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BIFURCATION FROM INTERVAL AND POSITIVE SOLUTIONS FOR SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. We give a global description of the branches of positive solutions of second order periodic boundary value problems

$$u'' - q(t)u + \lambda a(t)f(u) = 0, \quad 0 < t < 2\pi,$$
$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

which are not necessarily linearizable. Our approach based on topological degree and global bifurcation techniques.

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1. INTRODUCTION

Krasnosel'skii's theorem in a cone has often been used to study the existence and multiplicity of positive solutions of periodic boundary value problems over forty years, see Krasnosel'skiĭ [1], Gustafson and Schmitt [2], Nussbaum [3], Atici and Guseinov [4], Jiang et al. [5], O'Ragan and Wang [6], Torres [7], Zhang and Wang [8] and the references therein. Very recently, Graef et al [9] considered the the following periodic boundary value problems

$$u'' - \rho^2 u + ra(t)f(u) = 0, \quad 0 < t < 2\pi, \tag{1.1}$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{1.2}$$

where $\rho > 0$ is a constant and r is a positive parameter, a and f satisfy the assumptions:

(H1) $a : [0, 2\pi] \to [0, \infty)$ is continuous and $\int_0^{2\pi} a(t)dt > 0$;

(H2) $f : [0, \infty) \to [0, \infty)$ is continuous and $f(u) > 0$ for $u > 0$.

Furthermore, let

$$f_0 = \lim_{s \to 0^+} \frac{f(s)}{s}, \qquad f_\infty = \lim_{s \to +\infty} \frac{f(s)}{s}.$$

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Under different combinations of superlinearity and sublinearity of the function f, using the cone expansion/compression fixed-point theorem, they established some existence results of positive solutions which are derived in terms of different values of r.

Of course the natural question is what would happen if the limits $\lim\limits_{s\to 0^+} \frac{f(s)}{s}$ and/or $\lim\limits_{s\to+\infty} \frac{f(s)}{s}$ do no exist?

It is the purpose here that we shall obtain a global description of the branches of positive solutions of second order periodic boundary value problems

$$(1.3)\qquad \begin{aligned} &u'' - q(t)u + \lambda a(t)f(u) = 0, \quad 0 < t < 2\pi, \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{aligned}$$

which are not necessarily linearizable, where $\lambda \in [0, \infty)$ is a parameter, q, a and f satisfy the assumptions:

(A0) $q \in C([0, 2\pi], [0, \infty))$ is of period 2π and $q \not\equiv 0$ on $[0, 2\pi]$;
(A1) $a \in C([0, 2\pi], [0, \infty))$ is of period 2π and $a(t) \not\equiv 0$ in any subinterval of $[0, 1]$;
(A2) $f : [0, \infty) \to [0, \infty)$ is continuous and differentiable, and the limits

$$f_0 = \liminf_{s\to 0^+} \frac{f(s)}{s}, \quad f^\infty = \limsup_{s\to+\infty} \frac{f(s)}{s},$$

$$f_\infty = \liminf_{s\to+\infty} \frac{f(s)}{s}, \quad f^0 = \limsup_{s\to 0^+} \frac{f(s)}{s}$$

exist;
(A3) $f_\infty > 0$.

To state our main results, we need the spectrum theory of the linear eigenvalue problem

$$(1.4)\qquad \begin{aligned} &- u'' + q(t)u = \lambda a(t)u, \quad 0 < t < 2\pi, \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{aligned}$$

Lemma 1.1 ([10])**.** *Let* (A0), (A1) *hold. Then the weight linear eigenvalue problem* (1.4) *has an infinite sequences of eigenvalue*

$$0 < \lambda_0 < \lambda_1 \le \lambda_2 < \cdots$$

such that the eigenfunction φ_n *corresponding to* λ_n *has exactly* $[\frac{n+1}{2}]$ *zeros on the interval* $[0, 2\pi]$.

Remark 1.2. In the Lemma 1.1, λ_0 are simple eigenvalue with positive eigenfunction.

The main results of this paper are as follows

Theorem 1.3. *Let* (A0)–(A3) *hold. Suppose* $f(0) = 0$ *and* $f_0 > 0$. *Then*

(i) $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right]$ *is a bifurcation interval from infinity for positive solutions, and there exists no bifurcation interval from infinity which is disjointed with* $\left(\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right)$. *More precisely, there exists a component* Σ_∞ *of positive solutions which meets* $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \times \{\infty\}$.

(ii) $\left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right]$ *is a bifurcation interval from the trivial solution, and there exists no bifurcation interval from the trivial solution which is disjointed with* $\left(\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right)$. *More precisely, there exists a unbounded component* Σ_0 *of positive solutions which meets* $\left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right] \times \{0\}$.

(iii) *If* $f(s) > 0$ *for all* $s > 0$, *then there is a number* $\lambda^* > 0$ *such that problem* (1.3) *admits no solutions with* $\lambda > \lambda^*$. *In this case,* $\Sigma_\infty = \Sigma_0$.

(iv) *If* $f(s_0) \leq 0$ *for some* $s_0 > 0$, *there exists no positive solution* (λ, u) *with* $\|u\| = s_0$. *Hence the components* Σ_0 *and* Σ_∞ *are disjoint, and problem* (1.3) *admits at least two positive solutions for all* $\lambda > \max\left\{\frac{\lambda_0}{f_0}, \frac{\lambda_0}{f_\infty}\right\}$.

Theorem 1.4. *Let* (A0)–(A3) *hold. Suppose* $f(0) = 0$ *and* $f^0 = 0$. *Then*

(i) *Assertion* (i) *of Theorem* 1.3 *holds.*

(ii) *There is no bifurcation of positive solutions from the line of trivial solutions* $\mathbb{R}^+ \times \{0\}$.

Theorem 1.5. *Let* (A0)–(A3) *hold. Suppose* $f(0) > 0$. *Then*

(i) *Assertion* (i) *of Theorem* 1.3 *holds.*

(ii) *There exists a unbounded component* Σ_0 *of positive solutions meeting* $(0, 0)$. *If* $f(s) > 0$ *for all* $s > 0$, *then* $\Sigma_\infty = \Sigma_0$.

(iii) *If* $f(s_0) \leq 0$ *for some* $s_0 > 0$, *there exists no positive solution* (λ, u) *with* $\|u\| = s_0$. *Hence the components* Σ_0 *and* Σ_∞ *are disjoint, and problem* (1.3) *admits at least two positive solutions for all* $\lambda > \frac{\lambda_0}{f_\infty}$.

Our main tools in the proof of Theorems 1.3–1.5 are topological arguments and the global bifurcation theorems for mappings which are not necessarily smooth.

Theorem 1.6 ([11, Rabinowitz]). *Let* V *be a real reflexive Banach space. Let* $F : \mathbb{R} \times V \to V$ *be completely continuous such that* $F(\lambda, 0) = 0$, $\forall \lambda \in \mathbb{R}$. *Let* $a, b \in \mathbb{R}$ $(a < b)$ *be such that* $u = 0$ *is an isolated solution of the equation*

$$u - F(\lambda, u) = 0, \qquad u \in V, \tag{1.5}$$

for $\lambda = a$ *and* $\lambda = b$, *where* $(a, 0)$, $(b, 0)$ *are not bifurcation points of* (1.5). *Furthermore, assume that*

$$d(I - F(a, \cdot), B_r(0), 0) \neq d(I - F(b, \cdot), B_r(0), 0),$$

where $B_r(0)$ *is an isolating neighborhood of the nontrivial solution. Let*

$$\mathcal{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ is a solution of } (1.5) \text{ with } u \neq 0\}} \cup ([a, b] \times \{0\}),$$

and let $\mathcal{C}$ be the connected component of $\mathcal{S}$ containing $[a,b]\times\{0\}$. Then, either

(i) *$\mathcal{C}$ is unbounded, or*
(ii) *$\mathcal{C}\cap[(\mathbb{R}\setminus[a,b])\times\{0\}]\neq\emptyset$.*

Theorem 1.7 ([12, Schmitt])**.** *Let V be a real reflexive Banach space. Let $F:\mathbb{R}\times V\to V$ be completely continuous, and let $a,b\in\mathbb{R}$ $(a<b)$ be such that the solutions of (1.5) are, a priori, bounded in V for $\lambda=a$ and $\lambda=b$, i.e., there exists an $R>0$ such that*

$$F(a,u)\neq u\neq F(b,u)$$

for all u with $\|u\|\geq R$. Furthermore, assume that

$$d(I-F(a,\cdot),B_R(0),0)\neq d(I-F(b,\cdot),B_R(0),0),$$

for $R>0$ large. Then there exists a closed connected set $\mathcal{C}$ of solutions of (1.5) that is unbounded in $[a,b]\times V$, and either

(i) *$\mathcal{C}$ is unbounded in λ direction, or else*
(ii) *there exists an interval $[c,d]$ such that $(a,b)\cap(c,d)=\emptyset$ and $\mathcal{C}$ bifurcates from infinity in $[c,d]\times V$.*

The rest of the paper is organized as follows: In Section 2, we state some notations and preliminary results. Section 3 and Section 4 are devoted to study the bifurcation from infinity and from the trivial solution for a nonlinear problem which are not necessarily linearizable, respectively. In Section 5, we briefly look at the continuum without bifurcation. Finally in Section 6, we concerns the intertwining of the branches bifurcating from infinity and from the trivial solution, showing that the essential role played by the fact whether f has zeros in $(0,\infty)$ or not.

2. NOTATION AND PRELIMINARY RESULTS

We shall work in the Banach space $E=C[0,2\pi]$ with sup norm $\|\cdot\|$. By a *positive solution* of Problem (1.3) we mean a pair (λ,u), where $\lambda>0$ and u is a solution of (1.3) with $u>0$ (i.e., $u\geq 0$ in $(0,2\pi)$ and $u\not\equiv 0$). Let $\Sigma\subset\mathbb{R}^+\times E$ be the closure of the set of positive solutions of (1.3).

Let $H:=L^2(0,2\pi)$, with inner product $\langle\cdot,\cdot\rangle_{L^2}$ and norm $\|\cdot\|_{L^2}$. Further Define the linear operator $L:D(L)\subset E\to E$

$$Lu=-u''+q(t)u,\qquad u\in D(L)$$

with

$$D(L)=\{u\in C^2[0,2\pi]\mid u(0)=u(2\pi),\ u'(0)=u'(2\pi)\}.$$

Then L is a closed operator with compact resolvent, and $0\in\rho(L)$.

We extend the function f to a continuous function $\bar{f}$ defined on $\mathbb{R}$ in such a way that $\bar{f} > 0$ for all $s < 0$. For $\lambda \geq 0$ we then look at arbitrary solutions u of second order periodic boundary value problems

$$\begin{aligned} &u'' - q(t)u + \lambda a(t)\bar{f}(u) = 0, \qquad t \in (0, 2\pi), \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{aligned} \tag{2.1}$$

Let $G(t, s)$ be the Green's function of the homogeneous boundary value problem

$$\begin{aligned} &-u'' + q(t)u = 0, \quad 0 < t < 2\pi, \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{aligned}$$

From Theorem 2.5 of [13], we know that $G(t, s) > 0$, $\forall t, s \in [0, 2\pi]$ under condition (A0).

Let

$$m = \min G(t, s), \quad M = \max G(t, s), \quad t, s \in [0, 2\pi].$$

Then $m > 0$, $M > 0$.

Remark 2.1. The problem (2.1) is equivalent to the operator equation $A : E \to E$,

$$Au := u = \lambda \int_0^{2\pi} G(t, s)a(s)\bar{f}(u(s))ds, \quad t \in [0, 2\pi].$$

For $\lambda > 0$, if u is solution of (2.1), from the positivity of $G(t, s)$ and $\bar{f}$, we have that $u > 0$, $t \in [0, 2\pi]$, and it is solution of (1.3). Therefore, the closure of the set of nontrivial solutions (λ, u) of (2.1) in $\mathbb{R}^+ \times E$ is exactly Σ.

Let $N : E \to E$ be the Nemytskii operator associated with $\bar{f}$:

$$N(u)(t) = a(t)\bar{f}(u(t)), \qquad u \in E.$$

The problem (2.1), with $\lambda \geq 0$, is now equivalent to the functional equation

$$u = \lambda L^{-1}N(u), \qquad u \in E. \tag{2.2}$$

In the following we shall apply the Leray-Schauder degree theory, mainly to the mapping $\Phi_\lambda : E \to E$,

$$\Phi_\lambda(u) = u - \lambda L^{-1}N(u).$$

For $R > 0$, let $B_R = \{u \in E : \|u\| < R\}$, let $\deg(\Phi_\lambda, B_R, 0)$ denote the degree of Φ_λ on B_R with respect to 0.

3. BIFURCATION FROM INFINITY

In order to investigate the bifurcation from infinity, we follow the standard pattern and perform the change of variable $z = \|u\|^{-2}u$ $(u \neq 0)$.

Lemma 3.1. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \cap \Lambda = \emptyset$. Then there exists a number $R_1 > 0$ such that*

$$\Phi_\lambda(u) \neq 0, \qquad \forall \lambda \in \Lambda, \quad \forall u \in E : \|u\| \geq R_1.$$

Proof. Let $r = \mathrm{dist}(\Lambda, \left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right])$. Then $r > 0$. Suppose to the contrary that there exists $\{(\mu_n, u_n)\} \subset \Lambda \times E$ with $\|u_n\| \to \infty$ $(n \to \infty)$, such that $\Phi_{\mu_n}(u_n) = 0$. We may assume $\mu_n \to \bar{\mu} \in \Lambda$. By Remark 2.1, $u_n > 0$ in $[0, 2\pi]$. Set $v_n := \|u_n\|^{-1}u_n$. Then

$$v_n = \mu_n L^{-1}\frac{N(u_n)}{\|u_n\|}.$$

Since $\|u_n\|^{-1}N(u_n)$ is bounded in E, $\{v_n\}$ is a relatively compact set in E by the compactness of L^{-1}. Suppose $v_n \to \bar{v}$ in E. Then $\|\bar{v}\| = 1$ and $v > 0$ in $[0, 2\pi]$. Further

$$Lv_n = \mu_n \|u_n\|^{-1} N(u_n). \tag{3.1}$$

From (A1), (A2), we have

$$\min_{0 \leq t \leq 2\pi} u_n(t) \geq \sigma \|u_n\|, \tag{3.2}$$

where $\sigma = \frac{m}{M} > 0$. For arbitrary

$$\epsilon \in \left(0, \min\left\{\frac{\lambda_0 f^\infty}{\lambda_0 - r f^\infty} - f^\infty, f_\infty - \frac{\lambda_0 f_\infty}{\lambda_0 + r f_\infty}\right\}\right), \tag{3.3}$$

there exists $K > 0$, such that for $n \geq K$,

$$\mu_n (f_\infty - \epsilon) a v_n \leq L v_n \leq \mu_n (f^\infty + \epsilon) a v_n, \quad t \in [0, 2\pi],$$

which implies that for all $\epsilon \in \left(0, \min\left\{\frac{\lambda_0 f^\infty}{\lambda_0 - r f^\infty} - f^\infty, f_\infty - \frac{\lambda_0 f_\infty}{\lambda_0 + r f_\infty}\right\}\right)$,

$$\bar{\mu}(f_\infty - \epsilon) a \bar{v} \leq L\bar{v} \leq \bar{\mu}(f^\infty + \epsilon) a \bar{v}, \quad t \in [0, 2\pi]. \tag{3.4}$$

Multiplying both sides of (3.4) with φ_0 and integrating from 0 to 2π and we using the fact $\langle \bar{v}, a\varphi_0 \rangle_{L^2} > 0$, we get that

$$\bar{\mu}(f_\infty - \epsilon) \leq \lambda_0 \leq \bar{\mu}(f^\infty + \epsilon), \quad t \in [0, 2\pi]. \tag{3.5}$$

If Λ is in the left side of $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right]$, then we have from (3.3) and the second inequality in (3.5) that

$$\bar{\mu} \geq \frac{\lambda_0}{f^\infty + \epsilon} > \frac{\lambda_0}{f^\infty} - r.$$

This contradicts the definition of r.

If Λ is in the right side of $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right]$, then we have from (3.3) and the first inequality in (3.5) that

$$\bar{\mu} \leq \frac{\lambda_0}{f_\infty - \epsilon} < \frac{\lambda_0}{f_\infty} + r.$$

This contradicts the definition of r again. □

Corollary 3.2. *For $\mu \in (0, \frac{\lambda_0}{f^\infty})$ and $R \geq R_1$, $\deg(\Phi_\mu, B_R, 0) = 1$.*

Proof. Lemma 3.1, applied to the interval $\Lambda = [0, \mu]$, guarantees the existence of $R_1 > 0$ such that for $R \geq R_1$

$$u - \tau\mu L^{-1}N(u) \neq 0, \quad u \in E : \|u\| \geq R, \quad \tau \in [0, 1].$$

Hence for any $R \geq R_1$,

$$\deg(\Phi_\mu, B_R, 0) = \deg(I, B_R, 0) = 1,$$

which implies the assertion. □

On the other hand, we have

Lemma 3.3. *Suppose $\lambda > \frac{\lambda_0}{f_\infty}$. Then there exists $R_2 > 0$ with the property that $\forall u \in E$ with $\|u\| \geq R_2$, $\forall \tau \geq 0$,*

$$\Phi_\lambda(u) \neq \tau\varphi_0.$$

Proof. Let us assume that for some sequence $\{u_n\}$ in E with $\|u_n\| \to \infty$ and numbers $\tau_n \geq 0$, $\Phi_\lambda(u_n) = \tau_n\varphi_0$. Then

$$Lu_n = \lambda N(u_n) + \tau_n\lambda_0 a\varphi_0,$$

and since $\tau_n\lambda_0 a\varphi_0 \geq 0$ in $[0, 2\pi]$, it follows that $u_n > 0$ in $[0, 2\pi]$.

Let $v_n = \frac{u_n}{\|u_n\|}$. Then

$$v_n(t) \geq \lambda L^{-1}\left(\frac{N(u_n)}{u_n}(t) \cdot v_n(t)\right) = \lambda \int_0^{2\pi} G(t,s)\frac{N(u_n(s))}{u_n(s)} \cdot v_n(s)ds, \tag{3.6}$$

and consequently,

$$\begin{aligned} \underline{\lim}\, v_n &\geq \lambda\, \underline{\lim}\left(L^{-1}\left(\frac{N(u_n)}{u_n} \cdot v_n\right)\right) \\ &\geq \lambda \int_0^{2\pi} G(t,s)(\underline{\lim}\,\frac{N(u_n)}{u_n} \cdot v_n)ds \\ &= \lambda\left(L^{-1}\,\underline{\lim}\left(\frac{N(u_n)}{u_n} \cdot v_n\right)\right). \end{aligned} \tag{3.7}$$

Since $\min\limits_{0\leq t\leq 2\pi} u_n(t) \geq \sigma\|u_n\|$, we have that

$$\underline{\lim}\, v_n(t) \geq \sigma > 0, \qquad t \in [0, 2\pi]. \tag{3.8}$$

Moreover,

$$\begin{aligned}
\langle \underline{\lim}\, v_n, a\varphi_0\rangle_{L^2} &\geq \lambda \left\langle \underline{\lim} \left(L^{-1} \left(\frac{N(u_n)}{u_n} \cdot v_n \right)\right), a\varphi_0 \right\rangle_{L^2} \\
&\geq \lambda \left\langle L^{-1} \left(\left(\underline{\lim} \frac{N(u_n)}{u_n} \cdot v_n \right)\right), a\varphi_0 \right\rangle_{L^2} && \text{(by (3.7))} \\
&\geq \lambda \left\langle L^{-1} \left(\underline{\lim} \frac{N(u_n)}{u_n} \cdot \underline{\lim}\, v_n \right), a\varphi_0 \right\rangle_{L^2} \\
&\geq \lambda f_\infty \langle L^{-1}(a\, \underline{\lim}\, v_n), a\varphi_0\rangle_{L^2} && \text{(by (3.8))} \\
&= \lambda f_\infty \langle L^{-1}(a\, \underline{\lim}\, v_n), \frac{1}{\lambda_0} L\varphi_0\rangle_{L^2} \\
&= \frac{\lambda}{\lambda_0} f_\infty \langle LL^{-1}(a\, \underline{\lim}\, v_n), \varphi_0\rangle_{L^2} \\
&= \frac{\lambda}{\lambda_0} f_\infty \langle \underline{\lim}\, v_n, a\varphi_0\rangle_{L^2}.
\end{aligned}$$

Therefore

$$\lambda \leq \frac{\lambda_0}{f_\infty}.$$

This is a contradiction. □

Corollary 3.4. *For $\lambda > \frac{\lambda_0}{f_\infty}$ and $R \geq R_2$,* $\deg(\Phi_\lambda, B_R, 0) = 0$.

Proof. By Lemma 3.3, there exists $R_2 > 0$ such that

$$\Phi_\lambda(u) \neq \tau\varphi_0, \qquad u \in E : \|u\| \geq R_2, \quad \tau \in [0,1].$$

Then

$$\deg(\Phi_\lambda, B_R, 0) = \deg(\Phi_\lambda - \varphi_0, B_R, 0) = 0$$

for all $R \geq R_2$. The assertion follows. □

We are now ready to prove

Proposition 3.5. $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right]$ *is a bifurcation interval from infinity for positive solutions. There exists an unbounded component Σ_∞ of positive solutions which meets $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \times \{\infty\}$ and is unbounded in λ direction. Moreover, there exists no bifurcation interval from infinity which is disjointed with $\left(\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right)$.*

Proof. For fixed $n \in \mathbb{N}$ with $\frac{\lambda_0}{f^\infty} - \frac{1}{n} > 0$, let us take that $a_n = \frac{\lambda_0}{f^\infty} - \frac{1}{n}$, $b_n = \frac{\lambda_0}{f_\infty} + \frac{1}{n}$ and $\hat{R} = \max\{R_1, R_2\}$. It is easy to check that for $R > \hat{R}$, all of the conditions of Theorem B are satisfied. So there exists a closed connected set $\mathcal{C}_n$ of solutions of (2.2) that is unbounded in $[a_n, b_n] \times E$, and either

(i) $\mathcal{C}_n$ is unbounded in λ direction, or else

(ii) $\exists\ [c,d]$ s.t. $(a_n, b_n) \cap (c,d) = \emptyset$ and $\mathcal{C}_n$ bifurcates from ∞ in $[c,d] \times E$. By Lemma 3.1, the case (ii) cannot occur. Thus $\mathcal{C}_n$ bifurcates from ∞ in $[a_n, b_n] \times E$ and is unbounded in λ direction. Furthermore, we have from Lemma 3.1 that for any closed interval $I \subset [a_n, b_n] \setminus \left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right]$, the set $\{u \in E \mid (\lambda, u) \in \Sigma, \lambda \in I\}$ is bounded in E. So $\mathcal{C}_n$ must be bifurcated from ∞ in $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \times E$ and is unbounded in λ direction.

□

Assertion (i) of Theorems 1.3–1.5 follows directly.

4. BIFURCATION FROM THE TRIVIAL SOLUTION

We suppose now $f(0) = 0$ and investigate the first case $f_0 > 0$.

Lemma 4.1. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $\left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right] \cap \Lambda = \emptyset$. Then there exists a number $\delta_1 > 0$ with the property*

$$\Phi_\lambda(u) \neq 0, \qquad \forall u \in E : 0 < \|u\| \leq \delta_1, \quad \forall \lambda \in \Lambda.$$

Proof. Let

$$r_2 = \operatorname{dist}\left(\Lambda, \left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right]\right).$$

Then $r_2 > 0$.

Suppose to the contrary that there exist sequences $\{\mu_n\}$ in Λ and $\{u_n\}$ in E : $\mu_n \to \bar{\mu} \in \Lambda$, $u_n \to 0$ in E, such that $\Phi_{\mu_n}(u_n) = 0$ for all $n \in \mathbb{N}$. By Remark 2.1, $u_n > 0$ in $[0, 2\pi]$.

Set $v_n = \frac{u_n}{\|u_n\|}$. Then $Lv_n = \mu_n \|u_n\|^{-1} N(u_n)$. Since $\|u_n\|^{-1} N(u_n)$ is bounded in E, we infer that $\{v_n\}$ is relatively compact in E, hence (for a subsequence) $v_n \to \bar{v}$ with $\bar{v} > 0$ in E, $\|\bar{v}\| = 1$.

Now, for any

$$\epsilon \in \left(0, \min\left\{\frac{\lambda_0 f^0}{\lambda_0 - r_2 f^0} - f^0, f_0 - \frac{\lambda_0 f_0}{\lambda_0 + r_2 f_0}\right\}\right), \tag{4.1}$$

there exists $\bar{N} > 0$, such that

$$f_0 - \epsilon < \frac{f(u_n)}{u_n} < f^0 + \epsilon, \qquad t \in [0, 2\pi], \quad n \geq \bar{N}.$$

This together with the fact $Lv_n = \mu_n a(\cdot) \frac{f(u_n)}{u_n} v_n$ imply that

$$\bar{\mu}(f_0 - \epsilon)\langle \bar{v}, a\varphi_0\rangle_{L^2} \leq \lambda_0 \langle \bar{v}, a\varphi_0\rangle_{L^2} \leq \bar{\mu}(f^0 + \epsilon)\langle \bar{v}, a\varphi_0\rangle_{L^2}.$$

Thus

$$\frac{\lambda_0}{f^0 + \epsilon} \leq \bar{\mu} \leq \frac{\lambda_0}{f_0 - \epsilon}. \tag{4.2}$$

From (4.1) and (4.2), it follows that

$$\frac{\lambda_0}{f^0} - r_2 < \bar{\mu} < \frac{\lambda_0}{f_0} + r_2.$$

This contradicts the fact $\bar{\mu} \in \Lambda$. □

Corollary 4.2. *For* $\lambda \in (0, \frac{\lambda_0}{f^0})$ *and* $\delta \in (0, \delta_1)$, $\deg(\Phi_\lambda, B_\delta, 0) = 1$.

On the other hand, we have

Lemma 4.3. *Suppose* $\lambda > \frac{\lambda_0}{f_0}$. *Then there exists* $\delta_2 > 0$ *such that* $\forall u \in E$ *with* $0 < \|u\| \le \delta_2$, $\forall \tau \ge 0$,

$$\Phi_\lambda(u) \ne \tau\varphi_0.$$

Proof. We assume again to the contrary that there exists $\tau_n \ge 0$ and a sequence $\{u_n\}$ with $\|u_n\| > 0$ and $u_n \to 0$ in E such that $\Phi_\lambda(u_n) = \tau_n\varphi_0$ for all $n \in \mathbb{N}$. As $Lu_n = \lambda N(u_n) + \tau_n\lambda_0 a\varphi_0$ and $\tau_n\lambda_0 a\varphi_0 \ge 0$ in $[0, 2\pi]$, we conclude from Remark 2.1 that $u_n > 0$ in $[0, 2\pi]$.

Notice that $u_n \in D(L)$ has a unique decomposition

$$u_n = w_n + s_n\varphi_0, \tag{4.3}$$

where $s_n \in \mathbb{R}$ and $\langle aw_n, \varphi_0\rangle_{L^2} = 0$. Since $u_n > 0$ on $[0, 2\pi]$ and $\|u_n\| > 0$, we have from (4.3) that $s_n > 0$. Now

$$s_n\lambda_0\langle a\varphi_0, \varphi_0\rangle_{L^2} = \langle Lu_n, \varphi_0\rangle_{L^2} = \lambda\langle N(u_n), \varphi_0\rangle_{L^2} + \tau_n\lambda_0\langle a\varphi_0, \varphi_0\rangle_{L^2}.$$

Choose $\beta > 0$ such that $\beta < f_0 - \frac{\lambda_0}{\lambda}$. For all sufficiently large n,

$$N(u_n) \ge a(\cdot)(f_0 - \beta)u_n, \quad t \in [0, 2\pi].$$

Thus

$$\begin{aligned} s_n\lambda_0\langle a\varphi_0, \varphi_0\rangle_{L^2} &\ge \lambda(f_0 - \beta)s_n\langle a\varphi_0, \varphi_0\rangle_{L^2} + \lambda_0\tau_n\langle a\varphi_0, \varphi_0\rangle_{L^2} \\ &> \lambda_0 s_n\langle a\varphi_0, \varphi_0\rangle_{L^2} \end{aligned}$$

a contradiction. □

Corollary 4.4. *For* $\lambda > \frac{\lambda_0}{f_0}$ *and* $\delta \in (0, \delta_2)$, $\deg(\Phi_\lambda, B_\delta, 0) = 0$.

Proof. Let $0 < \epsilon \le \delta_2$, where δ_2 is the number asserted in Lemma 4.3. As Φ_λ is bounded in $\bar{B}_\epsilon$, there exists $c > 0$ such that $\Phi_\lambda(u) \ne c\varphi_0$, $\forall u \in \bar{B}_\epsilon$. By Lemma 4.3,

$$\Phi_\lambda(u) \ne tc\varphi_0, \qquad u \in \partial B_\epsilon, \quad t \in [0, 2\pi].$$

Hence

$$\deg(\Phi_\lambda, B_\epsilon, 0) = \deg(\Phi_\lambda - c\varphi_0, B_\epsilon, 0) = 0.$$

□

Now, using Theorem 1.6 and the similar method to prove Proposition 3.5 with obvious changes, we may prove the following

Proposition 4.5. $\left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right]$ *is a bifurcation interval from the trivial solution. There exists an unbounded component* Σ_0 *of positive solutions which meets* $\left[\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right] \times \{0\}$*. Moreover, there exists no bifurcation interval from the trivial solution which is disjointed with* $\left(\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right)$*.*

This is exactly the assertion (ii) of Theorem 1.3.

We conclude this section by considering the case $f(0) = f^0 = 0$.

Proposition 4.6. *In the case* $f(0) = f^0 = 0$*, there is no bifurcation of positive solutions from the trivial solution.*

Proof. Suppose to the contrary that there exist sequences $\{\mu_n\}$ in $(0,\infty)$ and $\{u_n\}$ in $E : \mu_n \to \bar{\mu} \in [0,\infty), u_n \to 0$ in E, such that $\Phi_{\mu_n}(u_n) = 0$ for all $n \in \mathbb{N}$. By Remark 2.1, $u_n > 0$ in $[0, 2\pi]$.

Set $v_n = \frac{u_n}{\|u_n\|}$. Then we infer that (for a subsequence) $v_n \to \bar{v}$ with $\bar{v} > 0$ in E, $\|\bar{v}\| = 1$.

On the other hand, we have from $\Phi_{\mu_n}(u_n) = 0$ that

$$\begin{aligned} &\bar{v}'' - q(t)\bar{v} + \bar{\mu}a(t)f^0\bar{v} = 0, \qquad t \in (0, 2\pi), \\ &\bar{v}(0) = \bar{v}(2\pi), \quad \bar{v}'(0) = \bar{v}'(2\pi), \end{aligned}$$

which implies $\bar{v} \equiv 0$ on $[0, 2\pi]$. This is a contradiction. □

This is exactly the assertion (ii) of Theorem 1.4.

5. CONTINUUM WITH BIFURCATION

We briefly look at the situation where $f(0) > 0$.

Lemma 5.1. *There is a unbounded component* Σ_0 *of positive solutions of* (1.3) *meeting* $(0, 0)$*.*

Proof. By Amann [14, Theorem 17.1], Σ contains an unbounded component Σ_0 containing $(0, 0)$. Note that $(0, u)$ is a solution if and only if for $u = 0$, while $(\lambda, 0)$ is a solution only for $\lambda = 0$. Thus $(\lambda, u) \in \Sigma_0 \setminus \{(0, 0)\}$ is a positive solution, by Remark 2.1. □

6. GLOBAL BEHAVIOR OF THE COMPONENT OF POSITIVE SOLUTIONS

Lemma 6.1. *Suppose* $f(s) > \alpha s \ \forall s \geq 0$*, with* $\alpha > 0$*. Then there exists a number* $\lambda^* > 0$ *such that there is no positive solution* (λ, u) *of* $\Phi_\lambda(u) = 0$ *with* $\lambda > \lambda^*$*.*

Proof. Let (λ, u) be a positive solution of $\Phi_\lambda(u) = 0$. Then

$$-u''(t) + q(t)u(t) = \lambda a(t)f(u(t)) \geq \lambda\alpha a(t)u(t), \quad t \in (0, 2\pi)$$

and hence

$$\lambda_0\langle u, a\varphi_0\rangle_{L^2} = \langle Lu, \varphi_0\rangle_{L^2} \geq \lambda\alpha\langle au, \varphi_0\rangle_{L^2}.$$

Since $\langle u, a\varphi_0\rangle_{L^2} > 0$, it follows that $\lambda < \alpha^{-1}\lambda_0$. □

Note that (i) if $f(0) = 0$, $f_\infty > 0$, and $f(s) > 0\ \forall s > 0$, or (ii) if $f(0) > 0$, $f_\infty > 0$ and $f(s) > 0\ \forall s \geq 0$, the hypothesis of Lemma 6.1 is satisfied.

The assertion that $\Sigma_0 = \Sigma_\infty$ in both Theorem 1.3 (iii) and Theorem 1.5 (ii) now easily follows. For, in both cases, Σ_0 and Σ_∞ are contained in $[0, \lambda^*] \times E$. Moreover, there exists no bifurcation interval from infinity which is disjointed with $\left(\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right)$, there exists no bifurcation interval from the trivial solution which is disjointed with $\left(\frac{\lambda_0}{f^0}, \frac{\lambda_0}{f_0}\right)$. In Theorem 1.3 (iii), the unbounded component Σ_0 has to meet $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \times \{\infty\}$. In Theorem 1.5 (ii), the unbounded component Σ_0 has to meet $\left[\frac{\lambda_0}{f^\infty}, \frac{\lambda_0}{f_\infty}\right] \times \{\infty\}$.

Lemma 6.2. *Let* (A0)–(A3) *hold. If* $f(s_0) \leq 0$ *for some* $s_0 > 0$, *then* $\Phi_\lambda(u) = 0$ *has no solution* $(\lambda, u) \in [0, \infty) \times E$ *with* $\|u\| = s_0$.

Proof. For $\lambda = 0$, the assertion is obviously true.

Suppose to the contrary that $\Phi_\lambda(u) = 0$ for some $\lambda > 0$ and $\|u\| = s_0$. By Remark 2.1, $u > 0$ and $0 < u(t) \leq s_0$, $\forall t \in [0, 2\pi]$. By Condition (A2), there exists $m \geq 0$ such that $a(t)f(s) + ms$ is monotone increasing in s for $s \in [0, s_0]$. Then

$$(L + \lambda m)u = \lambda(N(u) + mu)$$

and, since $Ls_0 = 0$ and $N(s_0) \leq 0$,

$$(L + \lambda m)s_0 \geq \lambda(N(s_0) + ms_0).$$

Subtracting and letting $w := s_0 - u$, we get

$$(L + \lambda m)w \geq 0, \qquad t \in (0, 2\pi),$$
$$w(0) = w(2\pi), \qquad w'(0) = w'(2\pi).$$

Let $e \in C([0, 2\pi], (0, \infty))$, such that

$$-w'' + (q(t) + \lambda m)w = e(t) > 0, \quad t \in (0, 2\pi),$$

Let $G_1(t, s)$ be the Green's function of the homogeneous boundary value problem

$$-w'' + (q(t) + \lambda m)w = 0, \quad 0 < t < 2\pi,$$
$$w(0) = w(2\pi), \quad w'(0) = w'(2\pi).$$

From (A0), $\lambda \geq 0$ and $m \geq 0$, applying the Theorem 2.5 of [13], we know that $G_1(t,s) > 0$, $\forall t, s \in [0, 2\pi]$. Since

$$w(t) = \int_0^{2\pi} G_1(t,s)e(s)ds, \quad \forall t \in [0, 2\pi],$$

from $G_1(t,s) > 0$, $\forall t, s \in [0, 2\pi]$ and $e(s) > 0$, $\forall s \in [0, 2\pi]$, we have

$$w > 0, \quad t \in [0, 2\pi],$$

i.e. $s_0 > u$, $t \in [0, 2\pi]$. Hence $\|u\| < s_0$, a contradiction. □

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