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OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Some new oscillation criteria are established for second order neutral partial functional differential equation of the form

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right)\right]=a(t)\triangle u(x,t)+\sum_{k=1}^{s}a_k(t)\triangle u(x,t-\rho_k(t))$$
$$-q(x,t)u(x,t)-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j)),\quad (x,t)\in\Omega\times[0,\infty)\equiv G$$

under the conditions $\int_{t_0}^{\infty} r^{-1}(s)ds=\infty$ and $\int_{t_0}^{\infty} r^{-1}(s)ds<\infty$, respectively. where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$ and $\triangle$ is the laplacian in the Euclidean $N-$space R^N.

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1. INTRODUCTION

In this paper, we are concerned with the oscillation problem of second order neutral partial functional differential equation of the form

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right)\right]$$
$$=a(t)\triangle u(x,t)+\sum_{k=1}^{s}a_k(t)\triangle u(x,t-\rho_k(t))-q(x,t)u(x,t)$$
$$-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j)),(x,t)\in\Omega\times[0,\infty)\equiv G. \tag{1.1}$$

Where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, and $\triangle u(x,t)=\sum_{r=1}^{N}\frac{\partial^2 u(x,t)}{\partial x_r^2}$.

We assume throughout this paper that the following conditions hold.

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(A_1) $r(t) \in C^1(R_+;[0,\infty))$, $\lambda_i \in C^2([0,\infty);[0,\infty))$, $0 \le \sum_{i=1}^{l} \lambda_i(t) \le 1$, τ_i are nonnegative constants, $i \in I_l = \{1,2,\ldots,l\}$;

(A_2) $a, a_k, \rho_k \in C([0,\infty);[0,\infty))$, $\lim_{t\to\infty}(t-\rho_k(t)) = \infty$, $k \in I_s = \{1,2,\ldots,s\}$;

(A_3) $q, q_j \in C(\bar{G},(0,\infty))$, $q(t) = \min_{x\in\bar{\Omega}} q(x,t)$, $q_j(t) = \min_{x\in\bar{\Omega}} q_j(x,t)$, σ_j are nonnegative constants, $j \in I_m = \{1,2,\ldots,m\}$;

(A_4) $f_j \in C(R,R)$ are convex in $[0,\infty)$, $uf_j(u) > 0$ and $\frac{f_j(u)}{u} \ge \alpha_j$ for $u \ne 0$, α_j are positive constants, $j \in I_m$.

We consider two kinds of boundary conditions,

$$\frac{\partial u(x,t)}{\partial \gamma} + g(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times [0,\infty)$, and

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{1.3}$$

As usual, a solution $u(x,t)$ of the problem (1.1), (1.2) (or (1.1), (1.3)) is called oscillatory in the domain $G = \Omega \times [0,\infty)$ if for any positive number μ there exists a point $(x_0,t_0) \in \Omega \times [\mu,\infty)$ such that $u(x_0,t_0) = 0$ holds.

Recently, the oscillation problem for the partial functional differential equation has been studied by many authors. We refer the reader to [2, 5, 10] for parabolic equations and to [1, 3, 4, 6–8, 11] for hyperbolic equations. We note that conditions for the oscillation of the solutions for $r(t) = 1$, $\lambda_i(t) = 0$, $f_j(u) = u$ have been obtained in the works of Cui et al. [3]. Very recently, Wang and Meng [11] have studied the oscillation conditions for $\lambda_i(t) = 0$, $f_j(u) = u$. Li and Cui [9] have observed some oscillation properties of (1.1) under the following assumption

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} \to \infty \quad (t \to \infty).$$

To the best of our knowledge, it seems to have few oscillation and asymptotic behavior results for (1.1) in the case

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} ds < \infty.$$

In the following two sections, by using a generalized Riccati transformation, we will further the investigation and offer some new sufficient conditions for the oscillation of the problem (1.1), (1.2) as well as for (1.1), (1.3) under the two conditions above, and give some examples to illustrate the superiority of our main results at the end of this paper.

2. OSCILLATION OF THE PROBLEM (1.1) AND (1.2)

2.1. The case of $\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)} = \infty$.

Theorem 2.1. *If there exist a $j_0 \in I_m$ and a function $\rho \in C^1(I, R^+)$ such that*

$$\int^{\infty} \left[\rho(t)\varphi(t) - \frac{r(t-\sigma_{j_0})\rho'^2(t)}{4\rho(t)}\right] dt = \infty, \tag{2.1}$$

where

$$\varphi(t) = q(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t)\right] + \alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\right], \tag{2.2}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.2) in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$, without loss of generality we assume that $u(x,t) > 0$, $u(x, t-\tau_i) > 0$, $u(x, t-\rho_k(t)) > 0$ and $u(x, t-\sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t_1 \ge t_0$, $i \in I_l$, $k \in I_s$, $j \in I_m$.

Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx + \sum_{i=1}^{l} \lambda_i(t)\int_\Omega u(x,t-\tau_i)dx\right)\right] \\
&\quad = a(t)\int_\Omega \triangle u(x,t)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \triangle u(x,t-\rho_k(t))dx - \int_\Omega q(x,t)u(x,t)dx \\
&\qquad - \sum_{j=1}^{m}\int_\Omega q_j(x,t) f_j(u(x,t-\sigma_j))dx, \ t \ge t_1.
\end{aligned} \tag{2.3}$$

From Green's formula and boundary condition (1.2), it follows that

$$\int_\Omega \triangle u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial\gamma} dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \le 0, \quad t \ge t_1, \tag{2.4}$$

and

$$\begin{aligned}
\int_\Omega \triangle u(x,t-\rho_k(t))dx &= \int_{\partial\Omega} \frac{\partial u(x,t-\rho_k(t))}{\partial\gamma} dS \\
&= -\int_{\partial\Omega} g(x,t-\rho_k(t))u(x,t-\rho_k(t))dS \le 0, \quad t \ge t_1, \quad k \in I_s,
\end{aligned} \tag{2.5}$$

where dS is the surface element on $\partial\Omega$. Moreover, from $(A_3), (A_4)$ and Jensen's inequality, it follows that

$$\int_\Omega q(x,t)u(x,t)dx \ge q(t)\int_\Omega u(x,t)dx, \quad t \ge t_1, \tag{2.6}$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))dx \geq q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))dx$$

$$\geq q_j(t)\int_\Omega dx\cdot f_j\left(\int_\Omega u(x,t-\sigma_j)dx\left(\int_\Omega dx\right)^{-1}\right),\quad t\geq t_1. \tag{2.7}$$

Let

$$V(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\quad t\geq t_1, \tag{2.8}$$

where $|\Omega|=\int_\Omega dx$.

In view of (2.4)–(2.8), (2.3) yields

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\right)\right]$$
$$+q(t)V(t)+\sum_{j=1}^{m}q_j(t)f_j(V(t-\sigma_j))\leq 0,\quad t\geq t_1. \tag{2.9}$$

Let $Z(t)=V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)$, we have $Z(t)>0$ and $[r(t)Z'(t)]'\leq 0$ for $t\geq t_1$. Hence $r(t)Z'(t)$ is a decreasing function in the interval $[t_1,\infty)$. We can claim that $r(t)Z'(t)>0$ for $t\geq t_1$. In fact, if there exist a $T>t_1$ such that $r(T)Z'(T)<0$, this implies that

$$Z'(t)\leq\frac{r(T)Z'(T)}{r(t)}\quad\text{for } t\geq T,$$

and

$$Z(t)-Z(T)\leq r(T)Z'(T)\int_T^t\frac{ds}{r(s)},\quad t\geq T.$$

Therefore $\lim_{t\to\infty}Z(t)=-\infty$, which contradicts the fact that $Z(t)>0$.

From (2.9), for the j_0 in (2.1) we obtain

$$[r(t)Z'(t)]'+q(t)V(t)+\alpha_{j_0}q_{j_0}(t)V(t-\sigma_{j_0})\leq 0,\quad t\geq t_1. \tag{2.10}$$

or

$$[r(t)Z'(t)]'+q(t)\left[Z(t)-\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\right]$$
$$+\alpha_{j_0}q_{j_0}(t)\left[Z(t-\sigma_{j_0})-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})V(t-\tau_i-\sigma_{j_0})\right]\leq 0,\quad t\geq t_1.$$

Since $Z(t)\geq V(t), Z(t)$ is increasing, it follows that

$$[r(t)Z'(t)]'+q(t)\left[1-\sum_{i=1}^{l}\lambda_i(t)\right]Z(t)$$
$$+\alpha_{j_0}q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right]Z(t-\sigma_{j_0})\leq 0,\quad t\geq t_1.$$

That is

$$[r(t)Z'(t)]' + \varphi(t)Z(t-\sigma_{j_0}) \leq 0, \quad t \geq t_1, \tag{2.11}$$

where $\varphi(t)$ is defined by (2.2).

Define

$$w(t) = \rho(t)\frac{r(t)Z'(t)}{Z(t-\sigma_{j_0})}, \quad t \geq t_1.$$

Then $w(t) > 0$, and

$$\begin{aligned} w'(t) &= \rho'(t)\frac{r(t)Z'(t)}{Z(t-\sigma_{j_0})} + \rho(t)\frac{[r(t)Z'(t)]'Z(t-\sigma_{j_0}) - r(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})} \\ &\leq \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\varphi(t) - \frac{\rho(t)r(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})}. \end{aligned}$$

Using the fact that $r(t)Z'(t)$ is decreasing, we have

$$Z'(t-\sigma_{j_0}) \geq \frac{r(t)Z'(t)}{r(t-\sigma_{j_0})}, \quad t \geq t_1. \tag{2.12}$$

Thus

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\varphi(t) - \frac{w^2(t)}{\rho(t)r(t-\sigma_{j_0})} \leq \frac{r(t-\sigma_{j_0})\rho'^2(t)}{4\rho(t)} - \rho(t)\varphi(t).$$

Integrating the above inequality from some T_0 to t ($T_0 \geq t_1$), we have

$$w(t) \leq w(T_0) - \int_{T_0}^{t}\left[\rho(s)\varphi(s) - \frac{r(s-\sigma_{j_0})\rho'^2(s)}{4\rho(s)}\right]ds \tag{2.13}$$

Letting $t \to \infty$ in (2.13), from (2.1) we get contradiction. This completes the proof of Theorem 2.1. □

Corollary 2.2. *If the inequality* (2.9) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Choosing $\rho(t) = R(t-\sigma_{j_0})$ in Theorem 2.1, we get

Corollary 2.3. *If there exists a* $j_0 \in I_m$ *such that for* $t_1 \geq t_0$,

$$\int^{\infty}\left[R(t-\sigma_{j_0})\varphi(t) - \frac{1}{4R(t-\sigma_{j_0})r(t-\sigma_{j_0})}\right]dt = \infty, \tag{2.14}$$

where $\varphi(t)$ *is defined by* (2.2). *Then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Corollary 2.4. *If there exists a* $j_0 \in I_m$ *such that for* $t_1 \geq t_0$,

$$\liminf_{t\to\infty}\frac{1}{\ln R(t-\sigma_{j_0})}\int_{t_1}^{t}R(s-\sigma_{j_0})\varphi(s)ds > \frac{1}{4}, \tag{2.15}$$

where $\varphi(t)$ *is defined by* (2.2). *Then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Proof. It is not hard to verify that (2.15) yields the existence $\varepsilon > 0$ such that for all large t,

$$\frac{1}{\ln R(t-\sigma_{j_0})}\int_{t_1}^{t} R(s-\sigma_{j_0})\varphi(s)ds \geq \frac{1}{4}+\varepsilon,$$

which follows that

$$\int_{t_1}^{t} R(s-\sigma_{j_0})\varphi(s)ds \geq \frac{1}{4}\ln R(t-\sigma_{j_0})+\varepsilon \ln R(t-\sigma_{j_0}),$$

then (2.14) holds, hence the assertion of Corollary 2.4 follows from Corollary 2.3. □

Corollary 2.5. *Assume that there exists a* $j_0 \in I_m$ *such that for* $t_1 \geq t_0$*,*

$$\liminf_{t\to\infty} R^2(t-\sigma_{j_0})\varphi(t)r(t-\sigma_{j_0}) > \frac{1}{4}, \tag{2.16}$$

where $\varphi(t)$ *is defined by* (2.2)*. Then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G*.*

Proof. Obviously, (2.16) yields the existence $\varepsilon > 0$ such that for all large t,

$$R^2(t-\sigma_{j_0})\varphi(t)r(t-\sigma_{j_0}) \geq \frac{1}{4}+\varepsilon,$$

that is

$$R(t-\sigma_{j_0})\varphi(t) - \frac{1}{4R(t-\sigma_{j_0})r(t-\sigma_{j_0})} \geq \frac{\varepsilon}{R(t-\sigma_{j_0})r(t-\sigma_{j_0})},$$

which implies (2.14) hold and Corollary 2.5 is evident by Corollary 2.3. □

Theorem 2.6. *Assume that there exist a* $j_0 \in I_m$ *and a function* $h \in C^1(I,R)$ *such that*

$$\int^{\infty}\left(\varphi(t)-\frac{h^2(t)}{r(t-\sigma_{j_0})}\right)\exp\left(2\int^{t}\frac{h(s)}{r(s-\sigma_{j_0})}ds\right)dt = \infty, \tag{2.17}$$

where $\varphi(t)$ *is defined by* (2.2)*. Then every solution* $u(x,t)$ *of Eqs.* (1.1), (1.2) *is oscillatory in* G*.*

Proof. Let $u(x,t)$ be a nonoscillatory solution of Eqs. (1.1), (1.2), without loss of generality, as in the proof of Theorem 2.1, we may assume that there exists a number $T_0 \geq t_0$ such that $u(x,t) > 0$, $u(x,t-\tau_i) > 0$, $u(x,t-\rho_k(t)) > 0$, and $u(x,t-\sigma_j) > 0$ in $\Omega\times[T_0,\infty)$, $i \in I_l, k \in I_s$, $j \in I_m$, and we get $Z'(t) > 0$ holds for $t \geq T_0$. Furthermore, (2.11) and (2.12) hold. Define

$$w(t) = \frac{r(t)Z'(t)}{Z(t-\sigma_{j_0})}, \quad t \geq T_0, \tag{2.18}$$

obviously, $w(t) > 0$ for $t \geq T_0$. Differentiating (2.18), in view of (2.11) and (2.12), we have

$$\begin{aligned} w'(t) &= \frac{[r(t)Z'(t)]'Z(t-\sigma_{j_0}) - r(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})} \leq -\varphi(t) - \frac{1}{r(t-\sigma_{j_0})}w^2(t) \\ &= -\left[\varphi(t) - \frac{1}{r(t-\sigma_{j_0})}h^2(t)\right] - \frac{1}{r(t-\sigma_{j_0})}[w^2(t) + h^2(t)] \\ &\leq -\left[\varphi(t) - \frac{1}{r(t-\sigma_{j_0})}h^2(t)\right] - \frac{2h(t)}{r(t-\sigma_{j_0})}w(t), \end{aligned}$$

which follows that

$$\left[\exp\left(\int_{T_0}^t \frac{2h(s)}{r(s-\sigma_{j_0})}ds\right) w(t)\right]' \leq -\left[\varphi(t) - \frac{1}{r(t-\sigma_{j_0})}h^2(t)\right] \exp\left(\int_{T_0}^t \frac{2h(s)}{r(s-\sigma_{j_0})}ds\right).$$

Integrating the above inequality from T_0 to t, we have

$$\begin{aligned} 0 &< \exp\left(\int_{T_0}^t \frac{2h(s)}{r(s-\sigma_{j_0})}ds\right) w(t) \\ &\leq w(T_0) - \int_{T_0}^t \left(\varphi(s) - \frac{h^2(s)}{r(s-\sigma_{j_0})}\right) \exp\left(\int_{T_0}^s \frac{2h(\tau)}{r(\tau-\sigma_{j_0})}d\tau\right) ds. \end{aligned}$$

Letting $t \to \infty$ in the above inequality, which contradicts (2.17). This completes the proof of Theorem 2.6. □

Since Theorems 2.1–2.6 are of a high degree of a generality, it is convenient for application to derive a number of oscillation criteria with the appropriate choice of the function h and ρ, here, we omit the details.

2.2. **The case of $\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^t \frac{ds}{r(s)} < \infty$.** For simplicity, we define

$$\pi(t) = \int_t^\infty \frac{ds}{r(s)},$$

and we will consider the oscillation problem of (1.1), (1.2) under the condition

$$\pi(t_0) = \int_{t_0}^\infty \frac{ds}{r(s)} < \infty.$$

Theorem 2.7. *Suppose that there exists a continuously differentiable function $\rho(t)$ such that $\rho(t) > 0$, $\rho'(t) \geq 0$, we also suppose that $\lambda_i'(t) \geq 0$ for $t \geq t_0$, $i \in I_l$, $\lim_{t\to\infty} \sum_{i=1}^l \lambda_i(t) = A$. If for some $j_0 \in I_m$,*

$$\int_0^\infty \left[\varphi(t)\pi(t-\sigma_{j_0}) - \frac{1}{4r(t-\sigma_{j_0})\pi(t-\sigma_{j_0})}\right] dt = \infty, \tag{2.19}$$

and

$$\int^\infty \frac{1}{\rho(t)r(t)} \left(\int^t \rho(s)(q(s) + \alpha_{j_0}q_{j_0}(s))ds\right) dt = \infty, \tag{2.20}$$

then every solution $u(x,t)$ *of Eqs.* (1.1), (1.2) *is oscillates or*

$$\lim_{t\to\infty}\int_\Omega u(x,t)dx = 0.$$

Proof. Suppose that (1.1), (1.2) has a nonoscillatory solution $u(x,t)$ which may be assumed to be positive in $\Omega\times[T,\infty)$ for some $T>0$. As in the proof of Theorem 2.1, we know that $Z(t)>0$ and $r(t)Z'(t)$ is nonincreasing for $t\geq T$, consequently, it is easy to conclude that there exist two possible cases of the sign of $Z'(t)$, namely, $Z'(t)>0$ or $Z'(t)<0$ for $t\geq T_1\geq T$.

Case (1): Suppose $Z'(t)>0$ for sufficiently large t, then we are back to the case of Theorem 2.1 by choosing $\rho(t)=\pi(t-\sigma_{j_0})$. Thus the proof of Theorem 2.1 goes through, and we get contradiction by (2.19).

Case (2): Suppose $Z'(t)\leq 0$ for $t\geq T_1$, from the following conditions

$$\lambda_i'(t)\geq 0,\quad Z'(t)=V'(t)+\sum_{i=1}^{l}\lambda_i'(t)V(t-\tau_i)+\sum_{i=1}^{l}\lambda_i(t)V'(t-\tau_i),$$

we have $V'(t)\leq 0$, it follows that $\lim_{t\to\infty}Z(t)=a\geq 0$. Now we claim that $a=0$. Otherwise, $\lim_{t\to\infty}Z(t)=a>0$, so $\lim_{t\to\infty}V(t)=\frac{a}{1+A}>0$, there exists a constant $M>0$ such that $V(t)\geq M$, $V(t-\sigma_{j_0})\geq M$ for the j_0 in (2.20) and all $t\geq t_1\geq t_0$. From (2.10) we get

$$\begin{aligned}[r(t)Z'(t)]' &\leq -q(t)V(t)-\alpha_{j_0}q_{j_0}(t)V(t-\sigma_{j_0})\\ &\leq -Mq(t)-\alpha_{j_0}Mq_{j_0}(t)=-M(q(t)+\alpha_{j_0}q_{j_0}(t)),\quad t\geq t_1.\end{aligned}\tag{2.21}$$

Define $Q(t)=\rho(t)r(t)Z'(t)$, then $Q(t)\leq 0$, from (2.21) we get

$$\begin{aligned}Q'(t)&=\rho(t)[r(t)Z'(t)]'+\rho'(t)r(t)Z'(t)\leq\rho(t)[r(t)Z'(t)]'\\ &\leq -M\rho(t)(q(t)+\alpha_{j_0}q_{j_0}(t)).\end{aligned}$$

Integrating it from t_1 to t, we get

$$Q(t)\leq Q(t_1)-M\int_{t_1}^{t}\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds\leq -M\int_{t_1}^{t}\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds,$$

that is

$$\rho(t)r(t)Z'(t)\leq -M\int_{t_1}^{t}\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds,$$

so that

$$Z'(t)\leq -\frac{M}{\rho(t)r(t)}\int_{t_1}^{t}\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds.$$

Integrating the above inequality from t_1 to t, we obtain

$$Z(t)\leq Z(t_1)-M\int_{t_1}^{t}\frac{1}{\rho(s)r(s)}\left(\int_{t_1}^{s}\rho(\xi)(q(\xi)+\alpha_{j_0}q_{j_0}(\xi))d\xi\right)ds.$$

We can easily obtain a contradiction. So that $\lim_{t\to\infty} Z(t) = 0$, then $\lim_{t\to\infty} V(t) = 0$. This completes the proof of Theorem 2.7. □

Corollary 2.8. *If one of* (2.14), (2.15), (2.16), (2.17) *holds, and* (2.20) *holds, then every solution* $u(x,t)$ *of Eqs.* (1.1), (1.2) *is oscillatory or* $\lim_{t\to\infty} \int_\Omega u(x,t)dx = 0$.

3. OSCILLATION OF THE PROBLEM (1.1) AND (1.3)

The following fact will be used.

The smallest eigenvalue β_0 of the Dirichlet problem

$$\begin{cases} \triangle w(x) + \beta w(x) = 0 & \text{in } \Omega \\ w(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

is positive and the corresponding eigenfunction $\psi(x)$ is positive in Ω.

3.1. The case of $\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^t \frac{ds}{r(s)} = \infty$.

Theorem 3.1. *If all conditions of Theorem* 2.1 *hold, then every solution* $u(x,t)$ *of problem* (1.1), (1.3) *oscillates in* G.

Proof. To the contrary, if there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.3) in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$, without loss of generality, we assume that $u(x,t) > 0$, $u(x, t-\tau_i) > 0$, $u(x, t-\rho_k(t)) > 0$ and $u(x, t-\sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t_1 \geq t_0$, $i \in I_l$, $k \in I_s$, $j \in I_m$. Multiplying both side of (1.1) by $\psi(x) > 0$ and integrating it with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)\psi(x)dx + \sum_{i=1}^{l} \lambda_i(t) \int_\Omega u(x,t-\tau_i)\psi(x)dx\right)\right] \\ &= a(t)\int_\Omega \triangle u(x,t)\psi(x)dx + \sum_{k=1}^{s} a_k(t) \int_\Omega \triangle u(x,t-\rho_k(t))\psi(x)dx \\ &\quad - \int_\Omega q(x,t)u(x,t)\psi(x)dx \\ &\quad - \sum_{j=1}^{m} \int_\Omega q_j(x,t) f_j(u(x,t-\sigma_j))\psi(x)dx, \quad t \geq t_1. \end{aligned} \tag{3.2}$$

From Green's formula and boundary condition (1.3), it follows that

$$\begin{aligned} \int_\Omega \triangle u(x,t)\psi(x)dx &= \int_\Omega u(x,t)\triangle\psi(x)dx \\ &= -\beta_0 \int_\Omega u(x,t)\psi(x)dx \leq 0, \quad t \geq t_1, \end{aligned} \tag{3.3}$$

and

$$\int_\Omega \triangle u(x,t-\rho_k(t))\psi(x)dx = \int_\Omega u(x,t-\rho_k(t))\triangle\psi(x)dx$$

$$= -\beta_0 \int_\Omega u(x,t-\rho_k(t))\psi(x)dx \le 0, \quad t \ge t_1, \quad k \in I_s. \tag{3.4}$$

From $(A_3), (A_4)$ and Jensen's inequality, it follows that

$$\int_\Omega q(x,t)u(x,t)\psi(x)dx \ge q(t)\int_\Omega u(x,t)\psi(x)dx, \quad t \ge t_1, \tag{3.5}$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\psi(x)dx \ge q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))\psi(x)dx$$

$$\ge q_j(t)\int_\Omega \psi(x)dx \cdot f_j\left(\int_\Omega u(x,t-\sigma_j)\psi(x)dx\left(\int_\Omega \psi(x)dx\right)^{-1}\right), \quad t \ge t_1. \tag{3.6}$$

Set

$$V(t) = \int_\Omega u(x,t)\psi(x)dx\left(\int_\Omega \psi(x)dx\right)^{-1}, \quad t \ge t_1, \tag{3.7}$$

combining (3.2)–(3.7) we obtain

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\right)\right] + q(t)V(t)$$

$$+ \sum_{j=1}^{m} q_j(t)f_j(V(t-\sigma_j)) \le 0, \quad t \ge t_1. \tag{3.8}$$

The remainder of the proof is similar to that of Theorem 2.1, we omit it. □

Corollary 3.2. *If the inequality* (3.8) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillatory in* G.

The following conclusions can be proved analogously.

Corollary 3.3. *Let the conditions of Corollary* 2.3 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

Corollary 3.4. *Let the conditions of Corollary* 2.4 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

Corollary 3.5. *Let the conditions of Corollary* 2.5 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

Theorem 3.6. *Let the conditions of Theorem* 2.6 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

3.2. **The case of** $\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)} < \infty$**.** As the case in 3.1, the following conclusions can be proved analogously.

Theorem 3.7. *If the conditions of Theorem* 2.7 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)\psi(x)dx = 0$ *in* G*, where* $\psi(x)$ *is as in* (3.1).

Corollary 3.8. *If the conditions of Corollary* 2.8 *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)\psi(x)dx = 0$ *in* G*, where* $\psi(x)$ *is as in* (3.1).

4. EXAMPLES

Example 4.1. Consider the equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{t+\pi}\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{t+\pi}u(x,t-2\pi)\right)\right]\\
&=\left(\frac{1}{(t+\pi)^2}-\frac{3}{(t+\pi)^4}\right)\triangle u(x,t)\\
&\quad+\left(\frac{3}{(t+\pi)^3}+\frac{1}{(t+\pi)^2}\right)\triangle u\left(x,t-\frac{3\pi}{2}\right)\\
&\quad+\left(\frac{1}{2t^3\ln t}+\frac{1}{t^3}-\frac{1}{(t+\pi)}\right)\triangle u(x,t-\pi)\\
&\quad-\left(\frac{1}{2t^3\ln t}+\frac{2}{t^3}\right)u(x,t)-\frac{1}{t^3}u(x,t-\pi),\quad (x,t)\in(0,\pi)\times[0,\infty)
\end{aligned}\tag{4.1}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\ge 0.\tag{4.2}$$

Here, $N=1$, $r(t)=\frac{1}{t+\pi}$, $l=1$, $\lambda_1(t)=\frac{1}{t+\pi}$, $\tau_1=2\pi$, $a(t)=\frac{1}{(t+\pi)^2}-\frac{3}{(t+\pi)^4}$, $s=2$, $a_1(t)=\frac{3}{(t+\pi)^3}+\frac{1}{(t+\pi)^2}$, $\rho_1(t)=\frac{3\pi}{2}$, $a_2(t)=\frac{1}{2t^3\ln t}+\frac{1}{t^3}-\frac{1}{t+\pi}$, $\rho_2(t)=\pi$, $q(x,t)=\frac{1}{2t^3\ln t}+\frac{2}{t^3}$, $m=1$, $q_1(x,t)=\frac{1}{t^3}$, $\sigma_1=\pi$, $f_1(u)=u$, it is easy to see that $q_{j_0}(t)=q_1(t)=\frac{1}{t^3}$, $\alpha_{j_0}=1$, $\lambda_1(t-\sigma_{j_0})=\lambda_1(t-\pi)=\frac{1}{t}$, $q(t)=q(x,t)=\frac{1}{2t^3\ln t}+\frac{2}{t^3}$, $r(t-\sigma_{j_0})=\frac{1}{t}$, $R(t)=\int_0^t\frac{ds}{r(s)}=\frac{t^2}{2}+\pi t$, then we have

$$\varphi(t)=\frac{1}{2t^3\ln t}+\frac{3}{t^3}-\frac{1}{2t^3(t+\pi)\ln t}-\frac{2}{t^3(t+\pi)}-\frac{1}{t^4},$$

and

$$\liminf_{t\to\infty}R^2(t-\sigma_{j_0})\varphi(t)r(t-\sigma_{j_0})=\frac{3}{4}>\frac{1}{4},$$

which shows that all conditions of Corollary 3.5 are verified. Thus every solutions of problem (4.1), (4.2) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u(x,t)=\sin x\cos t$ is such a solution.

Example 4.2. Consider the equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[(t+\pi)^2\frac{\partial}{\partial t}\left(u(x,t)+\left(1-\frac{1}{t+\pi}\right)u(x,t-2\pi)\right)\right]\\
&=\left(t^3+\frac{1}{t}+\frac{1}{(t+\pi)^2}\right)\triangle u(x,t)\\
&\quad+4(t+\pi)\triangle u(x,t-\frac{3\pi}{2})+\left(\pi+\frac{1}{t}\right)\triangle u(x,t-\pi)-\left(t^2+t+\pi\right)u(x,t)\\
&\quad-\left(t^3-t+\frac{1}{(t+\pi)^2}\right)u(x,t-\pi),\quad (x,t)\in(0,\pi)\times[0,\infty)
\end{aligned}
\tag{4.3}
$$

with the boundary condition

$$
u(0,t)=u(\pi,t)=0,\quad t\geq 0. \tag{4.4}
$$

Here, $N=1$, $r(t)=(t+\pi)^2$, $l=1$, $\lambda_1(t)=1-\frac{1}{t+\pi}$, $\tau_1=2\pi$, $a(t)=t^3+\frac{1}{t}+\frac{1}{(t+\pi)^2}$, $s=2$, $a_1(t)=4(t+\pi)$, $\rho_1(t)=\frac{3\pi}{2}$, $a_2(t)=\pi+\frac{1}{t}$, $\rho_2(t)=t-\pi$, $q(x,t)=t^2+t+\pi$, $q_1(x,t)=t^3-t+\frac{1}{(t+\pi)^2}$, $\sigma_1=\pi$, $f_1(u)=u$, it is easy to see that $q_{j_0}(t)=q_1(t)=q_1(x,t)=t^3-t+\frac{1}{(t+\pi)^2}$, $\alpha_{j_0}=1$, $\lambda_1(t-\sigma_{j_0})=\lambda_1(t-\pi)=1-\frac{1}{t}$, $q(t)=q(x,t)=t^2+t+\pi$, $r(t-\sigma_{j_0})=r(t-\pi)=t^2$, $R(t)=\int_0^t\frac{ds}{r(s)}=\int_0^t\frac{ds}{(s+\pi)^2}=\frac{1}{\pi}-\frac{1}{t+\pi}$, $\pi(t)=\frac{1}{t+\pi}$, $\pi(t-\sigma_{j_0})=\frac{1}{t}$, $\varphi(t)=t^2+\frac{t^2}{t+\pi}+\frac{1}{t(t+\pi)^2}$, let $\rho(s)=1$, then we have

$$
\begin{aligned}
&\int_0^\infty\left[\varphi(t)\pi(t-\sigma_{j_0})-\frac{1}{4r(t-\sigma_{j_0})\pi(t-\sigma_{j_0})}\right]dt\\
&=\int_0^\infty\left(t+\frac{t}{t+\pi}+\frac{1}{t^2(t+\pi)^2}-\frac{1}{4t}\right)dt=\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int^\infty\frac{1}{\rho(t)r(t)}\left(\int^t\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds\right)dt\\
&=\int^\infty\frac{1}{(t+\pi)^2}\left(\int^t(s^2+s+\pi+s^3-s+\frac{1}{(s+\pi)^2})ds\right)dt=\infty,
\end{aligned}
$$

which shows that (2.19), (2.20) hold, thus every solution of problem (4.3), (4.4) oscillates or $\lim_{t\to\infty}\int_\Omega u(x,t)\psi(x)dx=0$ in G from Theorem 3.7. However, the main results of [11] fails to the problem (4.3), (4.4) since $\lim_{t\to\infty}R(t)<\infty$.

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