R}$ then the "continuous integral" is obtained, i.e., $\int_T^t f(\zeta)\Delta\zeta = \int_T^t f(\zeta)d\zeta$.

For a function $f : \mathbb{T} \to \mathbb{C}$ such that $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$ (regressive), the solution of the initial value problem

$$y^{\Delta} = f(t)y \text{ and } y(t_0) = 1$$

exists and it is denoted as $y(t) = e_f(t, t_0)$ for $t \in \mathbb{T}$. Then,

$$e_f(t, s) = \exp\left(\int_s^t \xi_{\mu(\zeta)}(f(\zeta))\Delta\zeta\right), \tag{2.3}$$

where

$$\xi_h(z) = \begin{cases} \frac{\ln(1+hz)}{h} & \text{if } h > 0 \\ z & \text{if } h = 0, \end{cases}$$

for $z \in \mathbb{C}_h := \mathbb{C} - \{-\frac{1}{h}\}$.

Remark 2.2. For $z \in \mathbb{C}$,

$$\lim_{h\to 0^+} \xi_h(z) = z.$$

Notice that $\xi_h(z) = 0$ if and only if $z = 0$.

Some examples of (2.3) are:

1. If $\mathbb{T}$ is given by (1.2) then
$$e_f(t,T)=\exp\left(\int_T^{b_{n_T}} f(\zeta)d\zeta\right)\left[\prod_{j=n_T}^{n_t-1}\prod_{k=1}^{m_j}[1+f(p_{j,k})(p_{j,k}-p_{j,k-1})]\right]$$
$$\times\exp\left(\sum_{j=n_T+1}^{n_t-1}\int_{a_j}^{b_j} f(\zeta)d\zeta\right)\exp\left(\int_{a_{n_t}}^{t} f(\zeta)d\zeta\right),$$
where for $s=T$ or $s=t$ there is $n_s\in\mathbb{N}$ such that $a_{n_s}\leq s\leq b_{n_s}$. Extensions for $t=p_{j,k}$ or $T=p_{j,k}$ can be naturally obtained.
2. If $\mathbb{T}=\{T_n\}_{n=1}^{+\infty}$ is such that there is $r_0>0$ with $T_{n+1}>T_n$ for all $n\in\mathbb{N}$, then $e_f(t,T)=\displaystyle\prod_{T\leq T_j<t}(1+f(T_j)(T_{j+1}-T_j))$.
3. If $\mathbb{T}=\mathbb{R}$, then $e_f(t,T)=\exp\left(\int_T^t f(\zeta)d\zeta\right)$.

Now, it is created a metric space which will help to prove the main result. Let
$$\|\nu\|_{t_0,0}:=\sup_{t\geq t_0}\int_t^{\tau^{-1}(t)}\|\nu(\zeta)\|\Delta\zeta, \tag{2.4}$$
for all $\nu\in L^{loc}_{\mathbb{T}}([t_0,+\infty[,\mathbb{C})$ and
$$\mathcal{E}(t_0):=\left\{\nu\in L^{loc}_{\mathbb{T}}([t_0,+\infty[,\mathbb{C}):\|\nu\|_{t_0,0}<+\infty\right\},$$
where $t_0\in\mathbb{T}\cap[0,+\infty[$, r_0 is given in (**H2**). Then $(\mathcal{E}(t_0),\|\cdot\|_{t_0,0})$ is a Banach space. If
$$\mathcal{B}_{01}(t_0):=\{\nu\in\mathcal{E}(t_0):\|\nu\|_{t_0,0}\leq 1\},$$
$(\mathcal{B}_{01}(t_0),\|\cdot\|_{t_0,0})$ is a closed subset of the Banach space $(\mathcal{E}(t_0),\|\cdot\|_{t_0,0})$.

For $p\in[1,+\infty]$, assume that $L^p_{\mathbb{T}}(t_0)$ denotes the functions $f:[t_0,+\infty[\cap\mathbb{T}\to\mathbb{C}$ such that $\mathrm{ext}_1(f)\in L^p$ and

1. $$\|f\|_p=\left[\int_{t_0}^{+\infty}\|f(\zeta)\|^p\Delta\zeta\right]^{1/p}<+\infty, \tag{2.5}$$
for $p\in[1,+\infty[$
2. $\|f\|_\infty=\mathrm{esssup}\{\|f(t)\|:t\in[t_0,+\infty[\cap\mathbb{T}\}$.

Remark 2.3. Notice that $(L^p_{\mathbb{T}}(t_0),\|\cdot\|_p)\hookrightarrow(\mathcal{E}(t_0),\|\cdot\|_0)$ for $p\in[1,+\infty]$. For $p\in[1,+\infty]$, $(L^p_{\mathbb{T}},\|\cdot\|_p)$ is a Banach space. If a sequence converges to zero (almost everywhere) in the norm $\|\cdot\|_p$, such a sequence converges to zero (almost everywhere) in the norm $\|\cdot\|_{t_0,0}$.

It is useful to keep in mind some important facts as Hölder's inequality, i.e, if $p>1$, $f\in L^p_{\mathbb{T}}$ and if $g\in L^{\frac{p}{p-1}}_{\mathbb{T}}$ then
$$\|fg\|_1\leq\|f\|_p\|g\|_{\frac{p}{p-1}};$$

if $p > 1$, $f \in L^p_{\mathbb{T}}$ and if $g \in L^p_{\mathbb{T}}$, then

$$fg \in L_{\mathbb{T}}^{\max\left\{1,\frac{p}{2}\right\}};$$

if $p > 1$, $f \in L^p_{\mathbb{T}}$, then $\lim_{t\to+\infty} e_f(t,\tau(t)) - 1 = 0$ and $e_f(t,\tau(t)) - 1 \in L^p_{\mathbb{T}}$.

3. ASYMPTOTIC CONSTANCY AND MAIN RESULT

3.1. **Previous asymptotic assumptions and facts.** The following lemma is about asymptotic constancy and it is motivated by Atkinson-Haddock [3]. It is given in order to lead, via change of variables, to the asymptotic formula of the linear dynamic equation (1.3).

Lemma 3.1. *Consider the functional dynamic equation*

$$x^{\Delta}(t) = f(t, x(t), x(\tau(t))), \quad t \in \mathbb{T} \cap [0, +\infty[, \tag{3.1}$$

where $\mathbb{T}$ is a time scale which satisfies (1.4), τ satisfies (**H2**)*, $f : [0, +\infty[\times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a locally integrable function in the first variable such that*

$$|f(t, z_1, z_2)| \le \gamma(t)|z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{C}$ and $\gamma : [0, +\infty[\cap \mathbb{T} \to \mathbb{R}$ is a locally integrable function such that

$$\Theta = \sup_{t \ge t_0} \int_t^{\tau^{-1}(t)} \gamma(s)\Delta s < 1,$$

for t_0 large enough. Then every solution of the equation (3.1) *is asymptotically constant as $t \to +\infty$. Clearly, every constant function is a solution of* (3.1)*.*

Proof. Let x be a solution of the equation (3.1). By integrating equation (3.1) from t^* to T, it is obtained

$$\int_{t^*}^{T} |x^{\Delta}(t)|\Delta t \le \int_{t^*}^{T} \gamma(t) \int_{\tau(t)}^{t} |x^{\Delta}(s)|\Delta s \Delta t.$$

By an analogous procedure to the proof of [3, Lemma 2.1],

$$\int_{t^*}^{T} |x^{\Delta}(t)|\Delta t \le \int_{\tau(t^*)}^{T} |x^{\Delta}(s)| \int_{s}^{\tau^{-1}(s)} \gamma(t)\Delta t \Delta s.$$

So,

$$\int_{t^*}^{T} |x^{\Delta}(s)| \left[1 - \int_{s}^{\tau^{-1}(s)} \gamma(t)\Delta t\right] \Delta s \le \int_{\tau(t^*)}^{t^*} |x^{\Delta}(s)| \int_{s}^{\tau^{-1}(s)} \gamma(t)\Delta t\Delta s.$$

By making $t^* = t_0$ and $T \to +\infty$,

$$\limsup_{T\to+\infty} \mathcal{K}(T) \le \frac{1}{1-\Theta} \int_{\tau(t_0)}^{t_0} |x^{\Delta}(s)| \int_{s}^{\tau^{-1}(s)} \gamma(t)\Delta t\Delta s,$$

where $\mathcal{K}(T) = \displaystyle\int_{t_0}^{T} |x^{\Delta}(s)|\Delta s$. It can be noticed that $\mathcal{K}(T)$ is an increasing and bounded function of T. So, $\mathcal{K}(T)$ converges to a constant limit as $T \to +\infty$. Hence $x^{\Delta} \in L^1_{\mathbb{T}}$. Therefore x is asymptotically constant as $t \to +\infty$. □

3.2. **Main result.** The following result, uses Lemma 3.1 and a linear transformation involving a type of exponential function on the time scale to reduce the problem to that of showing asymptotic constancy (that every solution tends to a constant plus $o(1)$) as $t \to +\infty$.

Proposition 3.2. *Consider the linear functional dynamic equation* (1.3), *with the assumptions* (1.4), **(H1)** *and* **(H2)**. *Assume that*

$$\Theta = \sup_{t \geq t_0 + r_0} \int_{t}^{\tau^{-1}(t)} |b(\zeta)| \sup_{\nu \in \mathcal{B}_{01}(t_0)} e_{\|\nu\|}(\sigma(\zeta), \tau(\zeta))\Delta\zeta < 1, \tag{3.2}$$

where $t_0 \geq 0$ is such that and $[t_0 - r_0, t_0] \cap \mathbb{T} \neq \phi$. Then, every solution $y(t)$ of (1.3), *defined for $t \in \mathbb{T} \cap [t_0, +\infty[$, has the following asymptotic formula*

$$y(t) = e_{\nu_\infty}(t, t_0)(c + o(1)), \tag{3.3}$$

as $t \to +\infty$, where

$$\nu_\infty(t) = b(t) + \sum_{j=1}^{+\infty} \Delta_j(t), \tag{3.4}$$

$$\Delta_j(t) = b(t)[e_{\mu_j}(\tau(t), t) - e_{\mu_{j-1}}(\tau(t), t)],$$

$\mu_j(t) = b(t)e_{\mu_{j-1}}(\tau(t), t)$, for all $t \geq t_0 + jr_0$, $\mu_j(t) = 0$ for all $t \in [t_0, t_0 + jr_0[$, $\mu_0 = 0$, $j \in \mathbb{N}$ and the series (3.4) *are defined as the limit of partial sums in the norm $\|\cdot\|_{t_0,0}$ given by* (2.4). *Conversely, given $c \in \mathbb{C}$ there is a solution $y = y(t)$ of* (1.3) *satisfying* (3.3).

Proof. Let $\mathcal{T}$ be the expression $(\mathcal{T}\nu)(t) = 0$ if $t_0 \leq t < t_0 + r_0$ and

$$(\mathcal{T}\nu)(t) = b(t)e_{\nu}(\tau(t), t),$$

for $t \geq t_0 + r_0$. From (3.2), $\mathcal{T}(\mathcal{B}_{01}(t_0)) \subseteq \mathcal{B}_{01}(t_0)$. So, the restriction $\mathcal{T} : \mathcal{B}_{01}(t_0) \to \mathcal{B}_{01}(t_0)$ is well defined.

$\mathcal{T} : \mathcal{B}_{01}(t_0) \to \mathcal{B}_{01}(t_0)$ is a contraction in $(B_{01}(t_0), \|\cdot\|_{t_0,0})$. In fact, for $\nu_1, \nu_2 \in \mathcal{B}_{01}(t_0)$, by the Mean Value Theorem,

$$|e_{\nu_1}(t, s) - e_{\nu_2}(t, s)| \leq \sup_{\nu \in \mathcal{B}_{01}(t_0)} e_{\|\nu\|}(t, s)\|\nu_1 - \nu_2\|_{t_0,0},$$

for $\nu_1, \nu_2 \in \mathcal{B}_{01}(t_0)$ and $t \geq s$. Then,

$$\begin{aligned} |(\mathcal{T}\nu_1)(t) - (\mathcal{T}\nu_2)(t)| &\leq |b(t)||e_{\nu_1}(\tau(t), t) - e_{\nu_2}(\tau(t), t))| \\ &\leq |b(t)| \sup_{\nu \in \mathcal{B}_{01}(t_0)} e_{\|\nu\|}(t, \tau(t))\|\nu_1 - \nu_2\|_{t_0,0} \end{aligned}$$

$$\leq |b(t)| \sup_{\nu \in \mathcal{B}_{01}(t_0)} e_{\|\nu\|}(\sigma(t), \tau(t)) \|\nu_1 - \nu_2\|_{t_0,0},$$

if $t \geq t_0 + r_0$ and $(\mathcal{T}\nu_1)(t) - (\mathcal{T}\nu_2)(t) = 0$ if $t_0 \leq t < t_0 + r_0$ since $|\nu_1|, |\nu_2| \leq \nu_\infty$. So,

$$|(\mathcal{T}\nu_1)(t) - (\mathcal{T}\nu_2)(t)| \leq |b(t)| \sup_{\nu \in \mathcal{B}_{01}(t_0)} e_{\|\nu\|}(\sigma(t), \tau(t)) \|\nu_1 - \nu_2\|_{t_0,0},$$

for all $t \geq t_0$. Therefore,

$$\|(\mathcal{T}\nu_1) - (\mathcal{T}\nu_2)\|_{t_0,0} \leq \Theta \|\nu_1 - \nu_2\|_{t_0,0}, \tag{3.5}$$

for all $\nu_1, \nu_2 \in \mathcal{B}_{01}(t_0)$, where Θ is given in (3.2) and $\mathcal{T}$ is a contraction.

Since $(B_{01}(t_0), \|\cdot\|_{t_0,0})$ is a complete metric space, by the Banach Fixed Point theorem, there is a unique function $\nu_\infty \in \mathcal{B}_{01}(t_0)$ such that $\mathcal{T}\nu_\infty = \nu_\infty$. The change of variables

$$y(t) = e_{\nu_\infty}(t, t_0) z(t),$$

is made in (1.3) and the equation

$$\begin{aligned} z^\Delta(t) = \frac{1}{1 + \mu(t)\nu_\infty(t)} & [b(t) e_{\nu_\infty}(\tau(t), t)[z(\tau(t)) - z(t)] \\ & + [(\mathcal{T}\nu_\infty)(t) - \nu_\infty(t)] z(t)], \end{aligned}$$

is obtained for $t \geq t_0 + r_0$, i.e., $z = z(t)$ satisfies the equation

$$z^\Delta(t) = b(t) e_{\nu_\infty}(\tau(t), \sigma(t))[z(\tau(t)) - z(t)], \ t \geq t_0 + r_0. \tag{3.6}$$

Clearly every constant function is a solution of (3.6). By Lemma 3.1 and condition (3.2), every solution of (3.6) is asymptotically constant. Therefore, every solution $y = y(t)$ of the equation (1.3), defined for $t \in \mathbb{T} \cap [t_0, +\infty[$, has the asymptotic formula

$$y(t) = e_{\nu_\infty}(t, t_0)(c + o(1)), \tag{3.7}$$

as $t \to +\infty$. Conversely, given any $c \in \mathbb{C}$ there is a solution of (1.3) which satisfies (3.7). Since $\mathcal{T} : \mathcal{B}_{01}(t_0) \to \mathcal{B}_{01}(t_0)$ is a contraction,

$$\lim_{n \to +\infty} \|\mathcal{T}^n(0) - \nu_\infty\|_{t_0,0} = 0.$$

Notice that $\mathcal{T}^n(0)$ can be written as the partial sum

$$\mathcal{T}^n(0) = \mathcal{T}(0) + \sum_{j=2}^{n} (\mathcal{T}^j(0) - \mathcal{T}^{j-1}(0)).$$

Now $\mathcal{T}(0) = b(t)$, $\mathcal{T}^{j+1}(0) - \mathcal{T}^j(0) = \Delta_j(t)$, where

$$\Delta_j(t) = b(t)[e_{\mu_j}(\tau(t), t) - e_{\mu_{j-1}}(\tau(t), t)],$$

$\mu_j(t) = b(t) e_{\mu_{j-1}}(\tau(t), t)$, for all $t \geq t_0 + j r_0$, $\mu_j(t) = 0$ for all $t \in [t_0, t_0 + j r_0[$, $\mu_0 = 0$ and $j \in \mathbb{N}$. So, a formula of ν_∞ in (3.3) is obtained. $\square$

As it can be seen in [6, 18], condition (3.2) is a condition of non oscillation for equation (1.3).

Remark 3.3. For making easier the understanding of the asymptotic formula (3.3), consider $\nu_n(t) = 0$ for $t \in [t_0, t_0 + r_0[$ and

$$\nu_n(t) = b(t) + \sum_{j=1}^{n} \Delta_j(t),$$

for $t \geq t_0 + r_0$ and $n \in \mathbb{N}$ in Proposition 3.2. Then, given $n_0 \in \mathbb{N}$, $\nu_n(t) = \nu_{n_0}(t)$ for all $t \in [t_0, t_0 + n_0 r_0[$ and $n \geq t_0 + n_0$. So, (3.3) may be written as an step asymptotic function:

$$y(t) = e_{\nu_{n_t}}(t, t_0)(c + o(1)),$$

as $t \to +\infty$, where $n_t \in \mathbb{N}$ is such that $t_0 \leq t < t_0 + n_t r_0$.

Remark 3.4. If $b \in L^p_{\mathbb{T}}(t_0)$ for some $p > 1$ then (3.2) is satisfied for t_0 large enough. Let $n \in \mathbb{N} \cup \{0\}$ be such that $p \in]2^n, 2^{n+1}]$. By Remark 2.3, $\sum_{j=n+1}^{+\infty} \Delta_j \in L^1$. So, ν_∞ in (3.3) may be written as:

$$y(t) = e_{\nu_n}(t, t_0)(c + o(1)),$$

as $t \to +\infty$, where

$$\nu_n(t) = b(t) + \sum_{j=1}^{n} \Delta_j(t).$$

4. APPLICATIONS

4.1. **Large enough limit points in a time scale.** Assume that $\mathbb{T}$ is a time scale satisfying (1.4) such that given $t_0 \in \mathbb{T}$, there is a limit point $t_\infty \in \mathbb{T}$ with $t_\infty \geq t_0$. Then (3.2) becomes,

$$\Theta = e \sup_{t \geq r_0} \int_t^{\tau^{-1}(t)} |b(\zeta)| \Delta\zeta < 1. \tag{4.1}$$

In fact, given a limit point $t_\infty \in \mathbb{T}$, there is a sequence $(\nu_n)_{n=1}^{+\infty}$ where $t_\infty \in \bigcap_{n=1}^{+\infty} \mathrm{supp}(\nu_n)$ such that $\lim_{n \to +\infty} e_{\|\nu_n\|}(\sigma(t), \tau(t)) = e$. In this case, condition (4.1) can be expressed as

$$\sup_{t \geq t_0} \int_t^{\tau^{-1}(t)} |b(\zeta)| \Delta\zeta < \frac{1}{e}, \tag{4.2}$$

for t_0 large enough.

4.2. **Case $\mathbb{T}$ is given by (1.2).** If $\mathbb{T}$ is given by (1.2), $\mathbb{T}$ has large enough limit points. Then, (3.2) becomes (4.2). By applying of the formula in (2.2), the following result is obtained:

Corollary 4.1. *Consider the linear functional differential equation*

$$y^{\Delta}(t) = b(t)y(\tau(t)),\ t \geq 0 \tag{4.3}$$

where

$$b : [0, +\infty[\cap \mathbb{T} \to \mathbb{C}$$

is a locally integrable function such that

$$\sup_{t \geq t_0} \Bigg[\int_t^{b_{n_t}} b(\zeta)d\zeta + \sum_{j=n_t}^{n_{\tau^{-1}(t)}-1} \sum_{k=1}^{m_j} b(p_{j,k-1})(p_{j,k} - p_{j,k-1}) + \sum_{j=n_t+1}^{n_{\tau^{-1}(t)}-1} \int_{a_j}^{b_j} b(\zeta)d\zeta + \int_{a_{n_t}}^{\tau^{-1}(t)} b(\zeta)d\zeta \Bigg] < \frac{1}{e}, \tag{4.4}$$

for t_0 large enough, where for $s = t$ or $s = \tau^{-1}(t)$ there is $n_s \in \mathbb{N}$ such that $a_{n_s} \leq s \leq b_{n_s}$.

Then, every solution of (4.3) *has the following asymptotic formula*

$$\begin{aligned} y(t) = {} & \exp\left(\int_{t_0}^{b_{n_{t_0}}} [b(\zeta) + \Delta_j(\zeta)]d\zeta \right) \\ & \times \left[\prod_{j=n_{t_0}}^{n_{\tau^{-1}(t)}-1} \prod_{k=1}^{m_j} [1 + [b(p_{j,k-1}) + \Delta_j(p_{j,k-1})](p_{j,k} - p_{j,k-1})] \right] \\ & \times \exp\left(\sum_{j=n_{t_0}+1}^{n_{\tau^{-1}(t)}-1} \int_{a_j}^{b_j} [b(\zeta) + \Delta_j(\zeta)]d\zeta \right) \\ & \times \exp\left(\int_{a_{n_{t_0}}}^{\tau^{-1}(t)} [b(\zeta) + \Delta_j(\zeta)]d\zeta \right) (c + o(1)), \end{aligned} \tag{4.5}$$

as $t \to +\infty$, where $b_j = p_{j,0} < p_{j,1} < \cdots < p_{j,m_j} = a_{j+1}$,

$$\Delta_j(t) = b(t) \left[e_{\mu_j}(\tau(t), t) - e_{\mu_{j-1}}(\tau(t), t) \right],$$

$\mu_j(t) = b(t)e_{\mu_{j-1}}(\tau(t), t)$, for all $t \geq t_0 + jr_0$, $\mu_j(t) = 0$ for all $t < t_0 + jr_0$, $\mu_0 = 0$ and $j \in \mathbb{N}$. Conversely, given $c \in \mathbb{C}$ there is a solution $y = y(t)$ of (4.3) *satisfying* (4.5).

Extensions for $t = p_{j,k}$ or $\tau^{-1}(t) = p_{j,k}$ can be naturally obtained.

4.3. **Case** $\mathbb{T} = \mathbb{R}$**.** The following corollary is an obvious consequence of the Corollary 4.1.

Corollary 4.2 (See [10, (2004)])**.** *Consider the linear functional differential equation*

$$y'(t) = b(t)y(\tau(t)), t \geq 0 \tag{4.6}$$

where

$$b : [0, +\infty[\to \mathbb{C}$$

is a locally integrable function such that

$$\sup_{t \geq t_0} \int_t^{\tau^{-1}(t)} |b(s)| ds < \frac{1}{e}, \tag{4.7}$$

for t_0 *large enough. Then, every solution of* (4.6) *has the following asymptotic formula*

$$y(t) = \exp\left(\int_{r_0}^t \left[b(s) + \sum_{j=1}^{+\infty} \Delta_j(s)\right] ds\right)(c + o(1)), \tag{4.8}$$

as $t \to +\infty$*, where*

$$\Delta_j(t) = b(t)[e^{\int_t^{\tau(t)} \mu_j(\zeta)d\zeta} - e^{\int_t^{\tau(t)} \mu_{j-1}(\zeta)d\zeta}],$$

$\mu_j(t) = b(t)e^{\int_t^{\tau(t)} \mu_{j-1}(\zeta)d\zeta}$*, for all* $t \geq jr_0$*,* $\mu_j(t) = 0$ *for all* $t \in [0, jr_0[$*,* $\mu_0 = 0$ *and* $j \in \mathbb{N}$*. Conversely, given* $c \in \mathbb{C}$ *there is a solution* $y = y(t)$ *of* (4.6) *satisfying* (4.8)*.*

Remark 4.3. Haddock-Sacker [14] proposed a conjecture which they proved for a scalar delay differential equation of the form

$$x'(t) = \lambda_0(t)x(t) + \tilde{b}(t)x(t - r_0),$$

where $e^{-\int_{t-r_0}^t \lambda_0(s)ds}\tilde{b}(t) \in L^2$. By mean the change of variables $x = \exp\left(\int_{r_0}^t \lambda_0(s)ds\right) y$, it is obtained the delay differential equation

$$y'(t) = b(t)y(t - r_0),$$

where $b = e^{-\int_{t-r_0}^t \lambda_0(s)ds}\tilde{b}(t)$. From Remark 3.4, (4.8) can be written as

$$y(t) = \exp\left(\int_{r_0}^t b(s)ds\right)(c + o(1)),$$

as $t \to +\infty$ which is the scalar Haddock-Sacker result.

4.4. **Case $\mathbb{T} = \mathbb{Z}$.** Consider $\mathbb{T} = \mathbb{Z}$ and $\tau(n) = n - k$. Then,

$$\sup \left\{ \prod_{\zeta=n-k}^{n-1} [1 + v(\zeta)] : \sum_{\zeta=n}^{n+k-1} |v(\zeta)| \leq 1 \right\} = \left(1 + \frac{1}{k}\right)^k .$$

So, the relation (3.2) becomes

$$\sup_{n \geq n_0} \sum_{\zeta=n-k}^{n-1} |b(\zeta)| < \left(\frac{k}{k+1}\right)^{k+1} , \tag{4.9}$$

where $n_0 \in \mathbb{N}$ is so large as it would be necessary. So, the following corollary is obtained as a consequence of Proposition 3.2.

Corollary 4.4. *Consider the linear functional differential equation*

$$\Delta y(n) = b(n) y(n-k), \tag{4.10}$$

where $\Delta y(n) = y(n+1) - y(n)$, $k \in \mathbb{N}$, $(b(n))_{n=0}^{+\infty}$ *is a sequence of real numbers which satisfies* (4.9) *for* n_0 *large enough. Then, every solution of* (4.10) *has the following asymptotic formula*

$$y(n) = \left[\prod_{\zeta=N}^{n-1} \left[1 + \left(\prod_{l=j-k}^{\zeta-1} b(l) \right) + \sum_{j=1}^{+\infty} \Delta_j(\zeta) \right] \right] (c + o(1)), \tag{4.11}$$

as $n \to +\infty$, *where*

$$\Delta_j(n) = b(n) \left[\prod_{\zeta=n-k}^{n} \frac{1}{1 + \mu_j(\zeta)} - \prod_{\zeta=n-k}^{n} \frac{1}{1 + \mu_{j-1}(\zeta)} \right],$$

$\mu_j(n) = b(n) \prod_{\zeta=n-k}^{n} \frac{1}{1+\mu_{j-1}(\zeta)}$, *for all* $n \geq jk$, $\mu_j(n) = 0$ *for all* $n < jk$, $\mu_0 = 0$ *and* $j \in \mathbb{N}$. *Conversely, given* $c \in \mathbb{C}$ *there is a solution* $y = y(n)$ *of* (4.10) *satisfying* (4.11).

Remark 4.5. The reader can compare the smallness condition (4.9) with those ones by Győri-Pituk's [13, Theorem 1]. They ask in their Theorem 1 a condition like (4.9) and a ℓ^p smallness condition. They get a recursive asymptotic formula as it is done here. Fon N large enough, ℓ^p smallness condition implies (4.9).

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