Dynamic Systems and Applications 19 (2010) 271-288

NEW RESULTS ON CRITICAL OSCILLATION CONSTANTS DEPENDING ON A GRAININESS

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ABSTRACT. We establish criteria of Hille-Nehari type for the half-linear second order dynamic equation $(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0$, $\Phi(u)=|u|^{\alpha-1}\operatorname{sgn} u$, $\alpha>1$, on time scales, under the condition $\int^{\infty} r^{1/(1-\alpha)}(s)\,\Delta s<\infty$. As a particular important case we get that there is a (non-improvable) critical oscillation constant which may be different from the one known from the continuous case, and its value depends on the graininess of a time scale and on the coefficient r. Along with the results of the previous paper by the author, which dealt with the condition $\int^{\infty} r^{1/(1-\alpha)}(s)\,\Delta s=\infty$, a quite complete discussion on generalized Hille-Nehari type criteria involving the best possible constants is provided. To prove these criteria, appropriate modifications of the approaches known from the linear case ($\alpha=2$) or the continuous case ($\mathbb{T}=\mathbb{R}$) cannot be used in a general case, and thus we apply a new method. As applications of the main results we state criteria for strong (non)oscillation, examine a generalized Euler type equation, and establish criteria of Kneser type. Examples from q-calculus and h-calculus, and a Hardy type inequality are presented as well. Our results unify and extend many existing results from special cases, and are new even in the well-studied discrete case.

AMS (MOS) Subject Classification. 34C10, 39A11, 39A12, 39A13

1. INTRODUCTION

Consider the half-linear dynamic equation

$$(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0, \tag{1.1}$$

where $\Phi(u)=|u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha>1$, $1/r(t)>0$ and $p(t)$ are rd-continuous functions defined on an interval $[a,\infty)\subseteq\mathbb{T}$, $\mathbb{T}$ being a time scale. If $\mathbb{T}=\mathbb{R}$ and $\alpha=2$, then (1.1) reduces to the well-known Sturm-Liouville linear differential equation

$$(r(t)y')'+p(t)y=0. \tag{1.2}$$

One of the most famous (non)oscillation criteria, the so-called Hille-Nehari ones, say that (1.2) is oscillatory provided $\liminf_{t\to\infty}(\int_a^t 1/r(s)\,ds)\int_t^{\infty} p(s)\,ds>1/4$, and (1.2) is nonoscillatory provided $\limsup_{t\to\infty}(\int_a^t 1/r(s)\,ds)\int_t^{\infty} p(s)\,ds<1/4$, where we assume $\int_t^{\infty} p(s)\,ds\ge 0$ and $\int_a^{\infty} 1/r(s)\,ds=\infty$, see, e.g., [28, Chapter 2]. Many works have appeared in which these criteria were generalized or extended either to half-linear

Received April 24, 2009

differential equations or to difference equations or to dynamic equations on time scales, see, e.g., [5, 6, 7, 8, 9, 12, 14, 15, 16, 17, 18, 20, 24, 25, 29]. However, concerning an extension in the sense of time scales different from $\mathbb{R}$, the results presented there usually contain restrictions, typically of two types, that disable examination of many important cases: The constants on the right-hand sides are not the best possible and/or the choice of time scale is strictly limited. These restrictions were substantially removed only very recently, see [26, 27]; in particular, it was shown that the constant on the right-hand side may depend on time scales and on the coefficient r, and can be strictly less than 1/4 (or, more generally, than $((\alpha-1)/\alpha)^{\alpha-1}/\alpha)$ with being still the best possible. Somehow related oscillation result can be found also in [3]. Another interesting and kind of initiating paper is [6].

In the above criteria for (1.2) we required $\int_a^\infty 1/r(s)\,ds = \infty$. If we consider the complementary case, i.e., $\int_a^\infty 1/r(s)\,ds < \infty$, then the corresponding criteria can be obtained from the known ones using certain (linear) transformation of dependent variable which transforms (1.2) with the convergent integral condition into the equation of the same form with the divergent integral condition. In our general case, these conditions read as

$$\int_a^\infty r^{1-\beta}(s)\,\Delta s = \infty \tag{1.3}$$

and

$$\int_a^\infty r^{1-\beta}(s)\,\Delta s < \infty, \tag{1.4}$$

where β is the conjugate number to α, i.e., $1/\alpha + 1/\beta = 1$. In [27] we established a generalization of Hille-Nehari criteria for (1.1) under condition (1.3), see Theorem 2.2 below. Now we wish to derive corresponding criteria for (1.1) in the complementary case, i.e., under condition (1.4). However, in contrast to the linear theory, there is no suitable transformation, which transforms the case with (1.4) into the one with (1.3), at disposal in a nonlinear case, and so we cannot use Theorem 2.2 as in [26]. In the continuous case, i.e., $\mathbb{T} = \mathbb{R}$, desired criteria under condition (1.4) were already established, see [15] or [9], and they are based on a knowledge of oscillation behavior of certain generalized Euler differential equation, Hille-Wintner type comparison theorem, and a transformation of independent variable. This approach however cannot be used in a general time scale case (or even in the discrete case $\mathbb{T} = \mathbb{Z}$) since a behavior of a related Euler type equation (along with the right oscillation constant) is unknown and the required transformation is not available. Hence, for equation (1.1) we have to use an approach which is different from the ones known from the linear or continuous theory.

A suitable method which is applicable in a general case was developed in [19] very recently, see Lemma 2.1 in the next section, where we present also other preliminary results and introduce notations. Using this lemma, in Section 3, we prove Hille-Nehari type criteria for equation (1.1) provided (1.4) holds. In particular, these criteria contain the critical (thus the best possible) constant under certain mild assumptions, i.e., the constants on the right-hand sides are the same and, moreover, they exhibit the expected dependence on the graininess of a time scale and on the coefficient r. In special cases, especially if $\alpha = 2$ and/or $\mathbb{T} = \mathbb{R}$, they reduce to the classical (sharp) results. It is worthy of note that our results are new also in the well-studied case $\mathbb{T} = \mathbb{Z}$, i.e., for half-linear difference equations. In the last section we present applications of the main results: Criteria for strong (non)oscillation are derived, a conditional oscillation of a generalized Euler type dynamic equation is described, which is then used in establishing sharp Kneser type criteria. We also provide several important examples from q-calculus and h-calculus, and obtain a Hardy type inequality involving the best possible constant. Finally we indicate some directions for a future research.

2. NOTATION AND PRELIMINARIES

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, σ, f^σ, μ, f^Δ and $\int_a^b f(s)\,\Delta s$ stand for time scale, forward jump operator, $f\circ\sigma$, graininess, delta derivative of f, and delta integral of f from a to b, respectively. Recall that, for instance, $f^\Delta(t) = f'(t)$ when $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = \Delta f(t)$ when $\mathbb{T} = \mathbb{Z}$, and $f^\Delta(t) = D_q f(t)$ when $\mathbb{T} = \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, where D_q denotes the Jackson derivative. See [13], which is the initiating paper of the time scale theory written by Hilger, and the monograph [4] by Bohner and Peterson containing a lot of information on time scale calculus. Time scales intervals will be denoted as usual real intervals, and from the context it always be clear whether the interval under consideration is real or of time scale type.

A (nontrivial) *solution* to (1.1) is said to be *nonoscillatory* if it is eventually of one sign, otherwise it is said to be *oscillatory*. Thanks to the Sturm type separation theorem, see [23], either all (nontrivial) solutions of (1.1) are oscillatory or all (nontrivial) solutions of (1.1) are nonoscillatory. Hence the *equation* can easily be classified as *oscillatory* or *nonoscillatory*, similarly as in the linear theory. Basic qualitative properties of (1.1) were studied in [23] or in [2].

As we distinguish two cases depending on behavior of the coefficient r in (1.1), it is practical to introduce the notation:

$$R_D(t) := \int_a^t r^{1-\beta}(s)\,\Delta s, \quad R_C(t) := \int_t^\infty r^{1-\beta}(s)\,\Delta s$$

Subscripts D and C stand for the divergence case (condition (1.3)) and the convergence case (condition (1.4)), respectively. Similar convention is used also in below defined formulae.

The so-called *Riccati technique* plays a very important role in the qualitative theory of (1.1) and there are known several sophisticated modifications of this method. The basic statement says that (1.1) is nonoscillatory if and only if the *generalized Riccati dynamic inequality* $w^\Delta + p(t) + S(w,r)(t) \le 0$ is solvable with $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t)) > 0$ for large t, where

$$S(w,r) = \lim_{\lambda\to\mu} \frac{w}{\lambda}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda\Phi^{-1}(w))}\right),$$

see [23]. Recall that (1.1) and $w^\Delta + p(t) + S(w,r)(t) = 0$ are related by the substitution $w = r\Phi(y^\Delta/y)$. Under conditions (1.3) and $\int_t^\infty p(s)\,\Delta s \ge 0$, in the above characterization of nonoscillation, instead of Riccati type dynamic inequality we can take the integral inequality

$$w(t) \ge \int_t^\infty p(s)\,\Delta s + \int_t^\infty S(w,r)(s)\,\Delta s,$$

see [25]. This can be then used to show the equivalence between nonoscillation of (1.1) and the (pointwise) convergence of the sequence $\{\psi_k(t)\}$, where $\psi_0(t) = \int_t^\infty p(s)\,\Delta s$ and $\psi_k(t) = \psi_0(t) + \int_t^\infty S(\psi_{k-1}, r)(s)\,\Delta s$, $k = 1, 2, \dots$, see [25]. If, in addition, $p(t) \ge 0$, then $w(t) \le R_D^{1-\alpha}(t)$. The following two characterizations of nonoscillation to (1.1) can be understood as complements to the previous ones, now for the case (1.4), see [19]: Assume $p(t) \ge 0$. Equation (1.1) is nonoscillatory if and only if $\int^\infty p(s)(R^\sigma(s))^\alpha \Delta s$ converges and there is a function w satisfying $w(t) \ge -R_C^{1-\alpha}(t)$ and

$$R_C^\alpha(t)w(t) \ge \int_t^\infty p(s)(R_C^\sigma(s))^\alpha \Delta s - \int_t^\infty w(s)(R_C^\alpha(s))^\Delta \Delta s + \int_t^\infty S(w,r)(s)(R_C^\sigma(s))^\alpha \Delta s$$

for large t. In the if part, $p(t) \ge 0$ can be relaxed to $\int_t^\infty p(s)(R_C^\sigma(s))^\alpha \Delta s \ge 0$ for large t. The following lemma (we call it a *function sequence technique*) is based on the previous relation and plays a crucial role in the proofs of our main results. Denote $H(t) = R_C^{-\alpha}(t)\int_t^\infty p(s)(R_C^\sigma(s))^\alpha \Delta s$ and

$$\mathcal{G}(u)(t) = R_C^{-\alpha}(t)\int_t^\infty \left(-u(s)(R_C^\alpha(s))^\Delta + S(u,r)(s)(R_C^\sigma(s))^\alpha\right)\,\Delta s.$$

In [19] we showed that $u \mapsto \mathcal{G}(u)$ is increasing for $u \ge -R_C^{1-\alpha}$. Further, $\mathcal{G}(-R_C^{1-\alpha}) = -R_C^{1-\alpha}$. Define the sequence $\{\varphi_k(t)\}$ by

$$\varphi_0 = -R_C^{1-\alpha}, \quad \varphi_{k+1} = H + \mathcal{G}(\varphi_k), \quad k = 0, 1, 2, \dots. \tag{2.1}$$

Clearly, $\varphi_{k+1} \ge \varphi_k$, $k = 0, 1, 2, \dots$, provided $H \ge 0$.

Lemma 2.1 ([19]). *Let $p(t) \ge 0$ for large t. Equation* (1.1) *is nonoscillatory if and only if there exists $t_0 \in [a,\infty)$ such that $\lim_{k\to\infty}\varphi_k(t) = \varphi(t)$ for $t \ge t_0$, i.e., $\{\varphi_k(t)\}$ is well defined and pointwise convergent. In the if part, $p(t) \ge 0$ can be relaxed to the condition $H(t) \ge 0$ for large t.*

For comparison purposes we now recall the sharp Hille-Nehari criteria for the case when (1.3) hold, see [27]. To this aim we introduce the notation:

$$\mathfrak{R}_D(\lambda)(t) := \frac{\lambda(t) r^{1-\beta}(t)}{R_D(t)}$$

$$\gamma_D(x) := \lim_{t \to x} \left(\frac{(t+1)^{\frac{\alpha-1}{\alpha}} - 1}{t} \right)^{\alpha} \frac{t}{(t+1)^{\alpha-1} - 1}, \quad x \in [0, \infty) \cup \{\infty\},$$

and

$$\mathcal{A}_D(t) := R_D^{\alpha-1}(t) \int_t^{\infty} p(s)\, \Delta s.$$

The proof of the following theorem is based on the above mentioned equivalence between nonoscillation of (1.1) and the convergence of the sequence $\{\psi_k(t)\}$.

Theorem 2.2 ([27]). *Let $\int_t^{\infty} p(s)\, \Delta s$ exist, be eventually nonnegative and eventually nontrivial for large t, and* (1.3) *hold.*

(i) Let $M_* = \liminf_{t \to \infty} \mathfrak{R}_D(\mu)(t)$. If

$$\liminf_{t \to \infty} \mathcal{A}_D(t) > \gamma_D(M_*), \tag{2.2}$$

then (1.1) *is oscillatory.*

(ii) Let $M^* = \limsup_{t \to \infty} \mathfrak{R}_D(\mu)(t)$. If

$$\limsup_{t \to \infty} \mathcal{A}_D(t) < \gamma_D(M^*), \tag{2.3}$$

then (1.1) *is nonoscillatory.*

If $M := M_* = M^*$ in Theorem 2.2, then $\gamma_D(M)$ is the critical constant satisfying

$$\gamma_D(M) = \begin{cases} \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1} & \text{if } M = 0, \\ \left(\frac{(M+1)^{\frac{\alpha-1}{\alpha}} - 1}{M} \right)^{\alpha} \frac{M}{(M+1)^{\alpha-1} - 1} & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases}$$

Note that $x \mapsto \gamma_D(x)$ is decreasing on $[0, \infty)$. Thus the critical constant is not invariant with respect to time scales and it may be strictly less than the constant known from the continuous theory. If, in addition, $\alpha = 2$, then

$$\gamma_D(M) = \begin{cases} \frac{1}{4} & \text{if } M = 0, \\ \frac{1}{\left(\sqrt{M+1} + 1\right)^2} & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases}$$

This result matches the one known from the linear theory, see [26] and also [6].

In the next section we prove a complement to Theorem 2.2. For this purpose we introduce the following notation:

$$\mathfrak{R}_C(\lambda)(t) := \frac{\lambda(t) r^{1-\beta}(t)}{R_C(t)}$$

$$\gamma_C(x) := \lim_{t\to x}\left(\frac{1-(1-t)^{\frac{\alpha-1}{\alpha}}}{t}\right)^{\alpha}(1-t), \quad x\in[0,1],$$

and

$$\mathcal{A}_C(t) := R_C^{-1}(t)\int_t^\infty (R_C^\sigma(s))^\alpha p(s)\,\Delta s.$$

For $x\in(0,1)$ the function γ_C can be rewritten as $\gamma_C(x) = \left(\frac{(1-x)^{1/\alpha}-1}{x}+1\right)^\alpha$. Now it is not difficult to see that $x\mapsto\gamma_C(x)$ is decreasing and nonnegative on $[0,1]$. Further we have $\gamma_C(0)=\beta^{-\alpha}$ and $\gamma_C(1)=0$.

3. MAIN RESULTS

We are in a position to state and prove the main result of the paper, namely the complement of Theorem 2.2 in the sense of condition (1.4).

Theorem 3.1. *Let* (1.4) *hold and* $\int_a^\infty p(s)(R_C^\sigma(s))^\alpha\Delta s$ *be convergent.*

(i) Assume $p(t)\ge 0$ *for large* t*. Let* $N_* = \liminf_{t\to\infty}\mathfrak{R}_C(\mu)(t)$*. If*

$$\liminf_{t\to\infty}\mathcal{A}_C(t) > \gamma_C(N_*), \tag{3.1}$$

then (1.1) *is oscillatory.*

(ii) Assume $\int_t^\infty p(s)(R_C^\sigma(s))^\alpha\Delta s\ge 0$ *for large* t*. Let* $N^* = \limsup_{t\to\infty}\mathfrak{R}_C(\mu)(t)$*. If*

$$\limsup_{t\to\infty}\mathcal{A}_C(t) < \gamma_C(N^*), \tag{3.2}$$

then (1.1) *is nonoscillatory.*

Proof. Condition (3.1) can be rewritten as $\int_t^\infty (R_C^\sigma(s))^\alpha p(s)\,\Delta s\ge\gamma_0 R_C(t)$ for large t, say $t\ge t_0$, where $\gamma_0>\gamma_C(N_*)$. Recall that $\{\varphi_k(t)\}$ is defined by (2.1), where $u\mapsto\mathcal{G}(u)$ is increasing for $u\ge -R_C^{1-\alpha}$. Then $\varphi_1(t) = R_C^{-\alpha}(t)\int_t^\infty(R_C^\sigma(s))^\alpha p(s)\,\Delta s + \mathcal{G}(\varphi_0)(t)\ \ge\ R_C^{-\alpha}(t)\gamma_0 R_C(t) - R_C^{1-\alpha}(t) = \gamma_1 R_C^{1-\alpha}(t)$, $t\ge t_0$, where $\gamma_1=\gamma_0-1$. We have $R_C^{\alpha-1}\gamma_1 R_C^{1-\alpha}=\gamma_1>-1$ and $R_C^{\alpha-1}\varphi_1(t)\ge\gamma_1>-1$. Hence, $\varphi_2(t) = R_C^{-\alpha}(t)\int_t^\infty(R_C^\sigma(s))^\alpha p(s)\,\Delta s+\mathcal{G}(\varphi_1)(t)\ \ge\ \gamma_0 R_C^{1-\alpha}(t)+\mathcal{G}(\gamma_1 R_C^{1-\alpha})(t) = \gamma_0 R_C^{1-\alpha}(t) + R_C^{-\alpha}(t)\int_t^\infty\mathcal{T}(\gamma_1 R_C^{1-\alpha})(s)\,\Delta s$, $t\ge t_0$, where $\mathcal{T}(u) = -u(R_C^\alpha)^\Delta + S(u,r)(R_C^\sigma)^\alpha$. We distinguish two cases. For convenience we skip the argument t sometimes in the computations. At a right-dense t, $\mathcal{T}(\gamma_1 R_C^{1-\alpha}) = \alpha r^{1-\beta}\gamma_1 R_C^{1-\alpha}R_C^{\alpha-1} + (\alpha-1)r^{1-\beta}|\gamma_1 R_C^{1-\alpha}|^\beta R_C^\alpha =$

$r^{1-\beta}(\alpha\gamma_1 + (\alpha-1)|\gamma_1|^\beta)$, while at a right-scattered t,

$$\begin{aligned}\mathcal{T}(\gamma_1 R_C^{1-\alpha}) &= -\gamma_1 R_C^{1-\alpha}\frac{(R_C^\sigma)^\alpha - R_C^\alpha}{\mu} \\ &\quad + \frac{\gamma_1 R_C^{1-\alpha}(R_C^\sigma)^\alpha}{\mu}\left(1 - \frac{r}{\left(r^{\beta-1} + \mu\Phi^{-1}(\gamma_1 R_C^{1-\alpha})\right)^{\alpha-1}}\right) \\ &= r^{1-\beta}\left(\frac{\gamma_1 R_C}{\mu r^{1-\beta}} - \frac{r\gamma_1 R_C^{1-\alpha}(R_C + \mu R_C^\Delta)^\alpha}{\mu r^{1-\beta}\left(r^{\beta-1} + \mu R_C^{-1}\Phi^{-1}(\gamma_1)\right)^{\alpha-1}}\right) \\ &= r^{1-\beta}\left(\frac{\gamma_1}{\mathfrak{R}_C(\mu)} - \frac{\gamma_1(1-\mathfrak{R}_C(\mu))^\alpha}{\mathfrak{R}_C(\mu)(1+\Phi^{-1}(\gamma_1)\mathfrak{R}_C(\mu))^{\alpha-1}}\right).\end{aligned}$$

Denote

$$\Gamma_*(t,u) = \inf_{s\ge t}\lim_{\lambda\to\mu(s)}\left\{\frac{u}{\mathfrak{R}_C(\lambda)(s)} - \frac{u(1-\mathfrak{R}_C(\lambda)(s))^\alpha}{\mathfrak{R}_C(\lambda)(s)(1+\Phi^{-1}(u)\mathfrak{R}_C(\lambda)(s))^{\alpha-1}}\right\}$$

with noting that in this formula we have $\lim_{\lambda\to 0}\{\ \} = \alpha u + (\alpha-1)|u|^\beta$, which corresponds to the "right-dense" case. Thus, at any $t \ge t_0$, we get

$$\varphi_2(t) \ge \gamma_0 R_C^{1-\alpha}(t) + R_C^{-\alpha}(t)\Gamma_*(t_0,\gamma_1)\int_t^\infty r^{1-\beta}(s)\,\Delta s = \gamma_2 R_C^{1-\alpha}(t),$$

where $\gamma_2 = \gamma_0 + \Gamma_*(t_0,\gamma_1)$. Further note that $u \mapsto \Gamma_*(t,u)$ is increasing for $u \ge -1$ which can be easily verified by differentiation, and $\Gamma_*(t,-1) = -1$. Hence, $\gamma_2 \ge \gamma_0 + \Gamma_*(t_0,-1) = \gamma_0 - 1 = \gamma_1$. Similarly, by induction, $\varphi_k(t) \ge \gamma_k R_C^{1-\alpha}(t)$, where

$$\gamma_{k+1} = \gamma_0 + \Gamma_*(t_0,\gamma_k), \quad k = 1,2,\ldots, \tag{3.3}$$

and $\gamma_{k+1} \ge \gamma_k$, $k = 1,2,\ldots$. We claim that $\lim_{k\to\infty}\gamma_k = \infty$. If not, let $\lim_{k\to\infty}\gamma_k = L < \infty$ (a limit must exist). Then from (3.3)

$$L = \gamma_0 + \Gamma_*(t_0,L). \tag{3.4}$$

First assume that $N := N_* = N^*$. Letting t_0 tend to ∞ in (3.4), we get

$$L = \gamma_0 + \Omega(L,N), \tag{3.5}$$

where

$$\Omega(L,N) = \lim_{\lambda\to N}\left\{\frac{L}{\lambda} - \frac{L(1-\lambda)^\alpha}{\lambda(1+\Phi^{-1}(L)\lambda)^{\alpha-1}}\right\}. \tag{3.6}$$

Note that $\Omega(L,N) = \lim_{t_0\to\infty}\Gamma_*(t_0,L)$. Recall that $N \in [0,1]$ and it is easy to see that $\Omega(L,0) = \alpha L + (\alpha-1)|L|^\beta$ and $\Omega(L,1) = L$. We have

$$\frac{\partial\Omega(L,N)}{\partial L} = \lim_{\lambda\to N}\left\{\frac{1}{\lambda} - \frac{(1-\lambda)^\alpha}{\lambda(1+\Phi^{-1}(L)\lambda)^\alpha}\right\}.$$

The case with $N = 1$ is immediate since then from (3.5), $L = \gamma_0 + L$, a contradiction. Thus next we may assume $N \in [0,1)$. It is a matter of routine computations to see that the graph of $L \mapsto \gamma_C(N) + \Omega(L,N)$ is a parabola-like curve and touches the line $L \mapsto L$ at $L^\# = -\lim_{t\to N}\left((1-(1-t)^{(\alpha-1)/\alpha})/t\right)^{\alpha-1}$; the values of $L^\#$ and $\gamma_C(N)$

being found by solving $\partial\Omega(L,N)/\partial L = 1$ and $L^{\#} = \gamma + \Omega(L^{\#}, N)$ with respect to L and γ, respectively. Since in (3.5) we have $\gamma_0 > \gamma_C(N)$, there is no real solution of that equation, and we get a contradiction. This proves that $\lim_{k\to\infty}\gamma_k = \infty$ and so $\varphi_k(t) \to \infty$ as $k \to \infty$ for $t \geq t_0$. Consequently, (1.1) is oscillatory by Lemma 2.1. Now we analyse the case when $N_* < N^*$. A closer examination of the previous part shows that we are particularly interested in $L \in [-1, 0]$. It is not difficult to verify that $x \mapsto \Omega(L, x)$ is increasing on $[0, 1]$ for $L \in [-1, 0]$. Hence, letting t_0 tend to ∞ in (3.4) we have $L = \gamma_0 + \Omega(L, N_*)$. Since we assume $\gamma_0 > \gamma_C(N_*)$, we get that the last equation has no real solution, similarly as in the previous part. Thus oscillation of (1.1) follows from Lemma 2.1.

(ii) Condition (3.2) can be rewritten as $\int_t^\infty (R_C^\sigma(s))^\alpha p(s)\,\Delta s \leq \delta_0 R_C(t)$ for large t, say $t \geq t_0$, where $0 < \delta_0 < \gamma_C(N^*)$. Similarly as in part (i), we get

$$\varphi_k(t) \leq \delta_k R_C^{1-\alpha}(t), \tag{3.7}$$

where $\delta_1 = \delta_0 - 1$ and

$$\delta_{k+1} = \delta_0 + \Gamma^*(t_0, \delta_k), \tag{3.8}$$

$k = 1, 2, \ldots$, with

$$\Gamma^*(t,u) = \sup_{s\geq t} \lim_{\lambda\to\mu(s)} \left\{ \frac{u}{\mathfrak{R}_C(\lambda)(s)} - \frac{u(1-\mathfrak{R}_C(\lambda)(s))^\alpha}{\mathfrak{R}_C(\lambda)(s)(1+\Phi^{-1}(u)\mathfrak{R}_C(\lambda)(s))^{\alpha-1}} \right\}.$$

Thanks to monotone properties of $u \mapsto \Gamma^*(t,u)$, we have $\delta_{k+1} \geq \delta_k > -1$, $k = 1, 2, \ldots$. We need to show that $\{\delta_k\}$ converges. First assume $N := N_* = N^*$. Consider the fixed point problem $x = g(\omega, x)$, where $g(\omega, x) = \omega + \Omega(x, N)$, Ω being defined by (3.6), with a real parameter ω, and the perturbed problem $x = \tilde{g}(\omega, x, t_0)$, where $\tilde{g}(\omega, x, t_0) = \omega + \Gamma^*(t_0, x)$. The fixed point will be found by means of the iteration scheme $x_{k+1} = g(x_k)$, $k = 1, 2, \ldots$. The graph of $x \mapsto g(0, x)$ is a parabola-like curve with the minimum at the point $[-1, -1]$. The graph of $x \mapsto g(\gamma_C(N), x)$ touches the line $x \mapsto x$ at $x = x^{\#} := -\lim_{t\to N}\left((1-(1-t)^{(\alpha-1)/\alpha})/t\right)^{\alpha-1}$. Therefore, if we choose $x_1 = -1 + \gamma_C(N)$ (noting that $-1 < x_1 < x^{\#}$), then, as easily seen, the approximating sequence $\{x_k\}$ for the problem $x = g(\gamma_C(N), x)$ is strictly increasing, and converges to $x^{\#}$. Clearly, if $-1 < y_1 < -1 + \gamma_C(N)$, then the approximating sequence for the same problem, i.e., satisfying $y_{k+1} = g(\gamma_C(N), y_k)$, $k = 1, 2, \ldots$, is increasing as well and permits $y_k < x_k < x^{\#}$, $k \in \mathbb{N}$. Therefore $\{y_k\}$ converges. Now take into account that $\lim_{t_0\to\infty}\tilde{g}(\omega, x, t_0) = g(\omega, x)$. Hence the function $\tilde{g}$ in the perturbed problem can be made as close to g as we need (locally, on the interval under consideration) provided t_0 is sufficiently large. This closeness of g to $\tilde{g}$ along with the inequality $\delta_0 < \gamma_C(N)$ and the behavior of $\{y_k\}$ lead to the fact that the sequence $\{\delta_k\}$ in the original problem (3.8) converges for t_0 large. Thus $\{\varphi_k(t)\}$ converges by (3.8), and so (1.1) is nonoscillatory by Lemma 2.1. The case when $N_* < N^*$ can be treated similarly, taking into account also the ideas of the last part of (i). In particular, we take $0 < \delta_0 < \gamma_C(N^*)$. □

Remark 3.2. (i) A closer examination of the proof of (ii) of Theorem 3.1 shows that under the condition $\mathfrak{R}_C(t) \equiv N$ for large t, assumption (3.2) can be replaced by $\mathcal{A}_C(t) \leq \gamma_C(N)$ for large t. Indeed in that proof we then take δ_0 such that $0 < \delta_0 \leq \gamma_C(N)$. The statement then follows from the fact that $\tilde{g}(\omega, x, t) \equiv g(\omega, x)$ and the graph of $x \mapsto g(\gamma_C(N), x)$ touches the line $x \mapsto x$. This result will be very important to show nonoscillation in some critical cases, as presented below. We will provide also several important examples of $\mathbb{T}$ and r where $\mathfrak{R}_C(t) \equiv N$ is satisfied.

(ii) In the nonoscillation criterion, we assume $\int_t^\infty p(s)(R_C^\sigma(s))^\alpha \Delta s \geq 0$ for large t. We conjecture that the condition $p(t) \geq 0$ for large t can be relaxed to this inequality also in the oscillation criterion.

Corollary 3.3. *Let $N := N_* = N^*$ in Theorem 3.1. Then $\gamma_C(N)$ is the critical constant, i.e., the constants on the right-hand sides of (3.1) and (3.2) are the same. In particular, $N \in [0, 1]$ and*

$$\gamma_C(N) = \begin{cases} \beta^{-\alpha} & \text{if } N = 0, \\ (1-N)\left(\frac{1-(1-N)^{\frac{\alpha-1}{\alpha}}}{N}\right)^\alpha & \text{if } 0 < N < 1, \\ 0 & \text{if } N = 1. \end{cases}$$

Remark 3.4. In view of monotone properties of $x \mapsto \gamma_C(x)$ on $[0, 1]$, we have

$$(1-N)\left(\frac{1-(1-N)^{\frac{\alpha-1}{\alpha}}}{N}\right)^\alpha < \beta^{-\alpha} \tag{3.9}$$

for $N \in (0, 1]$. Therefore we see that the critical constant is not invariant with respect to time scales and it may be strictly less than the constant known from the continuous theory.

Corollary 3.5. *Let $N := N_* = N^*$ and $\alpha = 2$ in Theorem 3.1. Then $\gamma_C(N)$ is the critical constant satisfying*

$$\gamma_C(N) = \begin{cases} \frac{1}{4} & \text{if } N = 0, \\ \frac{1-N}{\left(\sqrt{1-N}+1\right)^2} & \text{if } 0 < N < 1, \\ 0 & \text{if } N = 1. \end{cases}$$

Remark 3.6. Using Theorem 2.2 with $\alpha = 2$ and a transformation of dependent variable, in [26, Theorem 3.3] we showed the linear version of Theorem 3.1 (i.e., all is considered with $\alpha = 2$), where the expression $\mathfrak{R}_C(\lambda)(t)$ is replaced by the expression $\tilde{\mathfrak{R}}_C(\lambda)(t) = \lambda(t)r^{-1}(t)/R_C^\sigma(t)$ and the function $\gamma_C(x)$ is replaced by $\tilde{\gamma}_C(x) = \lim_{t\to x}\left(\sqrt{t+1}+1\right)^{-2}$. We have $\mathfrak{R}_C(\mu) = 1/\left(1 + 1/\tilde{\mathfrak{R}}_C(\mu)\right)$ for $\mu > 0$ and

$\gamma_C \circ \vartheta = \tilde{\gamma}_C$ with $\vartheta(x) = 1/(1+1/x)$. From this it follows that Theorem 3.1 reduces to [26, Theorem 3.3] for $\alpha = 2$. In particular, if there exists $\tilde{N} = \lim_{t\to\infty} \tilde{\mathfrak{R}}_C(\mu)(t)$, then $\tilde{N} \in [0,\infty) \cup \{\infty\}$, $N = \lim_{t\to\infty} \mathfrak{R}_C(\mu)(t)$ exists, and $\gamma_C(N) = \tilde{\gamma}_C(\tilde{N})$. Note that also for our general case, i.e., $\alpha > 1$, the critical constant $\gamma(N)$ can be expressed in terms of $\tilde{\mathfrak{R}}_C$, $\tilde{N}$ and $\tilde{\gamma}$, and reads as $\tilde{\gamma}_C(\tilde{N}) = \lim_{t\to\tilde{N}} \left(\left((t+1)^{\frac{\alpha-1}{\alpha}} - 1\right)/t\right)^{\alpha} = (\gamma_C \circ \vartheta)(\tilde{N}) = \gamma_C(N)$, where $\tilde{N} = \lim_{t\to\infty} \tilde{\mathfrak{R}}_C(\lambda)(t)$ with $\tilde{\mathfrak{R}}_C(\lambda)(t) = \lambda(t) r^{1-\beta}(t)/R_C^{\sigma}(t)$.

4. APPLICATIONS, EXAMPLES, AND CONCLUDING REMARKS

(i) *Critical and oscillation constant.* If the constants on the right-hand sides of (3.1) and (3.2) are the same and equal to $\gamma_C(N)$ (in our case this happens when $N = N_* = N^*$), then $\gamma_C(N)$ is called a *critical constant* (or a *critical oscillation constant*). Sometimes in similar situations in the literature, this constant is said to be an *oscillation constant*. However, we prefer to use the former terminology since the latter one is usually used in a different meaning and concerns a conditional oscillation, see the next subsection. The term "critical constant" reflects the fact that this constant cannot be improved and forms a sharp border between oscillation and nonoscillation. Note that the strict inequalities in (3.1) and (3.2) cannot be replaced by nonstrict ones since no conclusion can be drawn if either $\liminf_{t\to\infty} \mathcal{A}_C(t)$ or $\limsup_{t\to\infty} \mathcal{A}_C(t)$ equals the critical constant; both oscillation and nonoscillation may happen, as shown already in the continuous linear theory. On the other hand, under additional conditions, the equality is "closer" to nonoscillation, see Remark 3.2. Our result also show that if $\liminf_{t\to\infty} \mathcal{A}_C(t) > \beta^{-\alpha}$, then (1.1) is oscillatory no matter what time scale is, because of (3.9). However, in addition, Theorem 3.1 says that $\beta^{-\alpha}$ is not the best possible constant which is universal for all time scales. In particular, it may not be critical at all. In fact, it depends on a time scale and with $\mu(t) \not\equiv 0$ it depends also on the coefficient r. In general it may happen $\gamma_C(N) \in [0, \beta^{-\alpha}]$; later we give examples where $\gamma_C(N) < \beta^{-\alpha}$. We conclude this subsection with the note that oscillation of (1.1) is still possible even when $\liminf_{t\to\infty} \mathcal{A}_C(t) < \gamma_C(N)$. This is implied the following theorem, and we emphasize that there is no need of an additional condition on a time scale.

Theorem 4.1 ([19])**.** *Let* $p(t) \geq 0$ *for large* t *and* (1.4) *hold. If* $\limsup_{t\to\infty} \mathcal{A}_C(t) > 1$, *then* (1.1) *is oscillatory.*

(ii) *Strong and conditional oscillation.* Consider the equation

$$(r(t)\Phi(y^{\Delta}))^{\Delta} + \lambda p(t)\Phi(y^{\sigma}) = 0, \tag{4.1}$$

where $r(t) > 0$, $p(t) > 0$, and λ is a real parameter. In the linear continuous case, the concept of strong and conditional oscillation was introduced by Nehari [22]. We say that (4.1) is *conditionally oscillatory* if there exists a constant $0 < \lambda_0 < \infty$

such that (4.1) is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The value λ_0 is called the *oscillation constant* of (4.1). If equation (4.1) is oscillatory (resp. nonoscillatory) for every $\lambda > 0$, then this equation is said to be *strongly oscillatory* (resp. *strongly nonoscillatory*). Next we apply the results from the previous section to derive necessary and sufficient conditions for strong (non)oscillation in the case when (1.4) hold.

Theorem 4.2. *Let $p(t) \geq 0$ for large t and* (1.4) *hold. Assume that $N^* < 1$. Then* (4.1) *is strongly oscillatory if and only if $\limsup_{t\to\infty} \mathcal{A}_C(t) = \infty$, and it is strongly nonoscillatory if and only if $\lim_{t\to\infty} \mathcal{A}_C(t) = 0$.*

Proof. If $\limsup_{t\to\infty} \mathcal{A}_C(t) = \infty$, then $\limsup_{t\to\infty} R_C^{-1}(t) \int_t^\infty (R_C^\sigma(s))^\alpha \lambda p(s)\,\Delta s > 1$ for every $\lambda > 0$, and so (4.1) is oscillatory for every $\lambda > 0$ by Theorem 4.1. Conversely, if (4.1) is strongly oscillatory, then

$$\limsup_{t\to\infty} R_C^{-1}(t) \int_t^\infty (R_C^\sigma(s))^\alpha \lambda p(s)\,\Delta s \geq \gamma_C(N^*) > 0 \tag{4.2}$$

for every $\lambda > 0$ by Theorem 3.1. This implies $\limsup_{t\to\infty} \mathcal{A}_C(t) = \infty$, otherwise (4.2) would be violated for sufficiently small λ. The proof of the part concerning strong nonoscillation is based on similar arguments. The details are left to the reader. □

One could ask whether the condition $N^* < 1$ in the last theorem may be dropped. In general, the answer is no. Realize that strong oscillation [strong nonoscillation] of (4.1) is nothing but $\lambda_0 = 0$ [$\lambda_0 = \infty$], where λ_0 is the oscillation constant. Now assume that $N^* = 1 = N_*$ and $\lim_{t\to\infty} \mathcal{A}_C(t) = L \in (0, \infty)$ exists. Then $\gamma_C(N_*) = 0$ and $\lim_{t\to\infty} R_C^{-1}(t) \int_t^\infty (R_C^\sigma(s))^\alpha \lambda p(s)\,\Delta s = \lambda L > 0$ for every $\lambda > 0$. This implies strong oscillation of (4.1), however the condition $\limsup_{t\to\infty} \mathcal{A}_C(t) = \infty$ does not hold. A particular example of such strongly oscillatory equation will be given later.

(iii) *Euler type equation.* Consider the equation

$$(r(t)\Phi(y^\Delta))^\Delta + \frac{\lambda r^{1-\beta}(t)}{(R_C^\sigma(t))^\alpha}\Phi(y^\sigma) = 0, \tag{4.3}$$

where λ is a positive parameter. We are interested only in positive λ's since for $\lambda = 0$, equation (4.3) is readily explicitly solvable, it is nonoscillatory, and thus for $\lambda < 0$ is nonoscillatory as well by the Sturm type comparison theorem. Equation is called of *Euler type* since for $\alpha = 2$, $\mathbb{T} = \mathbb{R}$, and after certain transformations it yields the well known Euler linear differential equation. Applying Theorem 3.1 we get that (4.3) is oscillatory provided $\lambda > \gamma_C(N_*)$ and nonoscillatory provided $\lambda < \gamma_C(N^*)$. If $N := N_* = N^*$, then $\gamma_C(N)$ is the critical constant and $\lambda_0 = \gamma_C(N)$ is the oscillation constant for (4.3), i.e., (4.3) is oscillatory provided $\lambda > \gamma_C(N)$ and it is nonoscillatory provided $\lambda < \gamma_C(N)$. In view of Remark 3.2, if $\mathfrak{R}_C(t) \equiv N$ eventually, then (4.3) is nonoscillatory also for $\lambda = \gamma_C(N)$. As shown below, the condition $\mathfrak{R}_C(t) \equiv N$ is satisfied in many important cases, e.g., $\mathbb{T}=\mathbb{R}$, or $\mathbb{T}=h\mathbb{Z}$ with $h>0$ and $r(t)=\vartheta^k$ with $\vartheta>1$, or $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and $r(t)=t^\delta$ with $\delta>\alpha-1$, or $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and $r(t)=\xi^{\log_q(1/t)}$ with $0<\xi<q^{1-\alpha}$. Finally recall that the Euler type equation corresponding to the case (1.3) takes the form

$$(r(t)\Phi(y^\Delta))^\Delta+\lambda(-R_D^{1-\alpha}(t))^\Delta\Phi(y^\sigma)=0 \tag{4.4}$$

with the oscillation constant $\lambda_0=\gamma_D(M)$, where $M=\lim_{t\to\infty}\mathfrak{R}_D(\mu)(t)$.

(iv) *Nonstandard behavior.* Let $\mathbb{T}=2^{\delta^{\mathbb{N}_0}}:=\{2^{\delta^k}:k\in\mathbb{N}_0\}$ with $\delta>1$. Then $\sigma(t)=t^\delta$ and $\mu(t)=t^\delta-t$. Assume $r(t)=(t\sigma(t))^{\alpha-1}$. Then $R_C(t)=1/t$ and $\mathfrak{R}_C(t)=(t^\delta-t)/t^\delta\to 1$ as $t\to\infty$. Thus $N_*=N^*=N=1$, and so $\gamma_C(N)=0$. Applying Theorem 3.1 we get that (4.3) is oscillatory for all $\lambda>0$, i.e., it is strongly oscillatory, in spite of the fact that $\limsup_{t\to\infty}\mathcal{A}_C(t)=\infty$ does not hold.

(v) *Kneser type criteria.* An application of the Sturm type theorem [23, Theorem 3], where (1.1) and (4.3) are compared, yields the following theorem (where we do not need to assume any sign restriction on p and $\int^\infty(R_C^\sigma(s))^\alpha p(s)\,\Delta s$ being convergent.)

Theorem 4.3. *Let the limit $N=\lim_{t\to\infty}\mathfrak{R}_C(t)$ exist. If*

$$\liminf_{t\to\infty} r^{\beta-1}(t)(R_C^\sigma(t))^\alpha p(t)>\gamma_C(N),$$

then (1.1) *is oscillatory. If*

$$\limsup_{t\to\infty} r^{\beta-1}(t)(R_C^\sigma(t))^\alpha p(t)<\gamma_C(N),$$

then (1.1) *is nonoscillatory.*

Clearly, $\gamma_C(N)$ is again non-improvable. A slight modification of this approach yields Kneser type criteria in the case when $N_*<N^*$. Similarly as the Sturm type theorem is applied to obtain Kneser type criteria, the Hille-Wintner comparison theorem ([19, Theorem 7]) can be used to obtain (back) Hille-Nehari type criteria provided we know the behavior of (4.3). In contrast to the Sturm type theorem where the coefficients are compared pointwise, the Hille-Wintner type theorem is based on an integral comparison. Thus this is an alternative approach how to prove Corollary 3.3 or Theorem 3.1. However, such an approach has a considerable disadvantage: It requires the knowledge of the oscillation constant of equation (4.3). But starting with equation (4.3), the oscillation constant can be acquired only in very special cases. In general, it is not known how to get it. On the other hand, the approach described in Section 3 does not require a knowledge of such constant; what's more, our Theorem 3.1 factually provides it. Recall that in [15], see also [9], the proof of Hille-Nehari criteria under conditions (1.4) and $\mathbb{T}=\mathbb{R}$ is based just on a knowledge of oscillation

behavior of certain generalized Euler differential equation, Hille-Wintner type comparison theorem, and a transformation of independent variable, which however is not at disposal in "non-continuous" ($\mathbb{T} \neq \mathbb{R}$) cases.

(vi) *Examples from q-calculus.* Let $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$. Then $\sigma(t) = qt$, $\mu(t) = (q-1)t$, and f^Δ reduces to the Jackson type derivative $D_q f$. We will compute the values of the critical constants for two different coefficients $r(t)$ to obtain important examples of conditionally oscillatory Euler type q-difference equations of the form

$$D_q(r(t)\Phi(D_q y(t))) + \frac{\lambda r^{1-\beta}(t)}{(R_C(qt))^\alpha}\Phi(y(qt)) = 0 \tag{4.5}$$

with the known oscillation constant that can be further used for comparison purposes. First assume that $r(t) = t^\delta$ with $\delta \in \mathbb{R}$, $\delta > \alpha - 1$. Then, with $t = q^n$, $n \in \mathbb{N}_0$, we have

$$\begin{aligned} N = \lim_{t\to\infty} \mathfrak{R}_C(t) &= \lim_{t\to\infty} \frac{(q-1)q^n((q^n)^\delta)^{1-\beta}}{\sum_{j=n}^\infty \mu(q^j)((q^j)^\delta)^{1-\beta}} \\ &= \lim_{t\to\infty} \frac{q^{n(1+\delta(1-\beta))}}{\sum_{j=n}^\infty q^{j(1+\delta(1-\beta))}} = \lim_{t\to\infty} \frac{q^{n(1+\delta(1-\beta))}(q^{1+\delta(1-\beta)}-1)}{-q^{n(1+\delta(1-\beta))}} \\ &= 1 - q^{1+\delta(1-\beta)}. \end{aligned}$$

We have used the q-L'Hospital rule. Alternatively we can immediately sum the geometric series $\sum_{j=n}^\infty q^{j(1+\delta(1-\beta))}$, which then proves that $\mathfrak{R}_C(t) \equiv 1 - q^{1+\delta(1-\beta)}$. Thus the associated oscillation (and critical) constant to equation (4.5) is equal to $\gamma_C(1 - q^{1+\delta(1-\beta)}) \in (0, \beta^{-\alpha})$. Moreover, in view of Remark 3.2, equation (4.5) is nonoscillatory provided $\lambda = \gamma_C(1 - q^{1+\delta(1-\beta)})$. Note that the complementary case $\delta \le \alpha - 1$ corresponds to condition (1.3) (and equation (4.4)) and was discussed in [27]. Now assume that $r(t) = \xi^{\log_q(1/t)}$ with $0 < \xi < q^{1-\alpha}$. Then, with $t = q^n$, $n \in \mathbb{N}_0$, we have $r(t) = \xi^{-n}$. Applying similar arguments as above, we get $N = 1 - q\xi^{\beta-1} \equiv \mathfrak{R}_C(t)$. So the associated oscillation (and critical) constant to equation (4.5) is equal to $\gamma_C(1 - q\xi^{\beta-1})$ and (4.5) is nonoscillatory provided λ is equal to the oscillation constant. The complementary case $\xi \ge q^{1-\alpha}$ corresponds to condition (1.3) (and equation (4.4)) and was discussed in [27]. Observe how the "limits" as $q \to 1$ in these results correspond to the continuous counterparts. If we set $\alpha = 2$, then as a special case we get observations presented in [6]; the results in that paper are based on explicit solving of a linear Euler type q-difference equation. Of course, that approach cannot be used in a general case.

(vii) *Example from h-calculus.* Let $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ with $h > 0$. Then $\sigma(t) = t + h$, $\mu(t) \equiv h$, and the delta-derivative reduces to the usual forward h-difference operator Δ_h. Assume $r(t) = \vartheta^t$ with $\vartheta > 1$. Then

$$\mathfrak{R}_C(t) = \frac{h\vartheta^{t(1-\beta)}}{h\sum_{j=\frac{t}{h}}^\infty \vartheta^{jh(1-\beta)}} \equiv 1 - \vartheta^{h(1-\beta)},$$

using the fact that the series is geometric. Now it is easy to see that the Euler type h-difference equation

$$\Delta_h(r(t)\Phi(\Delta_h y(t))) + \frac{\lambda r^{1-\beta}(t)}{(R_C(t+h))^\alpha}\Phi(y(t+h)) = 0$$

is oscillatory for $\lambda > \gamma_C(1-\vartheta^{h(1-\beta)})$ and nonoscillatory for $\lambda \le \gamma_C(1-\vartheta^{h(1-\beta)})$, in view of Theorem 3.1 and Remark 3.2. Observe how the "limit" as $h \to 0$ in this result corresponds to the continuous counterpart.

(viii) *Inequalities.* Our result can be utilized also in the theory of inequalities: We will derive an inequality which can be understood as a variant of the Hardy type inequality with weight functions involving the best possible constant. We have seen that equation (4.3) is nonoscillatory provided $\lambda < \gamma_C(N)$ with $N = \lim_{t\to\infty}\mathfrak{R}_C(t)$. Using now the variational principle established in [23], we conclude that there is $a \in \mathbb{T}$ such that

$$\int_a^\infty r(t)|\xi^\Delta(t)|^\alpha \Delta t > \lambda \int_a^\infty \frac{r^{1-\beta}(t)}{(R_C^\sigma(t))^\alpha}|\xi^\sigma(t)|^\alpha \Delta t, \tag{4.6}$$

where $\lambda < \gamma_C(N)$, for every nontrivial ξ, where ξ is a piecewise rd-continuously delta-differentiable function on $[a,\infty)$, and there is $b \in (a,\infty)$ with $\xi(t) = 0$ if $t \notin (a,b)$. The constant $\gamma_C(N)$ is the best possible. If $\mathfrak{R}_C(t) \equiv N$, then λ in (4.6) can be replaced by $\gamma_C(N)$. Note that inequality (4.6) can be viewed also as a generalization of the Wirtinger type inequality, see, e.g., [9, Chapters 2.1.2 and 9.5.1] and [14]. Using again the variational principle (but this time its opposite implication), simple inequalities involving integrals and absolute values, and our new result concerning equation (4.3), we can also show the following statement: If the Hardy type inequality

$$\int_a^\infty r(t)f^\alpha(t)\,\Delta t > \gamma_C(N)\int_a^\infty \frac{r^{1-\beta}(t)}{(R_C^\sigma(t))^\alpha}\left(\int_a^{\sigma(t)} f(s)\,\Delta s\right)^\alpha \Delta t \tag{4.7}$$

holds for all nonnegative nontrivial rd-continuous f (thus here we have a different boundary conditions with respect to those in (4.6)), then $\gamma_C(N)$ is the best possible constant, cf. [24]. Indeed, inequality (4.7) where $\gamma_C(N)$ is replaced by a bigger constant λ would imply nonoscillation of (4.3), which however is oscillatory for such λ.

(ix) *Conclusions and future directions.* The observations presented in this paper solve the open problem posed in [27]: To establish a generalized sharp Hille-Nehari criteria for equation (1.1) under condition (1.4). Our new results along with the ones from [27] provide somehow complete discussion on Hille-Nehari type criteria for (1.1). The theorems are new even in some special well-studied cases, and in view of the presence of the critical constant, they are non-improvable in a certain sense. The proofs utilize a new method developed recently in [19]. A demand after a new approach arose out because of the failing of the approaches known from the linear or the continuous theories.

We remark that the results in the above applications and examples may not be obtained by any existing criteria, as far as the author is aware.

There are several delicate open problems that are immediately related to our results: First, how about oscillation or nonoscillation of (1.1) when $N_* < N^*$, i.e., $\gamma_C(N^*) < \gamma_C(N_*)$, and the limit values of the expression $\mathcal{A}_C(t)$ as $t \to \infty$ remains between $\gamma_C(N^*)$ and $\gamma_C(N_*)$? Such a situation is not very common in applications but generally it may occur. Second, does the condition $\mathcal{A}_C(t) \le \gamma_C(N)$ for large t imply nonoscillation of (1.1) also in the case when $N \not\equiv \mathfrak{R}_C(\mu)(t)$ for large t, where $N = \lim_{t\to\infty} \mathfrak{R}_C(\mu)(t)$ (cf. Remark 3.2)? Third, is it possible to relax sign condition on the coefficient $p(t)$ in Theorem 3.1? Finally, except of some very special cases, it is practically impossible to find an exact solution to Euler type equation (4.3). But having information about its oscillation constant, there could be a chance to find at least a solution of the related inequality $(r(t)\Phi(y^\Delta))^\Delta + \lambda r^{1-\beta}(t)(R_C^\sigma(t))^{-\alpha}\Phi(y^\sigma) \le 0$. Why is this useful when we already know the oscillation constant? If we find an eventually positive solution to this inequality with λ equaling the oscillation constant, then original equation (4.3) is nonoscillatory, see [23], and thus the above described open problem with $N \not\equiv \mathfrak{R}_C(\mu)(t)$ could be solved, in view of Hille-Wintner type comparison theorem. Moreover, if we know that such a solution is positive on $[a, \infty)$, then we have obtained a concrete value of the lower limit a of integration in Hardy inequality (4.6).

As another future direction which can be pursued in the context of our research is finding suitable modifications of the function sequence technique along with applications. For instance, we believe that following the general ideas of the proof of Theorem 3.1, but with a modified function sequence, may lead to (non-improvable) generalized criteria of Willett type, see e.g. [9].

5. ACKNOWLEDGMENT

The author was supported by the Grants KJB100190701 of the Grant Agency of ASCR and 201/07/0145 of the Czech Grant Agency, and by the Institutional Research Plan AV0Z010190503.

REFERENCES

[1] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete Oscillation Theory*, Hindawi, 2005.

[2] R. P. Agarwal, M. Bohner, P. Řehák, Half-linear dynamic equations on time scales: IVP and oscillatory properties, In *Nonlinear Analysis and Applications: To V. Lakshmikantham on his*

80th Birthday, R. P. Agarwal, D. O'Regan editors, Dordrecht: Kluwer Academic Publishers, 2003, Vol. I, 1–56.

[3] J. Baoguo, L. Erbe, A. Peterson, New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, *Comput. Math. Appl.*, 56:2744–2756, 2008.

[4] M. Bohner, A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[5] M. Bohner and S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, *Rocky Mountain J. Math.*, 34:1239–1254, 2004.

[6] M. Bohner, M. Ünal, Kneser's theorem in q-calculus, *J. Phys. A: Math. Gen.*, 38:6729–6739, 2005.

[7] O. Došlý, Oscillation criteria for half-linear second order differential equations, *Hiroshima Math. J.*, 28:507–521, 1998.

[8] O. Došlý, P. Řehák, Nonoscillation criteria for second order half-linear difference equations, *Comput. Math. Appl.*, 42:453–464, 2001.

[9] O. Došlý, P. Řehák, *Half-linear Differential Equations*, Elsevier, North Holland, 2005.

[10] Á. Elbert, A half-linear second order differential equation, *Colloq. Math. Soc. János Bolyai*, 30:158–180, 1979.

[11] L. H. Erbe, S. Hilger, Sturmian theory on measure chains, *Diff. Equ. Dynam. Syst.*, 1:223–244, 1993.

[12] L. H. Erbe, A. C. Peterson, P. Řehák, Integral comparison theorems for second order linear dynamic equations, In Difference equations, special functions and orthogonal polynomials, Eds. S. Elaydi, J Cushing, R. Lasser, V. Papageorgiou, A. Ruffing, W. Van Assche, Munich, Germany, 25-30 July 2005. London, World Scientific, 2007, 497–506.

[13] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, *Res. Math.* 18:18–56, 1990.

[14] R. Hilscher, A time scales version of a Wirtinger type inequality and applications, *J. Comput. Appl. Math.*, 141:219–226, 2002.

[15] H. Hoshino, R. Imabayashi, T. Kusano, T. Tanigawa, On second-order half-linear oscillations, *Adv. Math. Sci. Appl.*, 8:199–216, 1998.

[16] N. Kandelaki, A. Lomtatidze, D. Ugulava, On the oscillation and nonoscillation of a second order half-linear differential equation, *Georgian Math. J.*, 7:329–346, 2000.

[17] T. Kusano, Y. Naito, A. Ogata, Strong oscillation and nonoscillation of quasilinear differential equations of second order, *Differential Equations Dynam. Systems*, 2:1–10, 1994.

[18] H. J. Li, C. C. Yeh, Existence of positive nondecreasing solutions of nonlinear difference equations, *Nonl. Anal. TMA*, 22:1271–1284, 1994.

[19] S. Matucci, P. Řehák, Nonoscillation of half-linear dynamic equations, submitted

[20] A. B. Mingarelli, *Volterra–Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, Lecture Notes in Mathematics, Vol. 989, Springer-Verlag, Berlin – Heidelberg – New York – Tokyo, 1983.

[21] J. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, *J. Math. Anal. Appl.*, 53:418–425, 1976.

[22] Z. Nehari, Oscillation criteria for second-order linear differential equations, *Trans. AMS*, 85:428–445, 1957.

[23] P. Řehák, Half-linear dynamic equations on time scales: IVP and oscillatory properties, *Nonl. Funct. Anal. Appl.*, 7:361–404, 2002.

[24] P. Řehák, Hardy inequality on time scales and its application to half-linear dynamic equations, *Arch. Ineq. Appl.*, 2005:495–507, 2005.

[25] P. Řehák, Function sequence technique for half-linear dynamic equations on time scales, *Panam. Math. J.*, 16:31–56, 2006.

[26] P. Řehák, How the constants in Hille-Nehari theorems depend on time scales, *Adv. Difference Equ.*, 2006:1–15, 2006.

[27] P. Řehák, A critical oscillation constant as a variable of time scales for half-linear dynamic equations, to appear in Math. Slov.

[28] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

[29] G. Zhang, S. S. Cheng, A necessary and sufficient oscillation for the discrete Euler equation, *Panam. Math. J.*, 9:29–34, 1999.