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MULTIPLE SOLUTIONS FOR A CLASS OF BOUNDARY–VALUE PROBLEMS WITH DEVIATING ARGUMENTS AND INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. This paper considers second order differential equations with integral boundary conditions. We establish sufficient conditions under which such boundary value problems with deviating arguments have positive solutions. To obtain the existence of at least three positive solutions, we use a fixed point theorem due to Avery and Peterson.

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1. INTRODUCTION

Put $J = [0,1]$, $J_0 = (0,1)$. Let us consider second order differential equations of type

$$\begin{cases} x''(t) + h(t)f(t, x(\alpha(t))) = 0, & t \in J_0, \\ x(0) = \gamma x'(0) - \displaystyle\int_0^1 g(s)x(s)ds, & \\ x(1) = \beta x(\eta), \beta, \gamma \in \mathbb{R}_+ = [0,\infty), & \eta \in (0,1), \end{cases} \tag{1.1}$$

where

$H_1:$ $f \in C(J \times \mathbb{R}, \mathbb{R}_+)$, $g \in C(J, \mathbb{R}_+)$, $\alpha \in C(J, J)$ and $t \leq \alpha(t) \leq 1$ on J,

$H_2:$ h is a nonnegative continuous function defined on $(0,1)$; h is not identically zero on any subinterval on $(0,1)$.

We have many fixed point theorems including corresponding theorems in a cone. Recently, there has been much attention on the existence of positive solutions for ordinary differential equations. There exists a vast literature devoted to the applications of fixed point theorems to obtain positive solutions of boundary value problems to second order differential equations, we mention, for example, only a few of papers [1]–[11]. Boundary value problems with integral conditions constitute an important class of such problems, see for example [3, 4]. A fixed point theorem in a cone can also be applied to second-order differential equations with deviating arguments but

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there is only a few papers when such techniques are applied, see for example [5, 7, 11]. In paper [10], the function f appearing on the right-hand-side depends on $x(t-1)$ for $t \in (0,1)$, where x is given on the initial interval $[0,1]$. It means that the corresponding problem from [10] has no delays. To obtain positive solutions of problem (1.1) we use a fixed point theorem due to Avery and Peterson [1]. Note that my paper is a first one when this fixed point theorem is applied to integral boundary problems with deviating arguments.

2. SOME LEMMAS

Let us consider the following problem

$$u''(t) + y(t) = 0, \quad t \in J_0, \tag{2.1}$$

$$u(0) = \gamma u'(0) - \int_0^1 g(s)u(s)ds, \tag{2.2}$$

$$u(1) = \beta u(\eta). \tag{2.3}$$

Put

$$g_1 = \int_0^1 g(s)ds, \quad g_2 = \int_0^1 sg(s)ds,$$
$$\Delta = (1-g_1)(1-\beta\eta) + (\gamma - g_2)(1-\beta), \quad G = \int_0^1 g(s)\int_0^s (s-\tau)y(\tau)d\tau ds.$$

We assume that:

$H_3:$ $\eta, g_1 \in (0,1)$, $\beta \in (0,1]$ and $\gamma \geq g_2$.

We require the following

Lemma 2.1. *Assume that $\eta \in (0,1)$, $\Delta \neq 0$ and $y \in C(J, \mathbb{R})$. Then problem (2.1)–(2.3) has the unique solution given by the following formula*

$$\begin{aligned} u(t) \;=\; & \frac{1}{\Delta}\Big\{[\gamma - g_2 + t(1-g_1)]\left[\int_0^1 (1-s)y(s)ds - \beta\int_0^\eta (\eta - s)y(s)ds\right] \\ & + [1-\beta\eta - t(1-\beta)]G\Big\} - \int_0^t (t-s)y(s)ds \end{aligned}$$

for $t \in J$.

Proof. Integrating two times differential equation (2.1) from 0 to t, we have

$$u(t) = u(0) + tu'(0) - \int_0^t (t-s)y(s)ds. \tag{2.4}$$

Using the boundary conditions (2.2)–(2.3), we obtain the system for $u(0)$ and $u'(0)$,

$$\begin{cases} u(0)(1-g_1) - u'(0)(\gamma - g_2) = G, \\ u(0)(1-\beta) + u'(0)(1-\beta\eta) = \displaystyle\int_0^1 (1-s)y(s)ds - \beta\int_0^\eta (\eta - s)y(s)ds + G. \end{cases}$$

Solving this system with respect to $u(0), u'(0)$ and substituting to formula (2.4) we have the assertion. This ends the proof. $\square$

Lemma 2.2. *Let Assumption H_3 hold. Assume that $y \in C(J, \mathbb{R}_+)$. Then the unique solution u of problem (2.1)–(2.3) satisfies the condition $u(t) \geq 0$ on $[0,1]$.*

Proof. Note that

$$\begin{aligned} u(0) &= \frac{1}{\Delta}\left\{(1-\beta\eta)G + (\gamma - g_2)\left[\int_0^1 (1-s)y(s)ds - \beta\int_0^\eta (\eta - s)y(s)ds\right]\right\} \\ &= \frac{1}{\Delta}\left\{(1-\beta\eta)G + (\gamma - g_2)\left[\int_0^\eta (1-\beta\eta - s(1-\beta))y(s)ds + \int_0^1 (1-s)y(s)ds\right]\right\} \\ &\geq 0, \end{aligned}$$

because $1 - \beta\eta - s(1-\beta) \geq 1 - \beta\eta - \eta(1-\beta) = 1 - \eta > 0$ for $s \in [0, \eta]$.

Moreover,

$$\begin{aligned} u(1) = &\frac{1}{\Delta}\left\{(1-\beta\eta - 1 + \beta)G + (\gamma - g_2 + 1 - g_1)\left[\int_0^1 (1-s)y(s)ds\right.\right. \\ &\left.\left. -\beta\int_0^\eta (\eta - s)y(s)ds\right]\right\} - \int_0^1 (1-s)y(s)ds \\ = &\frac{1}{\Delta}\left\{\beta(1-\eta)G + \beta(1-\eta)\int_0^\eta [\gamma - g_2 + s(1-g_1)]y(s)ds\right. \\ &\left. +\beta[\gamma - g_2 + \eta(1-g_1)]\int_\eta^1 (1-s)y(s)ds\right\} \geq 0. \end{aligned}$$

Since y is concave down and $y \in C(J, \mathbb{R}^+)$, then $u(t) \geq 0$, $t \in J$. This completes the proof. $\square$

Lemma 2.3. *Let Assumption H_3 hold. Assume that $y \in C(J, \mathbb{R}_+)$. Then the unique solution u of problem (2.1)–(2.3) satisfies the condition*

$$\min_{[\eta,1]} u(t) \geq \Gamma\|u\|,$$

where

$$\Gamma = \beta \min\left(\frac{1-\eta}{1-\beta\eta}, \eta\right).$$

Proof. Note that $u(1) = \beta u(\eta) \leq u(\eta)$. It means that

$$\min_{[\eta,1]} u(t) = u(1).$$

Put $u(t^*) = \|u\|$. If $t^* \leq \eta$, then

$$u(t^*) \leq u(\eta) + \frac{u(\eta) - u(1)}{1-\eta}(\eta - t^*) \leq u(\eta) + \frac{u(\eta) - u(1)}{1-\eta}\eta = \frac{1-\beta\eta}{\beta(1-\eta)}u(1).$$

It yields

$$\min_{[\eta,1]} u(t) \geq \frac{\beta(1-\eta)}{1-\beta\eta}\|u\|.$$

If $\eta < t^*$, then

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(0)}{\eta - 0}(t^* - \eta) \le u(\eta) + \frac{u(\eta)}{\eta}(1 - \eta) = \frac{1}{\beta\eta}u(1).$$

It yields

$$\min_{[\eta,1]} u(t) \ge \beta\eta\|u\|.$$

This ends the proof. □

Now, we present the necessary definitions from the theory of cones in Banach spaces.

Definition 2.4. Let E be a real Banach space. A nonempty convex set $P \subset E$ is said to be a cone provided that

(i) $ku \in P$ for all $u \in P$ and all $k \ge 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \le y$ if $y - x \in P$.

Definition 2.5. A map Λ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\Lambda : P \to \mathbb{R}_+$ is continuous and

$$\Lambda(tx + (1-t)y) \ge t\Lambda(x) + (1-t)\Lambda(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Similarly, we say the map φ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\varphi : P \to \mathbb{R}_+$ is continuous and

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.6. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let φ and Θ be nonnegative continuous convex functionals on P, Λ be a nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c and d, we define the following sets:

$$P(\varphi, d) = \{x \in P : \ \varphi(x) < d\},$$

$$P(\varphi, \Lambda, b, d) = \{x \in P : b \le \Lambda(x), \varphi(x) \le d\},$$

$$P(\varphi, \Theta, \Lambda, b, c, d) = \{x \in P : b \le \Lambda(x), \Theta(x) \le c, \varphi(x) \le d\},$$

and

$$R(\varphi, \Psi, a, d) = \{x \in P : a \le \Psi(x), \varphi(x) \le d\}.$$

We will use the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (1.1).

Theorem 2.7 (see [1]). *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative continuous convex functionals on P, Λ be a nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx) \le k\Psi(x)$ for $0 \le k \le 1$, such that for some positive numbers M and d,*

$$\Lambda(x) \le \Psi(x) \quad \text{and} \quad \|x\| \le M\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Suppose

$$T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$$

is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

(S_1): $\{x \in P(\varphi, \Theta, \Lambda, b, c, d) : \Lambda(x) > b\} \neq 0$ *and* $\Lambda(Tx) > b$ *for* $x \in P(\varphi, \Theta, \Lambda, b, c, d)$;

(S_2): $\Lambda(Tx) > b$ *for* $x \in P(\varphi, \Lambda, b, d)$ *with* $\Theta(Tx) > c$,

(S_3): $0 \notin R(\varphi, \Psi, a, d)$ *and* $\Psi(Tx) < a$ *for* $x \in R(\varphi, \Psi, a, d)$ *with* $\Psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$$\varphi(x_i) \le d, \quad \text{for} \quad i = 1, 2, 3,$$

$$b < \Lambda(x_1), \quad a < \Psi(x_2), \quad \text{with} \quad \Lambda(x_2) < b$$

and

$$\Psi(x_3) < a.$$

Let $X = C(J, \mathbb{R})$ be our Banach space with the norm $\|x\| = \max_{t \in J} |x(t)|$. Let

$$P = \{x \in X : x \text{ is nonnegative, concave on } J \text{ and } \min_{[\eta,1]} x(t) \ge \Gamma\|x\|\},$$

$$\bar{P}_r = \{x \in P : \|x\| \le r\},$$

where Γ is defined as in Lemma 2.3. We define the nonnegative continuous concave functional Λ on P by

$$\Lambda(x) = \min_{[\eta,1]} |x(t)|.$$

Note that $\Lambda(x) \le \|x\|$. Put $\Psi(x) = \Theta(x) = \|x\|$.

Theorem 2.8. *Let Assumptions H_1–H_3 hold. In addition, we assume that there exist positive constants a, b, c, d, $a < b$ and such that*

$$\mu > \frac{1}{\Delta}\left[(1 + \gamma - g_1 - g_2)\int_0^1 (1 - s)h(s)ds + (1 - \beta\eta)\int_0^1 g(s)\int_0^s (s - \tau)h(\tau)d\tau ds\right],$$

$$L = \frac{\beta}{\Delta}[\gamma - g_2 + \eta(1 - g_1)]\int_\eta^1 (1 - s)h(s)ds$$

and

(A_1): $f(t, u) \le \frac{d}{\mu}$ *for* $(t, u) \in J \times [0, d]$,

(A_2): $f(t,u) \geq \frac{b}{L}$ *for* $(t,u) \in [\eta,1] \times [b, \frac{b}{\Gamma}]$,
(A_3): $f(t,u) \leq \frac{a}{\mu}$ *for* $(t,u) \in J \times [0,a]$.

Then, problem (1.1) *has at least three positive solutions* x_1, x_2, x_3 *satisfying* $\|x_i\| \leq d$, $i = 1,2,3$ *and*

$$b \leq \Lambda(x_1), \quad a < \|x_2\| \quad \text{with} \quad \Lambda(x_2) < b$$

and $\|x_3\| < a$.

Proof. By T we denote the operator defined by

$$\begin{aligned}(Tx)(t) \;=\; & \frac{1}{\Delta}\Big\{[\gamma - g_2 + t(1-g_1)]\Big[\int_0^1 (1-s)h(s)f(s,x(\alpha(s)))ds \\ & -\beta\int_0^\eta (\eta - s)h(s)f(s,x(\alpha(s)))ds\Big] \\ & +[1-\beta\eta - t(1-\beta)]\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\Big\} \\ & -\int_0^t (t-s)h(s)f(s,x(\alpha(s)))ds\end{aligned}$$

for $t \in J$. Indeed, $T : X \to X$. Problem (1.1) has a solution x if and only if x solves the operator equation $x = Tx$.

Take $x \in P$. Then

$$(Tx)''(t) = -h(t)f(t,x(\alpha(t))) \leq 0, \quad t \in J$$

which implies that Tx is concave on J.

On the other hand, we obtain

$$\begin{aligned}(Tx)(0) = & \frac{1}{\Delta}\Big\{[\gamma - g_2]\Big[\int_0^\eta [1-\beta\eta - s(1-\beta)]h(s)f(s,x(\alpha(s)))ds \\ & +\int_\eta^1 (1-s)h(s)f(s,x(\alpha(s)))ds\Big] \\ & +(1-\beta\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\Big\} \\ \geq & \frac{1}{\Delta}\Big\{[\gamma - g_2]\Big[(1-\eta)\int_0^\eta h(s)f(s,x(\alpha(s)))ds \\ & +\int_\eta^1 (1-s)h(s)f(s,x(\alpha(s)))ds\Big] \\ & +(1-\beta\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\Big\} \geq 0,\end{aligned}$$

$$\begin{aligned}(Tx)(1) = & \frac{\beta}{\Delta}\Big\{(1-\eta)\int_0^\eta [\gamma - g_2 + s(1-g_1)]h(s)f(s,x(\alpha(s)))ds \\ & +[\gamma - g_2 + \eta(1-g_1)]\int_\eta^1 (1-s)h(s)f(s,x(\alpha(s)))ds\end{aligned}$$

$$+(1-\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\Bigg\} = \beta(Tx)(\eta) \geq 0.$$

It follows that $(Tx)(t) \geq 0$ on J. This and Lemma 2.1 show that $TP \subset P$.

Now we prove that the operator $T : P \to P$ is completely continuous. Let $x \in \bar{P}_r$. Then $|x| \leq r$. Note that h and f are continuous so h is bounded on J and f is bounded on $J \times [-r, r]$. It means that there exists a constant $K > 0$ such $\|Tx\| \leq K$. This proves that $T\bar{P}$ is uniformly bounded. On the other hand for $t_1, t_2 \in J$ there exists a constant $L_1 > 0$ such that

$$|(Tx)(t_1) - (Tx)(t_2)| \leq L_1|t_1 - t_2|.$$

This shows that $T\bar{P}$ is equicontinuous on J, so T is completely continuous.

Let $x \in \overline{P(\varphi, d)}$, so $0 \leq x(t) \leq d$, $t \in J$, and $\|x\| \leq d$. Note that also $0 \leq x(\alpha(t)) \leq d$, $t \in J$ because $0 \leq t \leq \alpha(t) \leq 1$ on J. By Assumption (A_1), we see that

$$\begin{aligned}
\varphi(Tx) &= \|Tx\| = \max_{t\in J}|(Tx)(t)| = \max_{t\in J}(Tx)(t) \\
&= \max_{t\in J}\frac{1}{\Delta}\Bigg\{[\gamma - g_2 + t(1-g_1)]\Bigg[\int_0^1 (1-s)h(s)f(s,x(\alpha(s)))ds \\
&\quad -\beta\int_0^\eta (\eta-s)h(s)f(s,x(\alpha(s)))ds\Bigg] \\
&\quad + [1-\beta\eta - t(1-\beta)]\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds \\
&\quad -\Delta\int_0^t (t-s)h(s)f(s,x(\alpha(s)))ds\Bigg\} \\
&\leq \frac{d}{\Delta\mu}\Bigg\{(1+\gamma-g_1-g_2)\int_0^1 (1-s)h(s)ds \\
&\quad +(1-\beta\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)d\tau ds\Bigg\} \leq d.
\end{aligned}$$

It proves that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

Now we need to show that condition (S_1) is satisfied. Take $x(t) = \frac{1}{2}\left(b + \frac{b}{\Gamma}\right)$ for $t \in J$. Then

$$\|x\| = \frac{b(\Gamma+1)}{2\Gamma} \leq \frac{b}{\Gamma}, \quad \text{so} \quad \Lambda(x) = \min_{[\eta,1]} x(t) = \frac{b(\Gamma+1)}{2\Gamma} > b = \frac{b}{\Gamma}\Gamma \geq \Gamma\|x\|.$$

It proves that

$$\left\{x \in P\left(\varphi, \Theta, \Lambda, b, \frac{b}{\Gamma}, d\right) : b < \Lambda(x)\right\} \neq \emptyset.$$

Let $b \leq u(t) \leq \frac{b}{\Gamma}$ for $t \in [\eta, 1]$. Then $\eta \leq t \leq \alpha(t) \leq 1$ on $[\eta, 1]$. It yields $b \leq u(\alpha(t)) \leq \frac{b}{\Gamma}$ on $[\eta, 1]$. Note that

$$\min_{[\eta,1]}(Tx)(t) = (Tx)(1),$$

see the proof of Lemma 2.3. Hence

$$\begin{aligned}\Lambda(Tx) &= \min_{[\eta,1]}(Tx)(t) = (Tx)(1)\\ &= \frac{\beta}{\Delta}\Bigg\{(1-\eta)\Bigg[\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\\ &\quad + \int_0^\eta [\gamma - g_2 + s(1-g_1)]h(s)f(s,x(\alpha(s)))ds\Bigg]\\ &\quad + [\gamma - g_2 + \eta(1-g_1)]\int_\eta^1 (1-s)h(s)f(s,x(\alpha(s)))ds\Bigg\}\\ &> \frac{\beta}{\Delta}[\gamma - g_2 + \eta(1-g_1)]\int_\eta^1 (1-s)h(s)f(s,x(\alpha(s)))ds\\ &\geq \frac{\beta}{\Delta}\frac{b}{L}[\gamma - g_2 + \eta(1-g_1)]\int_\eta^1 (1-s)h(s)ds \geq b,\end{aligned}$$

by Assumption (A_2). Consequently, $\Lambda(Tx) > b$, so condition (S_1) holds.

Now we need to prove that condition (S_2) is satisfied. Take $x \in P(\varphi, \Lambda, b, \frac{b}{\Gamma})$ and $\|Tx\| > \frac{b}{\Gamma} = d$. Then

$$\Lambda(Tx) = \min_{[\eta,1]}(Tx)(t) \geq \Gamma\|Tx\| > \Gamma\frac{b}{\Gamma} = b,$$

so condition (S_2) holds.

Indeed, $\varphi(0) = 0 < a$, so $0 \notin R(\varphi, \Psi, a, d)$. Suppose that $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = \|x\| = a$. Then

$$\begin{aligned}\Psi(Tx) &= \|Tx\| = \max_{t\in J}(Tx)(t)\\ &\leq \frac{1}{\Delta}\Bigg\{(1+\gamma-g_1-g_2)\int_0^1 (1-s)h(s)f(s,x(\alpha(s)))ds\\ &\quad + (1-\beta\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)f(\tau,x(\alpha(\tau)))d\tau ds\Bigg\}\\ &\leq \frac{a}{\Delta\mu}\Bigg\{(1+\gamma-g_1-g_2)\int_0^1 (1-s)h(s)ds\\ &\quad + (1-\beta\eta)\int_0^1 g(s)\int_0^s (s-\tau)h(\tau)d\tau ds\Bigg\} < a.\end{aligned}$$

It shows that condition (S_3) is satisfied.

By Theorem 2.7, there exist at least three positive solutions x_1, x_2, x_3 of problem (1.1) such that $\|x_i\| \leq d$ for $i = 1, 2, 3$,

$$b \leq \min_{[\eta,1]} x_1(t), \quad a < \|x_2\| \text{ with } \min_{[\eta,1]} x_2(t) < b$$

and $\|x_3\| < a$. This ends the proof. □

Example 2.9. We consider the following example

$$\begin{cases} x''(t) + Bf(x(\alpha(t))) = 0, \quad t \in (0,1), \\ x(0) = x'(0) - \dfrac{1}{2}\displaystyle\int_0^1 x(s)ds, \\ x(1) = \dfrac{1}{2}\, x\left(\dfrac{1}{2}\right), \end{cases} \tag{2.5}$$

where B is a positive constant and

$$f(u) = \begin{cases} \dfrac{u}{9}, & 0 \le u \le 1, \\ \dfrac{1}{9}(25u - 24), & 1 \le u \le \dfrac{3}{2}, \\ \dfrac{1}{33}(u + 48), & u \ge \dfrac{3}{2}. \end{cases}$$

Note that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $h(t) = B$, $g(t) = \frac{1}{2}$, $\gamma = 1$, $\beta = \eta = \frac{1}{2}$. In this case we have:

$$g_1 = \frac{1}{2}, \quad g_2 = \Gamma = \frac{1}{4}, \quad \Delta = \frac{3}{4}, \quad L = \frac{1}{6}B, \quad \mu > \frac{11}{12}B.$$

If we take $a = 1$, $b = \frac{3}{2}$, $c = d = 6$ and $\mu = B \ge 6$, then all assumptions of Theorem 2.8 hold, so problem (2.5) has at least three positive solutions.

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